A Relative Laplacian spectral recursion

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Dedicated to Richard Stanley on the occasion of his 60th birthday.

Abstract

The Laplacian spectral recursion, satisfied by matroid complexes and shifted complexes, expresses the eigenvalues of the combinatorial Laplacian of a simplicial complex in terms of its deletion and contraction with respect to vertex e, and the relative simplicial pair of the deletion modulo the contraction. We generalize this recursion to relative simplicial pairs, which we interpret as convex subsets of the Boolean algebra. The deletion modulo contraction term is replaced by the result of removing from the convex set Φ all pairs of faces in Φ that differ only by vertex e.

We show that shifted pairs and some matroid pairs satisfy this recursion. We also show that the class of convex sets satisfying this recursion is closed under a wide variety of operations, including duality and taking skeleta.

1 Introduction

There are two good reasons to extend the Laplacian spectral recursion from simplicial complexes to relative simplicial pairs.

The spectral recursion for simplicial complexes expresses the eigenvalues of the combinatorial Laplacian $\partial\partial^* + \partial^*\partial$ of a simplicial complex Δ in terms of the eigenvalues of its deletion $\Delta - e$, contraction Δ/e, and an "error term" $(\Delta - e, \Delta/e)$. This recursion does not hold for all simplicial complexes, but does hold for independence complexes of matroids and shifted simplicial complexes [2]. In each case, the deletion and contraction are again matroids or shifted complexes, respectively, but the error term is only a relative simplicial pair of the appropriate kind of complexes. Being able to apply the recursion to relative simplicial pairs, such as the error term, would make the spectral recursion truly recursive.

A more compelling reason comes from duality, the idea that a Boolean algebra looks the same upside-down as it does right-side-up. Many operations preserve the property of satisfying the spectral recursion [2], but the *dual* Δ^* (see equation (2)) of a simplicial complex Δ, which is an order filter instead of a simplicial complex, satisfies only a slightly modified version of the spectral recursion when Δ satisfies the spectral recursion [2, Theorem 6.3]. Relative simplicial pairs include both simplicial complexes and order filters as special cases, and so suggest a way to unify the two versions of the spectral recursion.

Furthermore, the Laplacian itself is self-dual (Section 3), and so we will state and prove most of our results in self-dual form. The first step is to think of relative simplicial pairs as *convex sets* in the Boolean algebra of subsets of the set of vertices, since the dual of a convex set is again convex, in a very natural way. To further emphasize this symmetry, we represent these convex sets by vertically symmetric capital Greek letters, such as Φ and Θ. When we extend the spectral recursion from simplicial complexes to convex sets, the ideas of deletion and contraction generalize easily and naturally. But, even with duality as a guide, it is not as clear what should replace $(\Delta - e, \Delta/e)$ as the error term.

The answer turns out to be to remove from Φ all the pairs $\{F, F \dot{\cup} e\}$ in Φ. This simple operation, which we will call the *reduction* of Φ with respect to e, and denote by $\Phi||e$, has a few remarkable (but easy to prove) properties that will allow us to show that it is the correct error term. To start, it is clear that this operation is self-dual, which goes nicely with deletion and contraction being more or less duals of one another. Somewhat more surprising is that $\Phi||e$ is still convex, albeit in two separate components (Lemma 2.3 and Proposition 2.4). Finally, it is necessary for the error term to have the same homology as Φ itself (see Lemma 3.3), and $\Phi||e$ satisfies this as well (equation (3)). Perhaps reduction deserves further investigation, beyond Laplacians, since it is easy to compute, preserves homology, and produces a smaller convex set. (Reduction is a special case of collapsing induced by a discrete Morse function coming from an acyclic, or Morse, matching, $F \leftrightarrow F \dot{\cup} e$, for all possible F; see [1, 5].)

Of course, the most important evidence that reduction is the right answer is that the spectral recursion, with $\Phi||e$ as the error term (equation (4)), holds for a variety of convex sets. We are able to prove (Theorem 5.12) that it does hold for shifted convex sets, that is, relative simplicial pairs of complexes, each of which is shifted on the same ordered vertex set. The analogue for matroids would be relative simplicial pairs of matroids connected by a strong map, and here our success is more limited. Although experimental evidence supports the conjecture that the spectral recursion holds for all such pairs (Conjecture 6.3), we are only able to prove it in the case where the difference in ranks between the matroids is 1 (Theorem 6.2). This does at least provide strong evidence that $\Phi||e$ is the correct error term. Further evidence is that the property of satisfying the spectral recursion is closed under many operations on convex sets (Section 3), including duality (Proposition 3.7).

We review convex sets and define operations on them, including reduction, in Section

2. We review Laplacians and introduce the spectral recursion for convex sets in Section 3. Our main results, that skeleta preserve the property of satisfying the spectral recursion (Theorem 4.7), and that shifted convex sets and certain matroid pairs satisfy the spectral recursion (Theorems 5.12 and 6.2), are the foci of Sections 4, 5, and 6, respectively.

2 Convex sets

In this section, we review convex sets, and extend many simplicial complex operations to convex sets. We also introduce the reduction operation $(\Phi||e)$, and establish some of its properties.

Definition. Let 2^E denote the Boolean algebra of subsets of finite set E. Recall that $\Phi \subseteq 2^E$ is *convex* if $F \subseteq G \subseteq H$ and $F, H \in \Phi$ together imply $G \in \Phi$. We will call the set E the *ground set* of Φ, individual members of E the *vertices* of Φ, and members of Φ the *faces* of Φ. Note that v may be a vertex of Φ without being in *any* face of Φ. In this case we call v a *loop* of Φ. (This is in analogy to a loop of a matroid.)

Convex sets are usually defined not just on $(2^E, \subseteq)$, as they are here, but on arbitrary partially ordered sets. (Indeed, the proof of Lemma 5.2 makes use of a "convex set" on 2^E with respect to a different partial order.) But what makes Laplacians work so well on convex sets of $(2^E, \subseteq)$ is that $(2^E, \subseteq)$ supports a chain complex (Lemma 2.6, and the preceding discussion), and so we restrict our attention to this case. Hereinafter, the word "convex" will only refer to convex sets of $(2^E, \subseteq)$.

An important special case of a convex set is a *simplicial complex*. As usual, $\Delta \subseteq 2^E$ is a simplicial complex if $G \subseteq H$ and $H \in \Delta$ together imply $G \in \Delta$. It is obvious that simplicial complexes may be defined as convex sets containing the empty face $\emptyset$. Of course, our motivation runs in the oppposite direction; convex sets are usually presented as pairs of simplicial complexes. If $\Delta' \subseteq \Delta$ are a pair of simplicial complexes on the same ground set, then the *relative simplicial pair* (Δ, Δ') is simply the set difference $\Delta - \Delta'$. It is easy to check that, if $\Phi \subseteq 2^E$, then Φ is convex precisely when

$$\Phi = (\Delta, \Delta') \tag{1}$$

for some simplicial complexes Δ, Δ', though the following example shows that Δ and Δ' are not unique.

Example 2.1. Let Φ be the convex set on ground set $\{1, \ldots, 6\}$ consisting of the faces $\{12456, 1245, 1246, 1356, 124, 135, 136\}$. (Here, we are omitting brackets and commas on individual faces, for clarity.) It is easy to check that Φ is convex (see also Example 2.2). In equation (1) we could set Δ to be the simplicial complex with facets (maximal faces) $\{12456, 1356\}$, and Δ' to be the simplicial complex with facets $\{1256, 1456, 2456, 356, 13\}$. But we could add the face 34 to both Δ and Δ', and they would still be simplicial complexes such that $\Phi = (\Delta, \Delta')$.

Although convex sets are the same as relative simplicial pairs, we will strive to put all of our results in the language of convex sets rather than relative simplicial pairs. One reason is the potential difficulty in describing properties of the convex set in terms of the pair of simplicial complexes which are not necessarily unique, as demonstrated in Example 2.1. Another, as alluded to in the Introduction, is to better take advantage of duality. The *dual* of a convex set Φ on ground set E is

$$\Phi^* = \{E - F \colon F \in \Phi\}. \tag{2}$$

It is easy to see that the dual of a convex set is again convex, and that $\Phi^{**} = \Phi$.

It is also easy to see the intersection of two convex sets is again convex, but we have to be more careful with union, even with disjoint union. If Φ and Θ are disjoint convex sets with faces $F \in \Phi$ and $G \in \Theta$ such that $F \subseteq G$, then $\Phi \mathbin{\dot\cup} \Theta$, the disjoint union of Φ and Θ, might not be convex. We thus define two convex sets Φ and Θ to be *totally unrelated* if $F \not\subseteq G$ and $G \not\subseteq F$ whenever $F \in \Phi$ and $G \in \Theta$, and, in this case, define the *direct sum* of Φ and Θ to be $\Phi \oplus \Theta = \Phi \mathbin{\dot\cup} \Theta$. It is easy to check that the direct sum of two convex sets is again convex.

Example 2.2. It is easy to see that the convex set Φ in Example 2.1 is a direct sum $\{12456, 1245, 1246, 124\} \oplus \{1356, 135, 136\}$. The components of the direct sum are indeed totally unrelated, even though they share many vertices.

The *join* of two convex sets Φ and Θ on disjoint ground sets is

$$\Phi * \Theta = \{F \mathbin{\dot\cup} G \colon F \in \Phi, G \in \Theta\}.$$

When Φ and Θ are simplicial complexes, this matches the usual definition of join. It is easy to see that the join of two convex sets is again convex. Some special cases of the join deserve particular attention. If Φ is convex and R is a set disjoint from the vertices of Φ, then define

$$R \circ \Phi = \{R\} * \Phi = \{R \mathbin{\dot\cup} F \colon F \in \Phi\},$$

the join of Φ with the convex set whose only face is R. If v a vertex not in Φ, then the *cone* of Φ is

$$v * \Phi = \{v, \emptyset\} * \Phi,$$

the join of Φ with the convex set whose two faces are v and the empty face. The *open star* of Φ is $v \circ \Phi$. Note that

$$v * \Phi = \Phi \mathbin{\dot\cup} (v \circ \Phi).$$

Deletion and contraction are well-known concepts from matroid theory, and were easily extended to simplicial complexes in [2]. Now we further extend to convex sets. If Φ is convex and e is a vertex of Φ, then the *deletion* and *contraction* of Φ by e are, respectively,

$$\begin{aligned} \Phi - e &= \{F \in \Phi \colon e \notin F\}; \\ \Phi/e &= \{F - e \colon F \in \Phi, e \in F\}. \end{aligned}$$

As opposed to the simplicial complex case, Φ/e is not necessarily a subset of $\Phi - e$. As with simplicial complexes, neither Φ/e nor $\Phi - e$ contains e in any of its faces, though we stil consider e to a vertex, albeit a loop, in each case. It is also easy to check that $\Phi - e$ and Φ/e are convex when Φ is convex. Note that

$$\begin{aligned}(\Phi - e)^* &= \{E - F \colon F \in \Phi, e \notin F\} = \{E - F \colon F \in \Phi, e \in E - F\} \\ &= e \circ (\Phi^*/e)\end{aligned}$$

and, similarly,

$$\begin{aligned}(\Phi/e)^* &= \{E - (F - e) \colon F \in \Phi, e \in F\} = \{(E - F) \dot\cup e \colon F \in \Phi, e \notin E - F\} \\ &= e \circ (\Phi^* - e).\end{aligned}$$

We are now ready to define reduction, which will be a focal point for most of the rest of our work.

Definition. If Φ is convex and e is a vertex of Φ, then the *star* of e in Φ is

$$\operatorname{st}_\Phi e = \bigcup_{F, F \dot\cup e \in \Phi} \{F, F \dot\cup e\} = e * ((\Phi - e) \cap (\Phi/e)),$$

and the *reduction* of Φ by e is

$$\Phi || e = \Phi - \operatorname{st}_\Phi e.$$

When Φ is a simplicial complex, $\operatorname{st}_\Phi e$ matches the usual definition. It is easy to check that $\operatorname{st}_\Phi e$ is convex when Φ is convex, but $\Phi||e$ takes a little more work.

Lemma 2.3. *If Φ is convex and e is a vertex of Φ, then $\Phi||e$ is again convex.*

Proof. Assume otherwise, so $F \subseteq G \subseteq H$, and $F, H \in \Phi||e$, but $G \notin \Phi||e$. Thus $F, H \in \Phi$, and, since Φ is convex, $G \in \Phi$.

If $e \notin G$, then $e \notin F$, and then $F \subseteq F \dot\cup e \subseteq G \dot\cup e$. But also $G \notin \Phi||e$ implies $G \dot\cup e \in \Phi$. Then, since Φ is convex, $F \dot\cup e \in \Phi$, which contradicts $F \in \Phi||e$.

Similarly, if instead $e \in G$, then $e \in H$, and then $G - e \subseteq H - e \subseteq H$. But also $G \notin \Phi||e$ implies $G - e \in \Phi$. Then since Φ is convex, $H - e \in \Phi$, which contradicts $H \in \Phi||e$. $\square$

Proposition 2.4. *If Φ is convex and e is a vertex of Φ, then $\Phi||e$ is the direct sum*

$$\begin{aligned}\Phi||e &= \{F \in \Phi||e \colon e \notin F\} \oplus \{G \in \Phi||e \colon e \in G\} \\ &= \{F \in \Phi \colon e \notin F, F \dot\cup e \notin \Phi\} \ \oplus \{G \in \Phi \colon e \in G, G - e \notin \Phi\}.\end{aligned}$$

Proof. To show $\Phi||e$ is the desired direct sum, let $F, G \in \Phi||e$ such that $e \notin F$, and $e \in G$; we must show F and G are unrelated. Since $e \in G \backslash F$, we know $G \not\subseteq F$, so assume $F \subseteq G$. Then $F \subseteq F \dot\cup e \subseteq G$. Since $F, G \in \Phi$, then also $F \dot\cup e \in \Phi$, which contradicts $F \in \Phi||e$. $\square$

Example 2.5. Let Θ be the convex set consisting of all faces $F \subseteq \{1, \ldots, 6\}$ such that F is a subset of 12356 or 12456, but also a superset of 12, 135, or 136. It is not hard to check that $\Theta||3$ is the convex set Φ of Examples 2.1 and 2.2. The direct sum decomposition of $\Phi = \Theta||3$ given in Example 2.2 is the one guaranteed by Proposition 2.4.

In the special case where Φ is a simplicial complex, $\{G \in \Phi||e \colon e \in G\}$ is empty and $\Phi||e = (\Phi - e, \Phi/e)$. It is easy to check that $(\mathrm{st}_\Phi\, e)^* = \mathrm{st}_{(\Phi^*)}\, e$, and so $(\Phi||e)^* = \Phi^*||e$.

We review our notation for boundary maps and homology groups of simplicial complexes (as in *e.g.*, [12, Chapter 1]). As usual, let Φ_i denote the set of i-dimensional faces of Φ, and let $C_i = C_i(\Phi; \mathbb{R}) := C_i(\Delta; \mathbb{R})/C_i(\Delta'; \mathbb{R})$ denote the i-dimensional oriented $\mathbb{R}$-chains of $\Phi = (\Delta, \Delta')$, *i.e.*, the formal $\mathbb{R}$-linear sums of oriented i-dimensional faces $[F]$ such that $F \in \Phi_i$. Let $\partial_{\Phi;i} = \partial_i \colon C_i \to C_{i-1}$ denote the usual (signed) *boundary operator*. Via the natural orthonormal bases Φ_i and Φ_{i-1} for $C_i(\Phi; \mathbb{R})$ and $C_{i-1}(\Phi; \mathbb{R})$, respectively, the boundary operator ∂_i has an adjoint map called the *coboundary operator*, $\partial_i^* \colon C_{i-1}(\Phi; \mathbb{R}) \to C_i(\Phi; \mathbb{R})$; *i.e.*, the matrices representing ∂ and ∂^* in the natural bases are transposes of one another.

As long as Φ is convex, $\mathcal{C}(\Phi) = C_\bullet(\Phi; \mathbb{R})$ supports an (algebraic) chain complex, *i.e.*, $\partial_{i-1}\partial_i = 0$. This simple observation is the key step to several results that follow. To start with, the usual homology groups $\tilde{H}_i(\Phi; \mathbb{R}) = \ker \partial_i / \operatorname{im} \partial_{i+1}$ are well-defined. Recall $\tilde{\beta}_i(\Phi) = \dim_{\mathbb{R}} \tilde{H}_i(\Phi; \mathbb{R})$.

Lemma 2.6. *If Φ is convex and e is a vertex of Φ, then*

$$\tilde{\beta}_i(\Phi||e) = \tilde{\beta}_i(\Phi)$$

for all i.

Proof. First note that $\mathrm{st}_\Phi\, e = e * ((\Phi - e) \cap (\Phi/e)) = e * (\Gamma, \Gamma') = (e * \Gamma, e * \Gamma')$ for some simplicial complexes Γ and Γ', and so is acyclic. Now, Φ, $\Phi||e$, and $\mathrm{st}_\Phi\, e$ are all convex, and thus support chain complexes; furthermore, by definition of $\Phi||e$,

$$0 \to \mathcal{C}(\mathrm{st}_\Phi\, e) \to \mathcal{C}(\Phi) \to \mathcal{C}(\Phi||e) \to 0$$

is a short exact sequence of chain complexes. The resulting long exact sequence in reduced homology (*e.g.*, [12, Section 24]),

$$\cdots \to \tilde{H}_i(\mathrm{st}_\Phi\, e) \to \tilde{H}_i(\Phi) \to \tilde{H}_i(\Phi||e) \to \tilde{H}_{i-1}(\mathrm{st}_\Phi\, e) \to \cdots,$$

becomes

$$\cdots \to 0 \to \tilde{H}_i(\Phi) \to \tilde{H}_i(\Phi||e) \to 0 \to \cdots,$$

and the result follows immediately. □

We collect here the easy facts we need about how direct sums and joins (and thus cones and open stars) of convex sets interact with deletion, contraction, stars, and reduction. Each fact is either immediate from the relevant definitions, or a routine calculation. For the identities with the join, we assume e is a vertex of Φ.

$$\begin{aligned}(\Phi \oplus \Theta) - e &= (\Phi - e) \oplus (\Theta - e) & \qquad (\Phi * \Theta) - e &= (\Phi - e) * \Theta \\ (\Phi \oplus \Theta)/e &= (\Phi/e) \oplus (\Theta/e) & (\Phi * \Theta)/e &= (\Phi/e) * \Theta \\ \mathrm{st}_{(\Phi\oplus\Theta)}\, e &= \mathrm{st}_\Phi\, e \oplus \mathrm{st}_\Theta\, e & \mathrm{st}_{(\Phi*\Theta)}\, e &= \mathrm{st}_\Phi\, e * \Theta \\ (\Phi \oplus \Theta)||e &= (\Phi||e) \oplus (\Theta||e) & (\Phi * \Theta)||e &= (\Phi||e) * \Theta\end{aligned}$$

3 Laplacians

In this section, we define the Laplacian operators and the spectral recursion, develop the tools we will need later to work with them, and show that several operations on convex sets, including duality (Proposition 3.7), preserve the property of satisfying the spectral recursion.

Definition. The (*i-dimensional*) *Laplacian* of a convex set Φ is the linear operator $L_i(\Phi)\colon C_i(\Phi;\mathbb{R}) \to C_i(\Phi;\mathbb{R})$ defined by

$$L_i = L_i(\Phi) := \partial_{i+1}\partial_{i+1}^* + \partial_i^*\partial_i.$$

It is not hard to see that $L_i(\Phi)$ maps each face $[F]$ to a linear combination of faces in Φ *adjacent* to F, that is, faces in Φ of the form $F - v \dot\cup w$ for some (not necessarily distinct) vertices v, w, and such that $F - v \in \Phi$ or $F \dot\cup w \in \Phi$. For details on the coefficients of these linear combinations (in the simplicial complex case, though the ideas are similar for convex sets), see [3, equations (3.2)–(3.4)], but we will not need that level of detail here. For more information on Laplacians, also see, *e.g.*, [6, 9, 11].

Each of $\partial_{i+1}\partial_{i+1}^*$ and $\partial_i^*\partial_i$ is positive semidefinite, since each is the composition of a linear map and its adjoint. Therefore, their sum L_i is also positive semidefinite, and so has only non-negative real eigenvalues. (See also [6, Proposition 2.1].) These eigenvalues do not depend on the arbitrary ordering of the vertices of Φ, and are thus invariants of Φ; see, *e.g.*, [3, Remark 3.2]. Define $\mathbf{s}_i(\Phi)$ to be the multiset of eigenvalues of $L_i(\Phi)$, and define $m_\lambda(L_i(\Phi))$ to be the multiplicity of λ in $\mathbf{s}_i(\Phi)$.

The first result of combinatorial Hodge theory, which goes back to Eckmann [4], is that

$$m_0(L_i(\Phi)) = \tilde{\beta}_i(\Phi). \tag{3}$$

Though initially stated only for the case where Φ is a simplicial complex, there is a simple proof that only relies upon Φ supporting a chain complex, and so applies to all convex sets Φ; see [6, Proposition 2.1].

A natural generating function for the Laplacian eigenvalues of a convex set Φ is

$$S_\Phi(t,q) := \sum_{i\geq 0} t^i \sum_{\lambda\in\mathbf{s}_{i-1}(\Phi)} q^\lambda = \sum_{i,\lambda} m_\lambda(L_{i-1}(\Phi))t^i q^\lambda.$$

We call S_Φ the *spectrum polynomial* of Φ. It was introduced (with slightly different indexing) for matroids in [9], and extended to relative simplicial pairs in [2]. Although S_Φ is defined for any convex Φ, it is only truly a polynomial when the Laplacian eigenvalues are not only non-negative, but integral as well. This will be true for the cases we are concerned with, primarily shifted convex sets [2], matroids [9], and matroid pairs $(M - e, M/e)$ [2].

Let F be a face in a convex set Φ. As usual, the *boundary* of F in Φ is the collection of faces $\{F - v \in \Phi \colon v \in F\}$. Similarly, the *coboundary* of F in Φ is the collection of faces $\{F \dot\cup w \in \Phi \colon w \notin F\}$. It is not hard to see that $\partial_{(\Phi^*)}$ and $(\partial_\Phi)^*$ each map $[F]$ to a linear combination of faces in the coboundary of F in Φ. In fact, [2, Lemma 6.1] states that $\partial_{(\Phi^*)}$ and $(\partial_\Phi)^*$ are isomorphic, up to an easy change of basis (multiplying some basis elements by -1). The easy corollary [2, Corollary 6.2] is that $L_i(\Phi)$ is, modulo that same change of basis, isomorphic to $L_{n-i-2}(\Phi^*)$. Therefore [2, equation (28)],

$$S_{\Phi^*}(t,q) = t^{|E|} S_\Phi(t^{-1}, q).$$

By [2, Corollary 4.3],

$$S_{\Phi * \Theta} = S_\Phi S_\Theta;$$

it follows then that

$$S_{R \circ \Phi} = t^{|R|} S_\Phi.$$

The following is the analogue for direct sums. It is simpler than the formula for disjoint union of simplicial complexes [2, Lemma 6.9], because even disjoint simplicial complexes share the empty face.

Lemma 3.1. *If Φ and Θ are convex sets such that $\Phi \oplus \Theta$ is well-defined, then*

$$\mathbf{s}_i(\Phi \oplus \Theta) = \mathbf{s}_i(\Phi) \cup \mathbf{s}_i(\Theta),$$

the multiset union of $\mathbf{s}_i(\Phi)$ and $\mathbf{s}_i(\Theta)$, and $S_{\Phi\oplus\Theta} = S_\Phi + S_\Theta$.

Proof. Since no face in Θ is related to any face in Φ, there are no adjacencies between faces in Φ and faces in Θ, nor do any of the faces in Θ change any adjacencies in Φ. Similarly, no faces in Φ change any adjacencies in Θ, and we conclude $L_i(\Phi \oplus \Theta) = L_i(\Phi) \oplus L_i(\Theta)$. Thus $\mathbf{s}_i(\Phi \oplus \Theta) = \mathbf{s}_i(\Phi) \cup \mathbf{s}_i(\Theta)$, and so $S_{\Phi\oplus\Theta} = S_\Phi + S_\Theta$. $\square$

Following [3], let the equivalence relation $\boldsymbol{\lambda} \mathrel{\mathring{=}} \boldsymbol{\mu}$ on multisets $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ denote that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ agree in the multiplicities of all of their *non-zero parts*, *i.e.*, that they coincide except for possibly their number of zeros.

Lemma 3.2. *If Φ and Θ are two convex sets such that $\Phi = \Theta \dot\cup \mathcal{N}$, where $\mathcal{N}$ is a collections of faces with neither boundary nor coboundary in Φ, then $\mathbf{s}_i(\Phi) \mathrel{\mathring{=}} \mathbf{s}_i(\Theta)$.*

Proof. Since Φ is convex, the faces in $\mathcal{N}$ are not related to any other face in Φ. Thus $\Phi = \Theta \oplus \mathcal{N}$. Furthermore, since the faces in $\mathcal{N}$ are not related to each other, $L_i(\mathcal{N})$ is the zero matrix for all i, and so $\mathbf{s}_i(\mathcal{N})$ consists of all 0's. Now apply Lemma 3.1. $\square$

Definition. We will say that a convex set Φ *satisfies the spectral recursion with respect to* e if e is a vertex of Φ and

$$S_\Phi(t,q) = qS_{\Phi-e}(t,q) + qtS_{\Phi/e}(t,q) + (1-q)S_{\Phi||e}(t,q). \tag{4}$$

We will say Φ *satisfies the spectral recursion* if Φ satisfies the spectral recursion with respect to every vertex in its ground set. (Note that Lemma 3.5 below means we need not be too particular about the ground set of Φ.)

When Φ is a simplicial complex, $\Phi||e$ becomes $(\Phi - e, \Phi/e)$, and equation (4) immediately reduces to the spectral recursion for simplicial complexes in [2].

The statement and proof of the following lemma strongly resemble their simplicial complex counterparts [2, Theorem 2.4 and Corollary 4.8]. Here as there, specializations of the spectrum polynomial reduce it to nice invariants of the convex set, and reduce the spectral recursion to basic recursions for those invariants. We sketch the proof in order to state what the spectrum polynomial and spectral recursion reduce to in each case.

Lemma 3.3. *The spectral recursion holds for all convex sets when* $q = 0$*,* $q = 1$*,* $t = 0$*, or* $t = -1$*.*

Proof. If $q = 0$, then by equation (3), S_Φ becomes $\sum_i t^i \tilde{\beta}_{i-1}(\Phi)$, as in [2, Theorem 2.4]. The spectral recursion then reduces to the identity $\tilde{\beta}_i(\Phi) = \tilde{\beta}_i(\Phi||e)$, which we established in Lemma 2.6.

If $q = 1$, then S_Φ becomes $\sum_i t^i f_{i-1}(\Phi)$, as in [2, Theorem 2.4], where $f_i(\Phi) = |\Phi_i|$. The spectral recursion then reduces to the easy identity

$$f_i(\Phi) = f_i(\Phi - e) + f_{i-1}(\Phi/e). \tag{5}$$

If $t = 0$, then S_Φ becomes $q^{f_0(\Phi)}$ if $\emptyset \in \Phi$ (as in [2, Theorem 2.4]), but becomes 0 otherwise. If $\emptyset \notin \Phi$, then every term in the spectral recursion becomes 0; if, on the other hand, $\emptyset \in \Phi$, then, as in [2, Theorem 2.4], the spectral recursion reduces to the trivial observation that $f_0(\Phi) = f_0(\Phi - e)$ if e is not a face of Φ, but $f_0(\Phi) = 1 + f_0(\Phi - e)$ if e is a face of Φ.

If $t = -1$, then S_Φ becomes $\chi(\Phi) = \sum_i (-1)^i f_i(\Phi) = \sum_i (-1)^i \tilde{\beta}_i(\Phi)$, the Euler characteristic of Φ, by [2, Corollary 4.8]. The spectral recursion now reduces to two easy identities about Euler characteristic: that $\chi(\Phi) = \chi(\Phi||e)$, which follows from Lemma 2.6; and that $\chi(\Phi) = \chi(\Phi - e) - \chi(\Phi/e)$, which follows from the identity (5) above. □

If Φ is convex and e is a vertex of Φ, define

$$\mathcal{S}_i(\Phi, e) = [t^i](S_\Phi - qS_{\Phi-e} - qtS_{\Phi/e} - (1-q)S_{\Phi||e}),$$

where $[t^i]p$ denotes the coefficient of t^i in polynomial p. Clearly, Φ satisfies the spectral recursion with respect to e precisely when $\mathcal{S}_i(\Phi, e) = 0$ for all i.

Lemma 3.4. *Let Φ and Θ be convex sets, each with vertex e, such that*

$$s_i(\Phi) \mathrel{\mathring{=}} \mathbf{s}_j(\Theta), \qquad s_i(\Phi - e) \mathrel{\mathring{=}} \mathbf{s}_j(\Theta - e),$$
$$s_i(\Phi||e) \mathrel{\mathring{=}} \mathbf{s}_j(\Theta||e), \text{ and} \qquad s_{i-1}(\Phi/e) \mathrel{\mathring{=}} \mathbf{s}_{j-1}(\Theta/e).$$

Then $\mathcal{S}_i(\Phi, e) = \mathcal{S}_j(\Theta, e)$.

Proof. Translating the $\mathring{=}$ assumptions to generating functions,

$$[t^i]S_\Phi = [t^j]S_\Theta + C_1, \qquad [t^i]S_{\Phi-e} = [t^j]S_{\Theta-e} + C_2,$$
$$[t^i]S_{\Phi||e} = [t^j]S_{\Theta||e} + C_3, \text{ and} \qquad [t^{i-1}]S_{\Phi/e} = [t^{j-1}]S_{\Theta/e} + C_4,$$

where C_1, C_2, C_3, and C_4 are constants. It is then easy to compute

$$\mathcal{S}_i(\Phi, e) - \mathcal{S}_j(\Theta, e) = (C_1 - C_3) + q(C_3 - C_2 - C_4).$$

This makes $\mathcal{S}_i(\Phi, e) - \mathcal{S}_j(\Theta, e)$ a linear polynomial in q. But by Lemma 3.3, $\mathcal{S}_i(\Phi, e) - \mathcal{S}_j(\Theta, e) = 0$ when $q = 0$ and when $q = 1$. Therefore $\mathcal{S}_i(\Phi, e) - \mathcal{S}_j(\Theta, e)$ must be identically 0, as desired. □

The following two results are easy to verify directly; the third is not much harder.

Lemma 3.5. *If Φ is convex and e is a loop, then Φ satisfies the spectral recursion with respect to e.*

Lemma 3.6. *The convex set with only a single face, and the convex set whose only two faces are a single vertex and the empty face, each satisfy the spectral recursion.*

Proposition 3.7. *Let Φ be a convex set with vertex e. If Φ satisfies the spectral recursion with respect to e, then so does Φ^*.*

Proof. Calculate

$$\begin{aligned}
S_{\Phi^*}(t, q) &= t^n S_\Phi(t^{-1}, q) \\
&= t^n (q S_{\Phi-e}(t^{-1}, q) + q t^{-1} S_{\Phi/e}(t^{-1}, q) + (1-q) S_{\Phi||e}(t^{-1}, q)) \\
&= q S_{(\Phi-e)^*}(t, q) + q t^{-1} S_{(\Phi/e)^*}(t, q) + (1-q) S_{(\Phi||e)^*}(t, q) \\
&= q S_{e\circ(\Phi^*/e)}(t, q) + q t^{-1} S_{e\circ(\Phi^*-e)}(t, q) + (1-q) S_{\Phi^*||e}(t, q) \\
&= q t S_{\Phi^*/e}(t, q) + q S_{\Phi^*-e}(t, q) + (1-q) S_{\Phi^*||e}(t, q).
\end{aligned}$$

□

Similar routine calculations establish the following two lemmas.

Lemma 3.8. *If Φ and Θ are convex sets that satisfy the spectral recursion with respect to e, and such that $\Phi \oplus \Theta$ is well-defined, then $\Phi \oplus \Theta$ satisfies the spectral recursion with respect to e.*

Lemma 3.9. *If Φ is a convex set that satisfies the spectral recursion with respect to e, and Θ is another convex set such that $\Phi * \Theta$ is well-defined, then $\Phi * \Theta$ satisfies the spectral recursion with respect to e.*

Corollary 3.10. *Let Φ be a convex set. If Φ satisfies the spectral recursion, then so do $v * \Phi$ and $R \circ \Phi$.*

Proof. Combine Lemmas 3.6 and 3.9 □

4 Skeleta

The main goal of this section is to show that taking skeleta preserves the property of satisfying the spectral recursion (Theorem 4.7). A key step is to show that skeleta and reduction interact reasonably well (Corollary 4.3).

Definition. We will say a convex set Φ is (i, j)-*dimensional* when $i \leq \dim F \leq j$ for all $F \in \Phi$. Note that it is *not* necessary for there to be a face of every dimension between i and j. If Φ is convex, we define the (i, j)-*skeleton* to be

$$\Phi^{(i,j)} = \{F \in \Phi \colon i \leq \dim F \leq j\}.$$

It is immediate that

$$\begin{aligned} \Phi^{(i,j)} - e &= (\Phi - e)^{(i,j)}, \\ \Phi^{(i,j)}/e &= (\Phi/e)^{(i-1,j-1)}. \end{aligned}$$

The corresponding statement with reduction instead of deletion or contraction is not true. For instance, in Example 2.5, $1256 \notin (\Theta||3)^{(1,3)}$ (since $12356 \in \Theta$), but $1256 \in \Theta^{(1,3)}||3$ (since 12356 is 4-dimensional, and so is not in $\Theta^{(1,3)}$). On the other hand, it will not be hard to show that at least the non-zero eigenvalues of $\Phi^{(i,j)}||e$ and $(\Phi||e)^{(i,j)}$ coincide. We first need two easy technical lemmas.

Lemma 4.1. *Let Φ be a convex set with vertices e and v. If $F, F \dot\cup v \in \Phi^{(i,j)}||e$ for some $i < j$, then $F, F \dot\cup v \in \Phi||e$.*

Proof. First note that $v \neq e$, since, otherwise, $F, F \dot\cup v \in \Phi^{(i,j)}||e$ would be impossible. Thus, either e is a vertex of both F and $F \dot\cup v$, or e is a vertex of neither.

First assume $e \notin F, F \dot\cup v$. Then $F \in \Phi^{(i,j)}||e$ implies $F \dot\cup e \notin \Phi^{(i,j)}$, and so $F \dot\cup e \notin \Phi$ (note $\dim F < j$). But then $F \dot\cup \{v, e\} \notin \Phi$, since Φ is convex and $F \in \Phi$. Now, with $F \dot\cup e, F \dot\cup \{v, e\} \notin \Phi$, we conclude $F, F \dot\cup v \in \Phi||e$.

Next assume $e \in F, F \dot\cup v$. Then $F \dot\cup v \in \Phi^{(i,j)}||e$ implies $F \dot\cup v - e \notin \Phi^{(i,j)}$, and so $F \dot\cup v - e \notin \Phi$ (note $\dim F \dot\cup v > i$). But then $F - e \notin \Phi$, since Φ is convex and $F \dot\cup v \in \Phi$. Now, with $F - e, F \dot\cup v - e \notin \Phi$, we conclude $F, F \dot\cup v \in \Phi||e$. □

Lemma 4.2. *If Φ is convex and e is a vertex of Φ, then*

$$\Phi^{(i,j)}||e = (\Phi||e)^{(i,j)} \dot\cup \mathcal{N},$$

where $\mathcal{N}$ is a set of faces with neither boundary nor coboundary in $\Phi^{(i,j)}||e$.

Proof. First we show $(\Phi||e)^{(i,j)} \subseteq \Phi^{(i,j)}||e$. Let $F \in (\Phi||e)^{(i,j)}$, so $F \in \Phi||e$ and $F \in \Phi^{(i,j)}$. If $e \notin F$, then $F \dot\cup e \notin \Phi$, so $F \dot\cup e \notin \Phi^{(i,j)}$, and so $F \in \Phi^{(i,j)}||e$. If, on the other hand, $e \in F$, then $F - e \notin \Phi$, so $F - e \notin \Phi^{(i,j)}$, and so $F \in \Phi^{(i,j)}||e$.

Now let $G \in \Phi^{(i,j)}||e$, $G \notin (\Phi||e)^{(i,j)}$. By Lemma 4.1, for every $v \in G$, we have $G - v \notin \Phi^{(i,j)}||e$, and, for every $w \notin G$, we have $G \dot\cup w \notin \Phi^{(i,j)}||e$. Therefore, G has neither boundary nor coboundary in $\Phi^{(i,j)}||e$, as desired. □

Corollary 4.3. *If Φ is convex, e is a vertex of Φ, and $i < j$, then*

$$\mathbf{s}_k(\Phi^{(i,j)}||e) \stackrel{\circ}{=} s_k((\Phi||e)^{(i,j)}),$$

for all k.

Proof. Apply Lemma 3.2 to Lemma 4.2. □

The following two equations are from [3, equation (3.6)], where they are established for simplicial complexes, but they are just easy consequences of Φ supporting a chain complex.

$$\mathbf{s}_i(\Phi) \stackrel{\circ}{=} \mathbf{s}_i(\Phi^{(i-1,i)}) \cup \mathbf{s}_i(\Phi^{(i,i+1)}), \tag{6}$$

$$\mathbf{s}_{i-1}(\Phi^{(i-1,i)}) \stackrel{\circ}{=} \mathbf{s}_i(\Phi^{(i-1,i)}). \tag{7}$$

As a result of this second equation, if Φ is $(i-1,i)$-dimensional, we will let $\mathbf{s}(\Phi)$ refer to the $\stackrel{\circ}{=}$ equivalence class of $\mathbf{s}_{i-1}(\Phi) \stackrel{\circ}{=} \mathbf{s}_i(\Phi)$.

Lemma 4.4. *If Φ is an $(i-1,i)$-dimensional convex set with vertex e, then $\mathcal{S}_i(\Phi, e) = \mathcal{S}_{i-1}(\Phi, e)$.*

Proof. By equation (7), since Φ is $(i-1,i)$-dimensional, $\mathbf{s}_{i-1} \stackrel{\circ}{=} \mathbf{s}_i$ for Φ, $\Phi - e$, and $\Phi||e$. Similarly, $\mathbf{s}_{i-2} \stackrel{\circ}{=} \mathbf{s}_{i-1}$ for Φ/e. Now apply Lemma 3.4. □

Lemma 4.5. *If Φ is convex and e is a vertex of Φ, then*

$$\mathcal{S}_i(\Phi, e) = \mathcal{S}_i(\Phi^{(i-1,i)}, e) + \mathcal{S}_i(\Phi^{(i,i+1)}, e).$$

Proof. Let b and t be two new vertices not in Φ, and let

$$\Theta = (b \circ \Phi^{(i-1,i)}) \oplus (t \circ \Phi^{(i,i+1)}).$$

It is immediate that Θ is well-defined, since $b \neq t$. (Indeed, b and t are introduced precisely to make a direct sum of out $\Phi^{(i-1,i)}$ and $\Phi^{(i,i+1)}$.) It is easy to verify that

$$\begin{aligned}
\mathbf{s}_{i+1}(\Theta) &= \mathbf{s}_i(\Phi^{(i-1,i)}) \cup \mathbf{s}_i(\Phi^{(i,i+1)}) \stackrel{\circ}{=} \mathbf{s}_i(\Phi), \\
\mathbf{s}_{i+1}(\Theta - e) &= \mathbf{s}_i((\Phi - e)^{(i-1,i)}) \cup \mathbf{s}_i((\Phi - e)^{(i,i+1)}) \stackrel{\circ}{=} \mathbf{s}_i(\Phi - e), \\
\mathbf{s}_i(\Theta/e) &= \mathbf{s}_{i-1}((\Phi/e)^{(i-2,i-1)}) \cup \mathbf{s}_{i-1}((\Phi/e)^{(i-1,i)}) \stackrel{\circ}{=} \mathbf{s}_{i-1}(\Phi/e), \\
\mathbf{s}_{i+1}(\Theta||e) &= \mathbf{s}_i(\Phi^{(i-1,i)}||e) \cup \mathbf{s}_i(\Phi^{(i,i+1)}||e) \\
&\stackrel{\circ}{=} \mathbf{s}_i((\Phi||e)^{(i-1,i)}) \cup \mathbf{s}_i((\Phi||e)^{(i,i+1)}) \stackrel{\circ}{=} \mathbf{s}_i(\Phi||e);
\end{aligned}$$

in each case, the last $\stackrel{\circ}{=}$-equivalence is by equation (6). Then, by Lemma 3.4, $\mathcal{S}_{i+1}(\Theta, e) = \mathcal{S}_i(\Phi, e)$, and so now it is easy to verify

$$\begin{aligned}
\mathcal{S}_i(\Phi, e) &= \mathcal{S}_{i+1}(\Theta, e) \\
&= \mathcal{S}_{i+1}(b \circ \Phi^{(i-1,i)} \oplus t \circ \Phi^{(i,i+1)}, e) \\
&= \mathcal{S}_{i+1}(b \circ \Phi^{(i-1,i)}, e) + \mathcal{S}_{i+1}(t \circ \Phi^{(i,i+1)}, e) \\
&= \mathcal{S}_i(\Phi^{(i-1,i)}, e) + \mathcal{S}_i(\Phi^{(i,i+1)}, e).
\end{aligned}$$

□

Lemma 4.6. *Let Φ be a convex set with vertex e. If every skeleton $\Phi^{(i-1,i)}$ satisfies the spectral recursion with respect to e then so does Φ.*

Proof. This is an immediate corollary to Lemma 4.5 □

Theorem 4.7. *Let Φ be a convex set with vertex e. If Φ satisfies the spectral recursion with respect to e, then so does every skeleton $\Phi^{(i,j)}$.*

Proof. By Lemma 4.6, it suffices to prove that every $\Phi^{(i,i+1)}$ satisfies the spectral recursion with respect to e, which we now do by induction on i.

If $i \leq -2$, then $\Phi^{(i,i+1)}$ is either the convex set whose only face is the empty face, or the empty convex set with no faces whatsoever. Either way, $\Phi^{(i,i+1)}$ trivially satisfies the spectral recursion.

If $i > -2$, then, by induction, $\mathcal{S}_i(\Phi^{(i-1,i)}, e) = 0$, and by hypothesis, $\mathcal{S}_i(\Phi, e) = 0$. Then by Lemma 4.5, $\mathcal{S}_i(\Phi^{(i,i+1)}, e) = 0$, and so $\Phi^{(i,i+1)}$ satisfies the spectral recursion with respect to e, by Lemma 4.4. □

5 Shifted convex sets

Our main goal in this section is to show that relative simplicial pairs that are shifted (on the same vertex order) satisfy the spectral recursion (Theorem 5.12). The key step is the construction of another convex set Φ^- that satisfies the spectral recursion when Φ does; this resembles, but is more involved than, a construction in the proof of the simplicial complex case [2, Lemma 4.22]. We first translate shifted relative simplicial pairs to shifted convex sets, and show that the dual of a shifted convex set is again convex and shifted (Proposition 5.6).

Definition. If $F = \{f_1 < \cdots < f_k\}$ and $G = \{g_1 < \cdots < g_k\}$ are k-subsets of integers, then $F \leq_C G$ under the *componentwise partial order* if $f_p \leq g_p$ for all p. A simplicial complex Δ on a ground set of integers is *shifted* if $G \leq_C H$ and $H \in \Delta$ together imply $G \in \Delta$. A convex set Φ is *shifted* when $\Phi = (\Delta, \Delta')$, for some shifted simplicial complexes Δ and Δ'.

We would like to replace this definition of shifted convex set, which depends on the simplicial complexes involved, to one that depends only on the convex set itself. In order to do this, we will need a single partial order that combines the (separate) conditions of Δ being shifted, and Δ being a simplicial complex, an idea implicit in the work of Klivans (see *e.g.*, [8, Figure 1] or [7, Figure 1]). If $F = \{f_1 < \cdots < f_k\}$ and $G = \{g_1 < \cdots < g_m\}$, then $F \leq_S G$ under the *shifted partial order* when $k \leq m$ and $f_{p+m-k} \leq g_p$ for all $1 \leq p \leq k$. In particular, it is easy to see that if $F \subseteq G$ or $F \leq_C G$, then $F \leq_S G$.

Lemma 5.1. *If $\Delta \subseteq 2^E$, then the following are equivalent:*

1. *Δ is a shifted simplicial complex; and*
2. *$F \leq_S H$ and $H \in \Delta$ together imply $F \in \Delta$.*

Proof. That statement 2 implies statement 1 is an easy exercise. To see the reverse implication, assume Δ is a shifted simplicial complex and that $F \leq_S H \in \Delta$, and let G consist of the last $|F|$ elements of H. Then it is easy to see that $F \leq_C G \subseteq H$. Therefore $G \in \Delta$, and, consequently, $F \in \Delta$. $\square$

Lemma 5.2. *If $\Phi \subseteq 2^E$, then the following are equivalent:*

1. *Φ is convex and shifted; and*
2. *$F \leq_S G \leq_S H$ and $F, H \in \Phi$ together imply $G \in \Phi$.*

Proof. Thanks to Lemma 5.1, the proof is entirely analogous to that of equation (1), but with $\leq_S$ instead of $\subseteq$. $\square$

We record here a few fundamental properties of the partial order $\leq_S$, omitting those proofs that are especially easy.

Lemma 5.3. *If $v \notin F, G$, then $F \leq_S G$ iff $F \dot{\cup} v \leq_S G \dot{\cup} v$.*

Corollary 5.4. *If $A \cap F = A \cap G = \emptyset$, then $F \leq_S G$ iff $F \dot{\cup} A \leq_S G \dot{\cup} A$.*

Lemma 5.5. *If $F, G \subseteq E$, then $F \leq_S G$ iff $E - G \leq_S E - F$.*

Proof. Let $A = F \cap G$ and $B = (E - F) \cap (E - G)$, and let $F' = F - A$ and $G' = G - A$. Thus $F = F' \dot{\cup} A$, $G = G' \dot{\cup} A$, $E - F = G' \dot{\cup} B$, and $E - G = F' \dot{\cup} B$. Then by Corollary 5.4 twice, $F \leq_S G$ iff $F' \leq_S G'$ iff $E - G \leq_S E - F$. $\square$

Proposition 5.6. *If Φ is a shifted convex set, then so is Φ^*.*

Proof. Assume $F \leq_S G \leq_S H$ and $F, H \in \Phi^*$. Then $E - H \leq_S E - G \leq_S E - F$, by Lemma 5.5, and $E - F, E - H \in \Phi$. Therefore $E - G \in \Phi$, and so $G \in \Phi^*$. $\square$

Lemma 5.7. *If $F \leq_S G$ and $\dim F < \dim G$, then $F \dot\cup 1 \leq_S G$.*

We now turn our attention to proving that shifted convex sets satisfy the spectral recursion. We start with a definition that does not rely upon Φ being shifted, but which will be very useful when Φ is shifted. If Φ is an $(i-1, i)$-dimensional convex set with vertex 1, then define

$$\Phi^- = \Phi - \mathcal{N}_\Phi,$$

where

$$\mathcal{N}_\Phi = \{F \in \Phi_i \colon 1 \in F, F - 1 \notin \Phi\} \dot\cup \{F \in \Phi_{i-1} \colon 1 \notin F, F \dot\cup 1 \notin \Phi\}.$$

Computing Φ^- dimension by dimension, we see that, equivalently,

$$\begin{aligned}
\Phi^- &= \{F \in \Phi_i \colon 1 \notin F\} \dot\cup \{F \in \Phi_i \colon 1 \in F, F - 1 \in \Phi\} \\
&\quad \dot\cup \{F \in \Phi_{i-1} \colon 1 \notin F, F \dot\cup 1 \in \Phi\} \dot\cup \{F \in \Phi_{i-1} \colon 1 \in F\} \\
&= (\Phi_i - 1) \dot\cup (1 \circ ((\Phi_i/1) \cap (\Phi_{i-1} - 1))) \\
&\quad \dot\cup ((\Phi_i/1) \cap (\Phi_{i-1} - 1)) \dot\cup (1 \circ (\Phi_{i-1}/1)) \\
&= (\Phi_i - 1) \dot\cup (1 * ((\Phi_i/1) \cap (\Phi_{i-1} - 1))) \dot\cup (1 \circ (\Phi_{i-1}/1)).
\end{aligned} \tag{8}$$

Lemma 5.8. *If Φ is a shifted $(i-1, i)$-dimensional convex set on ground set $\{1, \ldots, n\}$, then the faces of $\mathcal{N}_\Phi$ have neither boundary nor coboundary in Φ.*

Proof. Let $F \in \mathcal{N}_\Phi$. We split the proof into two cases, depending on the dimension of F.

First assume $\dim F = i - 1$. Then $F \in \Phi$ and $F \dot\cup 1 \notin \Phi$, which imply $F \dot\cup v \notin \Phi$ for any v, since $F \subseteq F \dot\cup 1 \leq_C F \dot\cup v$. Thus, F has no coboundary in Φ; F has no boundary in Φ simply because it has minimal dimension in Φ.

Now assume, on the other hand, $\dim F = i$. Then $F \in \Phi$ and $F - 1 \notin \Phi$, which imply $F - v \notin \Phi$ for any v, since $F - v \leq_C F - 1 \subseteq F$. Thus, F has no boundary in Φ; F has no coboundary in Φ simply because it has maximal dimension in Φ. $\square$

Lemma 5.9. *Let Φ be a shifted $(i-1, i)$-dimensional convex set on ground set $\{1, \ldots, n\}$, and let $1 \leq e \leq n$. Then Φ satisfies the spectral recursion with respect to e iff Φ^- does.*

Proof. By Lemma 3.4, it suffices to show $\mathbf{s}(\Phi^-) \doteq \mathbf{s}(\Phi)$, $\mathbf{s}(\Phi^- - e) \doteq \mathbf{s}(\Phi - e)$, $\mathbf{s}(\Phi^-/e) \doteq \mathbf{s}(\Phi/e)$, and $\mathbf{s}(\Phi^-||e) \doteq \mathbf{s}(\Phi||e)$. The main tools are Lemmas 3.2 and 5.8, which immediately show $\mathbf{s}(\Phi^-) \doteq \mathbf{s}(\Phi)$.

In order to show $\mathbf{s}(\Phi^-||e) \doteq \mathbf{s}(\Phi||e)$, we first claim that $\mathrm{st}_{\Phi^-} e = \mathrm{st}_\Phi e$. Indeed, $\mathrm{st}_{(\Phi - \mathcal{N})} e = \mathrm{st}_\Phi e$ for any set $\mathcal{N}$ of faces in Φ with neither boundary nor coboundary in Φ. Then

$$\begin{aligned}
\mathbf{s}(\Phi^-||e) &= \mathbf{s}((\Phi - \mathcal{N}_\Phi) - \mathrm{st}_{(\Phi - \mathcal{N}_\Phi)} e) = \mathbf{s}((\Phi - \mathrm{st}_\Phi e) - \mathcal{N}_\Phi) = \mathbf{s}((\Phi||e) - \mathcal{N}_\Phi) \\
&\doteq \mathbf{s}(\Phi||e),
\end{aligned}$$

by Lemmas 3.2 and 5.8, since the faces of $\mathcal{N}_\Phi$ have neither boundary nor coboundary in Φ, nor in any subset of Φ, such as $\Phi||e$.

To show $\mathbf{s}(\Phi^- - e) \doteq \mathbf{s}(\Phi - e)$ and $\mathbf{s}(\Phi^-/e) \doteq \mathbf{s}(\Phi/e)$, we split into two cases: $e = 1$; and $e \neq 1$. If $e \neq 1$, then equation (8) makes it easy to show that $\Phi^- - e = (\Phi - e)^-$ and $\Phi^-/e = (\Phi/e)^-$. Then Lemmas 3.2 and 5.8 show $\mathbf{s}(\Phi^- - e) \doteq \mathbf{s}((\Phi - e)^-) \doteq \mathbf{s}(\Phi - e)$ and $\mathbf{s}(\Phi^-/e) \doteq \mathbf{s}((\Phi/e)^-) \doteq \mathbf{s}(\Phi/e)$.

To address the $e = 1$ case, first note that $\Phi^- - 1 = (\Phi - \mathcal{N}_\Phi) - 1 = (\Phi - 1) - (\mathcal{N}_\Phi \cap (\Phi - 1))$. Let $\mathcal{N}' = \mathcal{N}_\Phi \cap (\Phi - 1)$. Since $\mathcal{N}' \subseteq \mathcal{N}_\Phi$, every face in $\mathcal{N}'$ has neither boundary nor coboundary in Φ, nor in any subset of Φ, such as $\Phi - 1$. Now apply Lemma 3.2 to see $\mathbf{s}(\Phi^- - 1) \doteq \mathbf{s}(\Phi - 1)$. The proof that $\mathbf{s}(\Phi^-/1) \doteq \mathbf{s}(\Phi/1)$ proceeds similarly. □

Definition. Let Φ be an $(i-1, i)$-dimensional convex set with vertex 1. Define

$$\Phi^+ = \Phi^- \mathbin{\dot\cup} \{F \mathbin{\dot\cup} 1 \colon 1 \notin F, F \in \Phi_i\} \mathbin{\dot\cup} \{F - 1 \colon 1 \in F, F \in \Phi_{i-1}\}.$$

Lemma 5.10. *If Φ is a shifted $(i-1, i)$-dimensional convex set on ground set $\{1, \ldots, n\}$, then $\Phi^+ = 1 * \Phi'$ for some shifted convex Φ' on ground set $\{2, \ldots, n\}$.*

Proof. First, by equation (8),

$$\begin{aligned} \Phi^+ &= (\Phi_i - 1) \mathbin{\dot\cup} (1 * ((\Phi_i/1) \cap (\Phi_{i-1} - 1))) \mathbin{\dot\cup} (1 \circ (\Phi_{i-1}/1)) \\ &\quad \mathbin{\dot\cup} (1 \circ (\Phi_i - 1)) \mathbin{\dot\cup} (\Phi_{i-1}/1) \\ &= 1 * ((\Phi_i - 1) \mathbin{\dot\cup} ((\Phi_i/1) \cap (\Phi_{i-1} - 1)) \mathbin{\dot\cup} (\Phi_{i-1}/1)). \end{aligned} \tag{9}$$

Now, coning preserves shiftedness of convex sets, since $1 * (\Delta, \Delta') = (1 * \Delta, 1 * \Delta')$ and, as is well-known and easy to prove, coning preserves shiftedness of simplicial complexes. Equation (9) thus reduces the proof of this lemma to showing that

$$\Phi' = (\Phi_i - 1) \mathbin{\dot\cup} ((\Phi_i/1) \cap (\Phi_{i-1} - 1)) \mathbin{\dot\cup} (\Phi_{i-1}/1) \tag{10}$$

is convex and shifted.

Equation (10) means $\Phi'_i = \Phi_i - 1$, $\Phi'_{i-1} = (\Phi_i/1) \cap (\Phi_{i-1} - 1)$, and $\Phi'_{i-2} = \Phi_{i-1}/1$, and so $G \in \Phi'$ precisely when the following conditions are met:

1. $i - 2 \leq \dim G \leq i$;
2. if $\dim G \leq i - 1$, then $G \mathbin{\dot\cup} 1 \in \Phi$; and
3. if $\dim G \geq i - 1$, then $G \in \Phi$.

We will use the characterization of shifted convex sets given in Lemma 5.2 to show that Φ' is convex and shifted. So assume $G \subseteq \{2, \ldots, n\}$; $F, H \in \Phi'$; and $F \leq_S G \leq_S H$. We need to show $G \in \Phi'$. Condition 1 follows directly from the hypotheses on G.

Next we establish condition 2; so assume $\dim G \leq i - 1$. First note that $\dim F \leq \dim G \leq i - 1$, so $F \mathbin{\dot\cup} 1 \in \Phi$, and $F \mathbin{\dot\cup} 1 \leq_S G \mathbin{\dot\cup} 1$, by Lemma 5.3. Now, if $\dim H = \dim G \leq i - 1$, then $H \mathbin{\dot\cup} 1 \in \Phi$ and $G \mathbin{\dot\cup} 1 \leq_S H \mathbin{\dot\cup} 1$, by Lemma 5.3, but if $\dim H > \dim G$, then $\dim H \geq i-1$, and so $H \in \Phi$ and, by Lemma 5.7, $G \dot{\cup} 1 \leq_S H$. Either way, for some $\widetilde{H}$ (either H or $H \dot{\cup} 1$), $F \dot{\cup} 1 \leq_S G \dot{\cup} 1 \leq_S \widetilde{H}$ and $F \dot{\cup} 1, \widetilde{H} \in \Phi$. Thus $G \dot{\cup} 1 \in \Phi$, as desired.

The proof that G satisfies condition 3 is similar; we start by assuming $\dim G \geq i-1$. First note that $\dim H \geq \dim G \geq i-1$, so $H \in \Phi$ while $G \leq_S H$. Now if $\dim F = \dim G \geq i-1$, then $F \in \Phi$ while $F \leq_S G$, but if $\dim F < \dim G$, then $\dim F \leq i$, and so $F \dot{\cup} 1 \in \Phi$ and, by Lemma 5.7, $F \dot{\cup} 1 \leq_S G$. Either way, for some $\widetilde{F}$ (either F or $F \dot{\cup} 1$), $\widetilde{F} \leq_S G \leq_S H$ and $\widetilde{F}, H \in \Phi$. Thus $G \in \Phi$, as desired. □

Lemma 5.11. *If Φ is a shifted $(i-1,i)$-dimensional convex set, then Φ satisfies the spectral recursion.*

Proof. By induction on the number of non-loop vertices. If Φ has no non-loop vertices, the result is trivially true. So assume Φ has ground set $\{1, \ldots, n\}$ with $n \geq 1$.

By Lemma 5.9, it suffices to show Φ^- satisfies the spectral recursion. Note that, by Lemma 5.10, $\Phi^- = (\Phi^+)^{(i-1,i)} = (1 * \Phi')^{(i-1,i)}$ and that Φ' is a shifted $(i-1,i)$-dimensional convex set with one less non-loop vertex (namely, vertex 1) than Φ^-, and hence fewer non-loop vertices than Φ. By induction, then, Φ' satisfies the spectral recursion. But since taking skeleta (Theorem 4.7) and coning (Corollary 3.10) preserve the property of satisfying the spectral recursion, Φ^- also satisfies the spectral recursion. □

Theorem 5.12. *If Φ is convex and shifted, then Φ satisfies the spectral recursion.*

Proof. It is immediate that, since Φ is shifted, so is $\Phi^{(i-1,i)}$ for all i. By Lemma 5.11, each $\Phi^{(i-1,i)}$ satisfies the spectral recursion. By Lemma 4.6, then, Φ satisfies the spectral recursion. □

Remark 5.13. It is an easy exercise to verify that, if Φ is shifted, then so are $\Phi - e$, Φ/e, and the two direct summands of $\Phi||e$ from Proposition 2.4.

6 Matroid pairs

In this section, we show that some matroid pairs satisfy the spectral recursion, and conjecture that many more do as well. We first set our notation for matroids. For more details, see, *e.g.*, [13]. We let $\mathcal{C}(M)$ denote the set of circuits of matroid M, and $IN(M)$ denote the independence complex, which is the simplicial complex consisting of the independent sets of M, and whose Laplacian was first studied in [9]. Our notation for deletion and contraction of convex sets and simplicial complexes is consistent with the notation for deletion and contraction of matroids, *e.g.*,

$$IN(M-e) = IN(M) - e, \text{ and}$$
$$IN(M/e) = IN(M)/e.$$

Similarly, e is a loop of M precisely when it is a loop of $IN(M)$.

The existence of a strong map $N \to N'$ is the natural condition on matroids N and N' to yield nice results about the convex set $(IN(N), IN(N'))$; see, *e.g.*, [10]. Roughly speaking, it means that the matroid structures of N and N' are compatible, comparable to demanding that Δ and Δ' are shifted on the *same* ordered ground set in order for (Δ, Δ') to be shifted pair. The factorization theorem (*e.g.*, [10, Theorem 8.2.8]) says that one characterization of the existence of such a strong map is that $N = M - A$ and $N' = M/A$ for some matroid M with ground set $E \dot\cup A$. The main result of this section is that, in the special case where $|A| = 1$, *i.e.*, $\operatorname{rank} N - \operatorname{rank} N' = \dim IN(N) - \dim IN(N') = 1$, the convex set $(IN(N), IN(N'))$ satisfies the spectral recursion. We need first one lemma.

Lemma 6.1. *If M is a matroid with ground element e, and e is not a loop, then*

$$(IN(M-e), IN(M/e)) = \bigoplus_{\substack{C \in \mathcal{C}(M) \\ e \in C}} (C - e) \circ IN(M/C).$$

Proof. This is essentially proved in [2, Lemmas 3.3 and 3.4]. We sketch the proof here, both for completeness, and to let the language of convex sets, not found in the original, simplify some of the steps.

Let $\Phi = (IN(M-e), IN(M/e))$. If $I \in \Phi$, then I is independent in M, but $I \dot\cup e$ is dependent in M, and so there is a *unique* circuit of M, which we denote by $\mathrm{ci}_M(e, I)$, contained in $I \dot\cup e$. For each circuit $C \in \mathcal{C}(M)$, let $M_C = \{I \in \Phi\colon\ \mathrm{ci}_M(e, I) = C\}$. Since each $I \in \Phi$ has a unique $\mathrm{ci}_M(e, I)$, the M_C's partition Φ.

In order to show that this partition is a direct sum, first note that, if $I_1 \in M_{C_1}$ and $I_2 \in M_{C_2}$, then $I_1 \dot\cup e$ cannot contain C_2, since $\mathrm{ci}_M(e, I)$ is the *unique* circuit of M contained in $I \dot\cup e$. Then, since $C_2 \subseteq I_2 \dot\cup e$, it follows that $I_2 \not\subseteq I_1$; similarly $I_1 \not\subseteq I_2$. We conclude that all the M_C's are totally unrelated, as desired.

Finally, as in [2],

$$\begin{aligned} M_C &= \{I \in IN(M-e)\colon C - e \subseteq I\} \\ &= (C - e) \circ IN((M-e)/(C-e)) \\ &= (C - e) \circ IN(M/C). \end{aligned}$$

□

Theorem 6.2. *If M is a matroid with ground element e, then the matroid pair $(IN(M - e), IN(M/e))$ satisfies the spectral recursion.*

Proof. If e is not a loop of M, then this is an immediate corollary to Lemmas 3.8 and 6.1, Corollary 3.10, and the fact [2, Theorem 3.18] that matroids satisfy the spectral recursion. If e is a loop of M, then it is a loop of $IN(M)$, and so

$$\begin{aligned} (IN(M-e), IN(M/e)) &= (IN(M) - e, IN(M)/e) = (IN(M), \emptyset) \\ &= IN(M), \end{aligned}$$

which satisfies the spectral recursion. □

We are unable to prove anything about $(IN(N), IN(N'))$ if $\operatorname{rank} N - \operatorname{rank} N' > 1$, because we don't have the analogue of Lemma 6.1 above. Still, experimental evidence on randomly chosen matroids supports the following natural conjecture.

Conjecture 6.3. *If there is a strong map $N \to N'$ between matroids N and N', then the convex set $(IN(N), IN(N'))$ has integral Laplacian eigenvalues, and satisfies the spectral recursion.*

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