h^*-vectors, Eulerian polynomials and stable polytopes of graphs

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Submitted: Jul 5, 2004; Accepted: Sep 22, 2004; Published: Oct 7, 2004
Mathematics Subject Classifications: Primary 52B20; Secondary 05C17, 05E99, 13F20.

Abstract

Conditions are given on a lattice polytope P of dimension m or its associated affine semigroup ring which imply inequalities for the h^*-vector $(h_0^*, h_1^*, \ldots, h_m^*)$ of P of the form $h_i^* \ge h_{d-i}^*$ for $1 \le i \le \lfloor d/2 \rfloor$ and $h_{\lfloor d/2 \rfloor}^* \ge h_{\lfloor d/2 \rfloor + 1}^* \ge \cdots \ge h_d^*$, where $h_i^* = 0$ for $d < i \le m$. Two applications to order polytopes of posets and stable polytopes of perfect graphs are included.

1 Introduction

Let P be an m-dimensional convex polytope in $\mathbb{R}^N$ having vertices with integer coordinates. It is a fundamental result due to Ehrhart [5, 6] that the function $i(P, r) = \#(rP \cap \mathbb{Z}^N)$, counting integer points in the r-fold dilate of P, is a polynomial in r of degree m, called the *Ehrhart polynomial* of P. Thus one can write

$$\sum_{r \ge 0} i(P, r)\, t^r = \frac{h_0^* + h_1^* t + \cdots + h_m^* t^m}{(1-t)^{m+1}} \tag{1}$$

for certain integers h_i^*. Following Stanley [21] we call $(h_0^*, h_1^*, \ldots, h_m^*)$ the *h^*-vector* of P and denote it by $h^*(P)$. It is known that $i(P, r)$ is the Hilbert function of a semistandard graded Cohen-Macaulay normal domain R_P called the semigroup ring of P; see [3, Chapter 6] and [9, Chapter X]. In particular the integers h_i^* are nonnegative. Recall that a sequence $(a_0, a_1, \ldots, a_n)$ of real numbers is said to be *unimodal* if $a_0 \le \cdots \le a_j \ge \cdots \ge a_n$ holds for some $0 \le j \le n$. Although h^*-vectors are not always unimodal, various results and conjectures concerning the unimodality of $h^*(P)$ have appeared in the literature [8, 9, 19, 20]. For instance it would follow from [8, Conjecture 1.5] and [19, Conjecture 4a] that $h^*(P)$ is unimodal if the semigroup ring R_P is standard and Gorenstein. Moreover, it was conjectured by Hibi [9, p. 111] that $h^*(P)$ is unimodal whenever it is symmetric, meaning that $h_i^* = h_{m-i}^*$ for $0 \le i \le m$.

General conditions on an integer polytope P, inspired by the work of V. Reiner and V. Welker on order polytopes of graded posets [15], which guarantee that $h^*(P)$ is unimodal were given in [1]. More precisely it was shown in [1] that if the pulling triangulation Δ_τ of P with respect to an ordering $\tau = (v_1, v_2, \ldots, v_p)$ of the vertices of P is unimodular (see Section 2 for basic definitions on triangulations) and for some n any facet of P contains exactly $n-1$ elements of $\{v_1, v_2, \ldots, v_n\}$ then $h^*(P)$ is equal to the h-vector [24, Section 8.3] of a simplicial d-dimensional polytope, where $d = m-n+1$, and hence that it is unimodal and satisfies $h_i^* = h_{d-i}^*$ for $0 \le i \le d$ by McMullen's g-theorem. The simplex with vertices $\{v_1, v_2, \ldots, v_n\}$ is called a *special simplex* for P in [1]. If $v_1, v_2, \ldots, v_n$ are any n vertices of P which are affinely independent we call their convex hull a *semispecial simplex* for P if any facet of P contains at least $n-1$ elements of $\{v_1, v_2, \ldots, v_n\}$. It is the main goal of this paper to prove the following more general statement.

Theorem 1.1 *Let P be an m-dimensional integer polytope with h^*-vector $(h_0^*, h_1^*, \ldots, h_m^*)$. If $\tau = (v_1, v_2, \ldots, v_p)$ is a linear ordering of the vertices of P such that*

(i) *the pulling triangulation Δ_τ of P is unimodular,*

(ii) *$\{v_1, v_2, \ldots, v_n\}$ is the vertex set of a semispecial simplex for P*

and $d = m - n + 1$, then $h_i^* \ge h_{d-i}^*$ for $0 \le i \le \lfloor d/2 \rfloor$, $h_{\lfloor d/2 \rfloor}^* \ge h_{\lfloor d/2 \rfloor + 1}^* \ge \cdots \ge h_d^*$ and $h_i^* = 0$ for $d < i \le m$.

The next corollary follows essentially from the case $n = 1$ of Theorem 1.1.

Corollary 1.2 *Let P be an m-dimensional integer polytope with h^*-vector $(h_0^*, h_1^*, \ldots, h_m^*)$. If P has a unimodular pulling triangulation then $h_i^* \ge h_{m-i}^*$ for $1 \le i \le \lfloor m/2 \rfloor$ and $h_{\lfloor m/2 \rfloor}^* \ge h_{\lfloor m/2 \rfloor + 1}^* \ge \cdots \ge h_m^*$.*

Pulling triangulations are specific examples of regular triangulations, so it is natural to ask for inequalities satisfied by $h^*(P)$ under the existence of a regular unimodular triangulation of P. I am grateful to Takayuki Hibi [10] for informing me that the following statement was also proved by himself and Richard Stanley in 1999 (unpublished) by essentially the same argument as the one given in Section 2.

Theorem 1.3 *Let P be an m-dimensional integer polytope with h^*-vector $(h_0^*, h_1^*, \ldots, h_m^*)$. If P has a regular unimodular triangulation then $h_i^* \ge h_{m-i+1}^*$ for $1 \le i \le \lfloor (m+1)/2 \rfloor$,*

$$h_{\lfloor (m+1)/2 \rfloor}^* \ge \cdots \ge h_{m-1}^* \ge h_m^*$$

and

$$h_i^* \le \binom{h_1^* + i - 1}{i}$$

for $0 \le i \le m$. In particular, if $h^*(P)$ is symmetric and P has a regular unimodular triangulation then $h^*(P)$ is unimodal.

It was observed in [9, Example 36.4] that the last inequality in Theorem 1.3 does not hold for some integer polytopes. An example of a 0/1 polytope P with no regular unimodular triangulations such that R_P is standard was given in [13].

This paper is structured as follows. Section 2 includes the necessary definitions and background on convex polytopes and their triangulations and h^*-vectors as well as the proof of Theorem 1.3. The notion of triangulation of a polytope P we will use does not require that all vertices of the triangulation are necessarily vertices of P unless the contrary is explicitly stated. In Section 3 we introduce the concept of a semispecial simplex for P and prove a slight generalization of Theorem 1.1 (see Corollary 3.4). Specifically we drop the assumption that all vertices of a (special or) semispecial simplex and the elements of the sequence τ which appears in Theorem 1.1 are vertices of P. Our proofs are based on a result of Kalai and Stanley (Lemma 2.2) on the h-vector of a Cohen-Macaulay subcomplex of the boundary complex of a simplicial polytope. Sections 4 and 5 include applications of Theorem 1.1 to order polytopes of (not necessarily graded) posets and to stable polytopes of perfect graphs, respectively. In Section 6 we state an analogue of Theorem 1.1 in the context of the affine semigroup ring of P.

2 Triangulations and h^*-vectors

Before proving Theorem 1.3 we review some basic definitions and background on simplicial complexes and convex polytopes. For undefined terminology and more information on these topics we refer the reader to [7, 9, 22, 23, 24]. A *polytopal complex* $\mathcal{F}$ [24, Section 8.1] is a finite, nonempty collection of convex polytopes in $\mathbb{R}^N$ such that (i) any face of a polytope in $\mathcal{F}$ is also in $\mathcal{F}$ and (ii) the intersection of any two polytopes in $\mathcal{F}$ is a (possibly empty) face of both. The elements of $\mathcal{F}$ are its *faces* and those of dimension 0 are its *vertices*. The *dimension* of $\mathcal{F}$ is the maximum dimension of a face. The complex $\mathcal{F}$ is *pure* if all maximal (with respect to inclusion) faces of $\mathcal{F}$ have the same dimension. The collection $\mathcal{F}(P)$ of all faces of a polytope P is a pure polytopal complex, called the *face complex* of P, as is the collection $\mathcal{F}(\partial P)$ of proper faces of P, called the *boundary complex*. The complex $\mathcal{F}$ is a (geometric) *simplicial comlpex* if all faces of $\mathcal{F}$ are simplices. Two simplicial complexes Δ and Δ' are said to be *combinatorially equivalent* if there exists a bijection ρ between the sets of faces of Δ and Δ' such that ρ and its inverse preserve inclusion. The h-vector $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of a simplicial complex Δ of dimension $d-1$ is defined by the formula

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i} \tag{2}$$

where f_i is the number of i-dimensional faces of Δ for $0 \leq i \leq d-1$ and $f_{-1} = 1$. We say that Δ is a *triangulation* of a polytopal complex $\mathcal{F}$ if the union of the faces of Δ is equal to the union of the faces of $\mathcal{F}$ and every face of Δ is contained in a face of $\mathcal{F}$. In particular we do not require that all vetrices of Δ are vertices of $\mathcal{F}$. A triangulation of the face complex $\mathcal{F}(P)$ of a convex polytope $P \subseteq \mathbb{R}^N$ is called a *triangulation* of P. Such a triangulation is *regular* if the faces of Δ are the projections, under the map $\mathbb{R}^{N+1} \to \mathbb{R}^N$ which forgets the last coordinate, of the *lower* faces of a convex polytope $Q \subseteq \mathbb{R}^{N+1}$, meaning those faces of Q which are visible from any point in $\mathbb{R}^{N+1}$ with sufficiently large negative last coordinate. A convex polytope $P \subseteq \mathbb{R}^N$ is called an *integer polytope* if all vertices of P have integer coordinates. A triangulation Δ of such a polytope P is called *unimodular* if all vertices of Δ have integer coordinates and the vertex set of any maximal simplex of Δ is a basis of the affine integer lattice $A \cap \mathbb{Z}^N$, where A is the affine span of P in $\mathbb{R}^N$. In the special case of pulling triangulations (which we discuss in the sequel) the following lemma appeared as [16, Corollary 2.5].

Lemma 2.1 ([2]) *If P is an integer polytope and Δ is any unimodular triangulation of P then $h^*(P) = h(\Delta)$.*

A convex polytope P is said to be *simplicial* if all its proper faces are simplices, so that $\mathcal{F}(\partial P)$ is a simplicial complex. The next lemma is a consequence of [21, Lemma 2.2] and the note following that lemma in [21]. The inequalities $h_i \geq h_{d-i}$ were first proved by Kalai [11].

Lemma 2.2 ([11, 21]) *If Δ is a Cohen-Macaulay subcomplex of the boundary complex of a d-dimensional simplicial polytope and $h(\Delta) = (h_0, h_1, \ldots, h_d)$ then $h_i \geq h_{d-i}$ for $0 \leq i \leq \lfloor d/2 \rfloor$ and $h_{\lfloor d/2 \rfloor} \geq h_{\lfloor d/2 \rfloor + 1} \geq \cdots \geq h_d$.*

Given a polytope Q and a sequence $\tau = (v_1, v_2, \ldots, v_p)$ of points containing the vertices of Q we can construct a simplicial polytope Q' of the same dimension as Q obtained from Q by a *sequence of pullings* with respect to τ. More specifically let $\mathrm{pull}_v(P)$ be the convex hull of the set of vertices of P and the point obtained by moving v beyond the hyperplanes supporting exactly those facets of P which contain v (see [24, Section 3.1]) if v lies on the boundary of P and let $\mathrm{pull}_v(P) = P$ otherwise. We define $Q' = Q_p$ where $Q_i = \mathrm{pull}_{v_i}(Q_{i-1})$ for $1 \leq i \leq p$ and $Q_0 = Q$. If τ is a linear ordering of the vertices of Q then Q' is the polytope obtained from Q by *pulling the vertices* of Q in the order τ [7, p. 80] [12, Section 2.5]. In this case the vertices of Q' can be labeled as $v'_1, v'_2, \ldots, v'_p$ so that if $v_{i_1}, v_{i_2}, \ldots, v_{i_j}$ are the vertices of a $(j-1)$-dimensional simplex which is a face of Q then $v'_{i_1}, v'_{i_2}, \ldots, v'_{i_j}$ are the vertices of a $(j-1)$-dimensional simplex which is a face of Q'.

Proof of Theorem 1.3. Let Δ be a regular unimodular triangulation of P. Being regular, Δ is combinatorially isomorphic to the complex of lower faces of an $(m+1)$-dimensional polytope Q and we may assume that Q has as many vertices as Δ. Pulling the vertices of Q in an arbitrary order produces an $(m+1)$-dimensional simplicial polytope Q' such that Δ is combinatorially isomorphic to a subcomplex Δ' of the boundary complex of Q'. Then $h(\Delta) = h(\Delta')$ and $h^*(P) = h(\Delta)$ by Lemma 2.1. Moreover Δ' is topologically a ball, being homeomorphic to Δ, and hence Cohen-Macaulay. Lemma 2.2 implies that $h(\Delta') = h^*(P)$ satisfies $h^*_{\lfloor (m+1)/2 \rfloor} \geq \cdots \geq h^*_m \geq h^*_{m+1} = 0$ and $h^*_i \geq h^*_{m-i+1}$ for $1 \leq i \leq \lfloor (m+1)/2 \rfloor$. Let $(h_0, h_1, \ldots, h_{m+1})$ be the h-vector of the boundary complex of Q'. The h-vector of Δ' satisfies $h^*_i \leq h_i$ for $0 \leq i \leq m$ by [21, Theorem 2.1]. Moreover $h^*_1 = h_1 = n - m - 1$, where n is the number of vertices of Δ, Δ', Q or Q' and the last inequality in the theorem follows from the Upper Bound Theorem $h_i \leq \binom{h_1+i-1}{i}$ for simplicial polytopes [24, Lemma 8.26]. The last statement in the theorem should be clear. $\square$

We conclude this section with the background on pulling triangulations of polytopal complexes needed for the proof of Theorem 1.1. For any polytopal complex $\mathcal{F}$ and set of points σ in $\mathbb{R}^N$ we denote by $\mathcal{F}\backslash\sigma$ the subcomplex of faces of $\mathcal{F}$ which do not contain any of the points in σ and write $\mathcal{F}\backslash v$ for $\mathcal{F}\backslash\sigma$ if σ consists of a single point v. Given a sequence $\tau = (v_1, v_2, \ldots, v_p)$ of points containing the vertices of $\mathcal{F}$ we define the *reverse lexicographic triangulation* or *pulling triangulation* $\Delta(\mathcal{F}) = \Delta_\tau(\mathcal{F})$ with respect to τ [16] [23, Chapter 8] as follows. We have $\Delta(\mathcal{F}) = \{v\}$ if $\mathcal{F}$ consists of a single vertex v and

$$\Delta(\mathcal{F}) = \Delta(\mathcal{F}\backslash v_1) \cup \bigcup_F \{\mathrm{conv}(\{v_1\} \cup G) : G \in \Delta(\mathcal{F}(F))\}$$

otherwise, where the union runs over the facets F not containing v_1 of the maximal faces of $\mathcal{F}$ which contain v_1. The triangulations $\Delta(\mathcal{F}\backslash v_1)$ and $\Delta(\mathcal{F}(F))$ are defined with respect to $(v_2, \ldots, v_p)$ by induction. Equivalently, for $i_0 > i_1 > \cdots > i_t$ the set $\{v_{i_0}, v_{i_1}, \ldots, v_{i_t}\}$ is the vertex set of a maximal simplex of $\Delta_\tau(\mathcal{F})$ if there exists a maximal flag $F_0 \subset F_1 \subset \cdots \subset F_t$ of faces of $\mathcal{F}$ such that v_{i_j} is the first element of τ in F_j for all j and v_{i_j} is not in F_{j-1} for $j \geq 1$. If $\mathcal{F}$ is the boundary complex of a polytope Q then $\Delta_\tau(\mathcal{F})$ is combinatorially isomorphic to the boundary complex of the simplicial polytope Q' obtained from Q by a sequence of pullings with respect to τ.

3 Semispecial simplices

Throughout this section P denotes an m-dimensional convex polytope in $\mathbb{R}^N$. We call an $(n-1)$-dimensional simplex Σ a *special simplex* for P if each facet of P contains exactly $n-1$ vertices of Σ. This definition is less restrictive than the one given originally in [1] since we do not require that all vertices of Σ are vertices of P. We call Σ a *semispecial simplex* if each facet of P contains at least $n-1$ vertices of Σ. Thus a semispecial simplex for P is special if and only if it is not contained in the boundary of P. Two examples are shown in Figure 1.

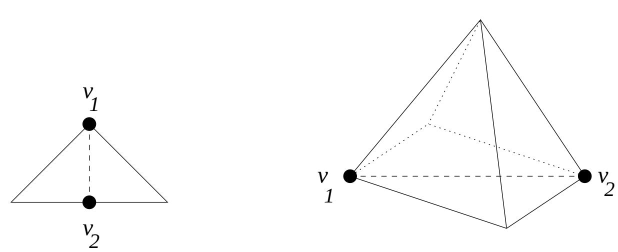

Figure 1: A special simplex for a triangle and a semispecial simplex for a square piramid.

Remark 3.1 Any set σ of n points in $\mathbb{R}^N$ having the property that any facet of P contains at least $n-1$ elements of σ must be affinely independent and hence it is the vertex set of a semispecial $(n-1)$-simplex for P. Indeed, if $v \in \sigma$ were in the affine span of $\sigma \backslash v$ then the affine span of any facet of P would have to contain v, which is impossible.

If V is any linear subspace of $\mathbb{R}^N$ then the *quotient polytope* $P/V \subseteq \mathbb{R}^N/V$ is the image of P under the canonical surjection $\mathbb{R}^N \to \mathbb{R}^N/V$. This is a convex polytope in $\mathbb{R}^N/V$ linearly isomorphic to the image $\pi(P)$ of P under any linear surjection $\pi : \mathbb{R}^N \to \mathbb{R}^{N-\dim V}$ with kernel V. Recall that the *simplicial join* $\Delta_1 * \Delta_2$ of two geometric simplicial complexes Δ_1 and Δ_2 in $\mathbb{R}^N$ is defined if no two line segments, each joining a point in a face of Δ_1 to a point in a face of Δ_2, intersect in their relative interiors. In this case the maximal faces of $\Delta_1 * \Delta_2$ are the simplices of the form $\mathrm{conv}(F_1 \cup F_2)$, where F_1 and F_2 are maximal faces of Δ_1 and Δ_2, respectively. The simplicial join satisfies $h(\Delta_1 * \Delta_2, x) = h(\Delta_1, x)\, h(\Delta_2, x)$, where $h(\Delta, x) = \sum_{i=0}^{d} h_i x^{d-i}$ if $h(\Delta) = (h_0, h_1, \ldots, h_d)$. In particular $h(\Delta_1 * \Delta_2) = h(\Delta_2)$ if Δ_1 is a simplex.

Lemma 3.2 *If P has a triangulation of the form $\Sigma * \Delta$ for some $(n-1)$-simplex Σ and simplicial complex Δ then Δ is combinatorially isomorphic to a triangulation of a pure $(m-n)$-dimensional shellable subcomplex of the boundary complex of a polytope of dimension $m-n+1$.*

Proof. Let V be the linear $(n-1)$-dimensional subspace of $\mathbb{R}^N$ parallel to the affine span of Σ and let Q be the corresponding $(m-n+1)$-dimensional quotient polytope P/V. If v is the point which is the image of Σ under the canonical surjection $\mathbb{R}^N \to \mathbb{R}^N/V$ then Q inherits a triangulation of the form $v * \Gamma$ for some simplicial complex Γ which is combinatorially isomorphic to Δ and triangulates a subcomplex of the boundary complex $\mathcal{F}(\partial Q)$. If Σ is not contained in the boundary of P then v is an interior point of Q and Γ triangulates $\mathcal{F}(\partial Q)$, which is pure and shellable [24, Section 8.2]. Otherwise v lies on the boundary of Q and Γ triangulates the subcomplex of $\mathcal{F}(\partial Q)$ consisting of all faces of Q which do not contain v. This is the complex of faces of Q which are not visible from a point beyond the hyperplanes supporting exactly those facets of Q which contain v and hence is pure $(m-n)$-dimensional and shellable (see Lemma 8.10 and Theorem 8.12 in [24]). $\square$

Lemma 3.3 *Suppose that $P = \mathrm{conv}\{v_1, v_2, \ldots, v_p\}$. Let $\Sigma = \mathrm{conv}\{v_1, v_2, \ldots, v_n\}$ and Δ be the pulling triangulation of $\mathcal{F}(P) \backslash \{v_1, v_2, \ldots, v_n\}$ with respect to $(v_{n+1}, \ldots, v_p)$. If Σ is a semispecial $(n-1)$-simplex for P and $\tau = (v_1, v_2, \ldots, v_p)$ then*

(i) *the pulling triangulation Δ_τ of P is combinatorially isomorphic to the simplicial join $\Sigma * \Delta$ and*

(ii) *Δ is combinatorially isomorphic to a Cohen-Macaulay subcomplex of the boundary complex of a simplicial polytope of dimension $m-n+1$.*

Proof. Part (i) can be proved exactly as part (i) of [1, Lemma 3.4], where Σ is assumed to be a special simplex for P. Lemma 3.2 implies that Δ is combinatorially isomorphic to a triangulation Γ of a pure $(m-n)$-dimensional shellable subcomplex of the boundary complex of a convex polytope Q of dimension $m-n+1$. Clearly Γ is homeomorphic to a ball or a sphere and hence Cohen-Macaulay. Let the isomorphism between Δ and Γ be induced by the map $v_i \to v'_i$ for $n+1 \le i \le p$ and let Q' be the simplicial $(m-n+1)$-dimensional polytope obtained from Q by any sequence of pullings starting with $(v'_{n+1}, \dots, v'_p)$. Since Δ is a pulling triangulation with respect to $(v_{n+1}, \dots, v_p)$, Γ is combinatorially isomorphic to a subcomplex of the boundary complex of Q'. This proves part (ii). □

Let $h^*(P) = (h^*_0, h^*_1, \dots, h^*_m)$ be the h^*-vector of P. Theorem 1.1 is a special case of the following corollary.

Corollary 3.4 *Suppose that* $P = \mathrm{conv}\{v_1, v_2, \dots, v_p\}$ *and* $v_i \in \mathbb{Z}^N$ *for* $1 \le i \le p$. *Let* $d = m-n+1$ *and* $\tau = (v_1, v_2, \dots, v_p)$. *If the pulling triangulation* Δ_τ *of* P *is unimodular and* $\Sigma = \mathrm{conv}\{v_1, v_2, \dots, v_n\}$ *is a semispecial* $(n-1)$*-simplex for* P *then* $h^*_i \ge h^*_{d-i}$ *for* $0 \le i \le \lfloor d/2 \rfloor$,

$$h^*_{\lfloor d/2 \rfloor} \ge h^*_{\lfloor d/2 \rfloor + 1} \ge \cdots \ge h^*_d$$

and $h^*_i = 0$ *for* $d < i \le m$. *Moreover* $h^*_d = 0$ *if* Σ *is not special.*

Proof. Let Δ denote the pulling triangulation of $\mathcal{F}(P) \setminus \{v_1, v_2, \dots, v_n\}$ with respect to $(v_{n+1}, \dots, v_p)$. Lemma 2.1 guarantees that $h^*(P) = h(\Delta_\tau)$. Then part (i) of Lemma 3.3 implies that $h(\Delta_\tau) = h(\Sigma * \Delta) = h(\Delta)$ and the result follows from part (ii) of the same lemma and Lemma 2.2. If Σ is not special then Δ is homeomorphic to a $(d-1)$-dimensional ball and hence $h^*_d = 0$. □

Proof of Corollary 1.2. It follows from the case $n = 1$ of Corollary 3.4 since a singleton $\{v_1\}$ is always a semispecial 0-dimensional simplex for P. □

Observe that if all vertices of the triangulation in the statement of Corollary 1.2 are vertices of P then $\{v_1\}$ is not special and hence $h^*_m = 0$. It was proved by F. Santos (unpublished) and Ohsugi and Hibi [14] that if P is a 0/1 polytope, meaning that its vertices are 0/1 vectors, defined by the system of inequalities

$$\begin{aligned} b_i \le \textstyle\sum_{j=1}^N a_{ij} x_j \le b_i + \varepsilon_i, \quad 1 \le i \le q \\ 0 \le x_j \le 1, \quad 1 \le j \le N \end{aligned} \tag{3}$$

for some integers a_{ij}, b_i and ε_i with $\varepsilon_i = 0$ or 1 then all pulling triangulations of P are unimodular. In view of Corollary 1.2 and the remark after its proof we get the following corollary.

Corollary 3.5 *If* P *is an* m*-dimensional* 0/1 *polytope in* $\mathbb{R}^N$ *defined by* (3) *and* $h^*(P) = (h^*_0, h^*_1, \dots, h^*_m)$ *then* $h^*_i \ge h^*_{m-i}$ *for* $0 \le i \le \lfloor m/2 \rfloor$ *and* $h^*_{\lfloor m/2 \rfloor} \ge h^*_{\lfloor m/2 \rfloor + 1} \ge \cdots \ge h^*_m = 0$.

4 Order polytopes and Eulerian polynomials

Let Ω be a poset (short for partially ordered set) on the ground set $[N] := \{1, 2, \ldots, N\}$ (see [17, Chapter 3] for an introduction to the theory of partially ordered sets). We will denote the partial order of Ω by $\leq_\Omega$. Recall that an (order) *ideal* of Ω is a subset $I \subseteq \Omega$ for which $a <_\Omega b$ and $b \in I$ imply that $a \in I$ and that b *covers* a in Ω if $a <_\Omega b$ but $a <_\Omega c <_\Omega b$ holds for no $c \in \Omega$. Let $\mathcal{L}(\Omega)$ be the set of *linear extensions* of Ω, meaning the set of permutations $w = (w_1, w_2, \ldots, w_N)$ of $[N]$ for which $w_i <_\Omega w_j$ implies $i < j$. The Ω-*Eulerian polynomial* is defined as

$$W(\Omega, t) = \sum_{w \in \mathcal{L}(\Omega)} t^{\mathrm{des}(w)}$$

where

$$\mathrm{des}(w) = \#\{i \in [N-1] : w_i > w_{i+1}\}$$

is the number of descents of w. Let Ω^0 be the poset obtained from Ω by adjoining a minimum element $\hat{0} = 0$. We define the *ideal height* of Ω to be the largest length e of a chain $I_0 \subset I_1 \subset \cdots \subset I_e$ of nonempty ideals of Ω^0 such that for $1 \leq i \leq e$ and for any $a \in I_{i-1}$ the set of elements covering a in Ω is a nonempty subset of I_i. Figure 2 shows the Hasse diagram of a poset Ω of ideal height 3. Observe that the cardinality of the shortest maximal chain in Ω in this example is equal to 4. The poset Ω is *naturally labeled* if the identity permutation $(1, 2, \ldots, N)$ is a linear extension. The following theorem is the main result of this section.

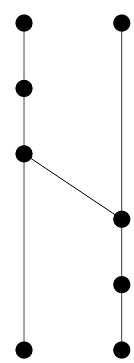

Figure 2: A poset of ideal height 3 with 8 elements.

Theorem 4.1 *Let Ω be a naturally labeled poset on $[N]$. If e is the ideal height of Ω, $W(\Omega, t) = q_0 + q_1 t + \cdots + q_N t^N$ is the Ω-Eulerian polynomial and $d = N - e$ then $q_i \geq q_{d-i}$ for $1 \leq i \leq \lfloor d/2 \rfloor$, $q_{\lfloor d/2 \rfloor} \geq q_{\lfloor d/2 \rfloor + 1} \geq \cdots \geq q_d$ and $q_i = 0$ for $d < i \leq N$.*

To prove this statement we will apply Theorem 1.1 to the order polytope of Ω. Let $\hat{\Omega}$ be the poset obtained from Ω^0 by adjoining a maximum element $\hat{1} = N + 1$. The *order polytope* [18] of Ω, denoted $O(\Omega)$, is the intersection of the hyperplanes $x_0 = 1$ and $x_{N+1} = 0$ in $\mathbb{R}^{N+2}$ with the cone defined by the inequalities $x_i \geq x_j$ for $i, j \in \hat{\Omega}$ with $i <_{\hat{\Omega}} j$. The vertices of $O(\Omega)$ are the characteristic vectors of the nonempty ideals of Ω^0 [18, Corollary 1.3] so, in particular, $O(\Omega)$ is an N-dimensional integer polytope. Moreover the facets of $O(\Omega)$ are defined exactly by the equalities of the form $x_i = x_j$ when j covers i in $\hat{\Omega}$ (see [18, Theorem 1.2] for a complete description of the facial structure of $O(\Omega)$) and the coefficients of the Ω-Eulerian polynomial

$$W(\Omega, t) = q_0 + q_1 t + \cdots + q_N t^N$$

are the entries of the h^*-vector $(h_0^*, h_1^*, \ldots, h_N^*)$ of $O(\Omega)$ [17, Section 4.5], that it $q_i = h_i^*$ for all i.

Proof of Theorem 4.1. Let P be the order polytope of Ω. It was shown by Ohsugi and Hibi [14, Example 1.3 (b)] that all pulling triangulations (with integer vertices) of order polytopes are unimodular. Moreover it follows easily from the description of the facets of P that if $I_0 \subset I_1 \subset \cdots \subset I_e$ is a chain of nonempty ideals of Ω^0 as in the definition of the ideal height for Ω then the characteristic vectors of the I_i are the vertices of a semispecial e-dimensional simplex for P. The result follows from Theorem 1.1. □

We close this section with a different characterization of ideal height. For $a \in \Omega^0$ let $\mathcal{C}_a$ denote the set of sequences $(a_0, a_1, \ldots, a_l)$ in $\hat{\Omega}$ such that $a_0 = \hat{0}$, $a_l = a$ and for $1 \leq i \leq l$ either a_i covers a_{i-1} in $\hat{\Omega}$ or a_{i-1} covers a_i. For each $\alpha = (a_0, a_1, \ldots, a_l) \in \mathcal{C}_a$ let $e(\alpha)$ denote the number of indices $1 \leq i \leq l$ for which a_i covers a_{i-1} and $a_i <_{\hat{\Omega}} \hat{1}$ and let $e(a)$ denote the minimum value of $e(\alpha)$ when α ranges over all sequences in $\mathcal{C}_a$.

Proposition 4.2 *The ideal height of Ω is equal to the maximum value of $e(a)$ for $a \in \Omega$.*

Proof. Let f denote the maximum value of $e(a)$ for $a \in \Omega$ and e denote the ideal height of Ω. Let a be any maximal element of Ω, let $\alpha = (a_0, a_1, \ldots, a_l) \in \mathcal{C}_a$ and let $I_0 \subset I_1 \subset \cdots \subset I_e$ be a chain of nonempty ideals of Ω^0 as in the definition of ideal height. Observe that (i) $a \notin I_{e-1}$, (ii) if $a_j \in \Omega$ and $a_j \notin I_i$ then $a_{j-1} \notin I_{i-1}$ and $a_{j-1} \in I_i$ is possible only when a_j covers a_{j-1} and (iii) if $a_j = \hat{1}$ then $a_{j-1} \notin I_{e-1}$. Since $a_0 \in I_0$ there must be at least e indices j for which a_j covers a_{j-1} in Ω^0. This shows that $e(a) \geq e$ and hence $f \geq e$.

For the other inequality let I_i denote the set of elements $a \in \Omega^0$ with $e(a) \leq i$ for $0 \leq i \leq f$. Observe that $e(\hat{0}) = 0$ and that if b covers a in Ω^0 then $e(a) \leq e(b) \leq e(a) + 1$. It follows that $I_0 \subset I_1 \subset \cdots \subset I_f$ is a chain of nonempty ideals of Ω^0. We will show that this chain satisfies the condition in the definition of ideal height, whence $f \leq e$. Let $1 \leq i \leq f$ and $a \in I_{i-1}$. Any element b in Ω covering a satisfies $e(b) \leq e(a) + 1 \leq i$ and hence $b \in I_i$. On the other hand if a were a maximal element of Ω and b is any element of Ω, which we may assume to be maximal, with $e(b) = f$ then the sequence of coverings $(b, \hat{1}, a)$ in $\hat{\Omega}$ shows that $e(b) = e(a)$, contradicting the hypothesis that $e(a) \leq i - 1 < f$. Hence a is covered by at least one element of Ω, which is necessarily in I_i. This completes the proof. □

5 Stable polytopes of perfect graphs

We consider simple (with no loops or multiple edges) graphs G on the finite set of nodes $[N] := \{1, 2, \ldots, N\}$. For $W \subseteq [N]$ we denote by $\rho(W)$ the 0/1 vector $\sum_{i \in W} e_i \in \mathbb{R}^N$, where e_i is the ith unit coordinate vector in $\mathbb{R}^N$, and call W a *stable set* for G if no two elements of W are joined by an edge in G. The *stable polytope* of G, introduced in [4] and denoted by $P(G)$, is the convex hull of all vectors $\rho(W)$, where W is a stable set for G. Observe that the empty set and all singleton subsets of $[N]$ are stable and hence $P(G)$ has dimension N. The *chromatic number* of G is the least number r of colors that can be assigned to the vertices of G, one color to each vertex, so that no two adjacent vertices of G are assigned the same color. The graph G is called *complete* if any two of its vertices are connected by an edge and *perfect* if for any induced subgraph H of G the chromatic number of H is equal to the number of vertices of the largest complete subgraph of H. We call G *semipure* if there exists a positive integer j such that all maximal complete subgraphs of G have either $j-1$ or j vertices. Thus if G is perfect and semipure with chromatic number r then all maximal complete subgraphs of G have either $r-1$ or r vertices. The following theorem applies in particular to all bipartite graphs and all perfect graphs of chromatic number three with no isolated vertices.

Theorem 5.1 *If G is a semipure perfect graph with N vertices and chromatic number r and $d = N - r + 1$ then the h^*-vector $(h_0^*, h_1^*, \ldots, h_N^*)$ of the stable polytope $P(G)$ satisfies $h_i^* \geq h_{d-i}^*$ for $1 \leq i \leq \lfloor d/2 \rfloor$, $h_{\lfloor d/2 \rfloor}^* \geq h_{\lfloor d/2 \rfloor + 1}^* \geq \cdots \geq h_{d-1}^*$ and $h_i^* = 0$ for $d \leq i \leq N$.*

Proof. It was proved in [14] that all pulling triangulations of stable polytopes of perfect graphs are unimodular. In view of Theorem 1.1 and the last statement in Corollary 3.4 it suffices to prove that there exists a semispecial $(r-1)$-simplex for $P(G)$ with integer vertices which is not special. Consider a coloring of G with colors $1, 2, \ldots, r$ and for $1 \leq i \leq r$ let W_i be the set of vertices of G colored with i, so that W_i is a stable set for G. We claim that $\Sigma = \mathrm{conv}\{\rho(W_i) : 1 \leq i \leq r\}$ is such a simplex for $P(G)$. Since G is perfect, by [4, Theorem 3.1] a facet of $P(G)$ is defined by a linear equality of the form $x_i = 0$ or $\sum_{j \in U} x_j = 1$ for some vertex set U of a maximal complete subgraph of G. A facet of the first form contains exactly $r-1$ of the points $\rho(W_i)$ since the sets W_i form a partition of $[N]$. Since G is also semipure, a facet of the second form contains either $r-1$ or r points $\rho(W_i)$, where the second case occurs, and the claim follows from Remark 3.1. $\square$

6 Affine semigroup rings and ideals

Let P be an m-dimensional integer polytope in $\mathbb{R}^N$ and $\mathbb{K}$ be a field. We denote by R_P the subalgebra of the algebra $\mathbb{K}[x_1, \ldots, x_N, x_1^{-1}, \ldots, x_N^{-1}, t]$ of Laurant polynomials over $\mathbb{K}$ generated by the monomials $x^\alpha t^r$ for positive integers r and $\alpha \in \mathbb{Z}^N$ such that $\alpha / r \in P$. The algebra R_P can be graded by letting $x^\alpha t^r$ have degree r. With this grading R_P is a semistandard graded Cohen-Macaulay normal domain, called the *semigroup ring* of P, whose Hilbert series is the Ehrhart series (1) of P. See [3, Chapter 6] for background on semigroup rings. Let $\tilde{P} = \{(1, x) : x \in P\}$ be the lift of P in the hyperplane $x_0 = 1$ in $\mathbb{R}^{N+1}$ and denote by $\mathcal{C}_P$ the cone in $\mathbb{R}^{N+1}$ generated by $\tilde{P}$ and by E_P the semigroup of integer points in $\mathcal{C}_P$. Let $\mathcal{L}$ be the linear span of $\mathcal{C}_P$ in $\mathbb{R}^{N+1}$ and $\mathcal{A}$ be a set of integer linear forms in $\mathbb{R}^{N+1}$, one for each facet of P, defining the cone $\mathcal{C}_P$ as the set of points $x \in \mathcal{L}$ satisfying $g(x) \geq 0$ for all $g \in \mathcal{A}$. For $\mathcal{G} \subseteq \mathcal{A}$ let $E_\mathcal{G}$ be the semigroup of elements α of E_P satisfying $g(\alpha) > 0$ for $g \in \mathcal{G}$ and $I_\mathcal{G}$ be the ideal of R_P generated by the monomials x^α with $\alpha \in E_\mathcal{G}$ (observe that the variable t has been replaced by x_0). We say that $E_\mathcal{G}$ has a *unique minimal element* β if $\beta + E_P = E_\mathcal{G}$ or, equivalently, if $I_\mathcal{G}$ is generated by the monomial x^β as an ideal of R_P. The proof of the following lemma is similar to that of [1, Corollary 4.1].

Lemma 6.1 *If $\mathcal{G} \subseteq \mathcal{A}$ and $v_1 + v_2 + \cdots + v_n$ is the unique minimal element of $E_\mathcal{G}$ for some integer points $v_1, v_2, \ldots, v_n$ in $\tilde{P}$ then $\{v_1, v_2, \ldots, v_n\}$ is the vertex set of a semispecial $(n-1)$-simplex for $\tilde{P}$.*

Proof. Let $\beta = v_1 + v_2 + \cdots + v_n$ and observe that $v_1, v_2, \ldots, v_n$ are necessarily distinct. Let F be a facet of $\tilde{P}$ with corresponding linear form $f \in \mathcal{A}$. If f is not in $\mathcal{G}$ then there exists a $v \in E_\mathcal{G}$ with $f(v) = 0$ and, by the minimality of β, $f(\beta) = 0$. Hence $f(v_i) = 0$, meaning that $v_i \in F$, for all $1 \leq i \leq n$. Suppose that $f \in \mathcal{G}$, so that $f(\beta) > 0$. We need to show that at most one of $v_1, v_2, \ldots, v_n$ satisfies $f(v_i) > 0$. Clearly at least one of them has this property. Assume that at least two of $v_1, v_2, \ldots, v_n$ satisfy $f(v_i) > 0$, say v_j is one of them, and let $f(\beta) = b$ and $f(v_j) = c$, so that $1 \leq c < b$. Since F is a facet of $\tilde{P}$ there exists a point x in the affine span of $\tilde{P}$, which we may assume to have rational coordinates, satisfying $f(x) < 0$ and $g(x) > 0$ for all $g \in \mathcal{A}$ other than f. By replacing x with a suitable positive integer multiple we find an integer point α in $\mathcal{L}$ satisfying $f(\alpha) < 0$ and $g(\alpha) > 0$ for all $g \in \mathcal{A}$ other than f. Letting $a = f(\alpha)$, we may choose a nonnegative integer t so that $0 < a + b + tc < b$. Then $\gamma = \alpha + \beta + tv_j$ is in $E_\mathcal{G}$ and satisfies $f(\gamma) < f(\beta)$, which contradicts the minimality of β. □

Recall that R_P is *standard* if it is generated by its homogeneous elements of degree one or, equivalently, if E_P is generated as a semigroup by the integer points in $\tilde{P}$. Clearly this holds if P has a unimodular triangulation. In the case $\mathcal{G} = \mathcal{A}$ assumption (ii) in the following corollary is equivalent to the statement that the ring R_P is Gorenstein [3, Corollary 6.3.8].

Corollary 6.2 *If*

(i) *all pulling triangulations of P with integer vertices are unimodular and*

(ii) *$I_\mathcal{G}$ is generated by one element as an ideal of R_P*

then the conclusion of Corollary 3.4 holds where, in the statement of the corollary, n is the x_0-coordinate of the unique minimal element of $E_\mathcal{G}$ and the equality $h^*_d = 0$ holds if $\mathcal{G} \subsetneq \mathcal{A}$.

Proof. Let β be the unique minimal element of $E_{\mathcal{G}}$, whose existence is guaranteed by (ii). Assumption (i) implies that R_P is standard and hence $\beta = v_1 + v_2 + \cdots + v_n$ for some integer points $v_1, v_2, \ldots, v_n$ in $\tilde{P}$, which are the vertices of a semispecial $(n-1)$-simplex for $\tilde{P}$ by Lemma 6.1. Because of (i) any sequence $(v_1, v_2, \ldots, v_p)$ of integer points containing the vertices of $\tilde{P}$ satisfies the assumptions of Corollary 3.4. The result follows from this corollary observing that β has x_0-coordinate equal to n. □

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