Triangulations and the Hajós Conjecture

Bojan Mohar*
Department of Mathematics,
University of Ljubljana,
1000 Ljubljana, Slovenia
bojan.mohar@fmf.uni-lj.si

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Abstract

The Hajós Conjecture was disproved in 1979 by Catlin. Recently, Thomassen showed that there are many ways that Hajós conjecture can go wrong. On the other hand, he observed that locally planar graphs and triangulations of the projective plane and the torus satisfy Hajós Conjecture, and he conjectured that the same holds for arbitrary triangulations of closed surfaces. In this note we disprove the conjecture and show that there are different reasons why the Hajós Conjecture fails also for triangulations.

1 Introduction

Hajós conjecture claims that every graph whose chromatic number is at least k contains a subdivision of K_k, the complete graph of order k. The conjecture has been proved for $k \leq 4$ by Dirac [3], while for $k = 5$, it yields a strengthening of the Four Color Theorem, which is still open. The conjecture was disproved for all $k \geq 7$ by Catlin [2]. Soon after that, Erdős and Fajtlowicz [4] proved that the conjecture is false for almost all graphs.

Recently, Thomassen [11] revived the interest in Hajós conjecture by showing that there is a great variety of reasons why Hajós conjecture can be wrong. At the end of this interesting work, Thomassen observed that the Hajós conjecture could be true to some limited extent. Maybe it holds in the setting under whose influence it was originally formulated (related to the Four Color Conjecture). For instance, it holds for graphs embedded in any fixed surface with sufficiently large edge-width (i.e., when all noncontractible cycles are long). Therefore, it seems plausible to propose:

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Conjecture 1.1 (Thomassen [11]) *Every graph that triangulates some surface satisfies Hajós conjecture.*

Every n-vertex triangulation of a nonplanar surface has at least $3n-3$ edges. Therefore, it contains a subdivision of K_5 by a theorem of Mader [6]. In particular, possible counterexamples to Conjecture 1.1 must have chromatic number at least 6. Thomassen [11] used a known list of 6- and 7-critical graphs on the projective plane and the torus [10] to prove that Conjecture 1.1 holds for triangulations on these two surfaces.

In this note we provide counterexamples to Conjecture 1.1. Additionally, we give some reasons showing that, in certain sense, almost all graphs should be close to some counterexamples.

2 A small counterexample

Let $\tilde{H}$ be the graph with vertex set

$$V = \{v_{i,j} \mid 0 \le i \le 4,\ 0 \le j \le 2\} \cup \{w_i \mid 0 \le i \le 4\}$$

in which distinct vertices $v_{i,j}$ and $v_{k,l}$ are adjacent if and only if $k \in \{i-1, i, i+1\}$ (where $i-1$ and $i+1$ are taken modulo 5), and w_i is adjacent to all vertices $v_{i,j}$ and $v_{i+1,j}$, $0 \le j \le 2$. In other words, $\tilde{H}$ is composed of 5 copies Q_i ($0 \le i \le 4$) of the graph K_7 with vertices $C_i^- = \{v_{i,0}, v_{i,1}, v_{i,2}\}$, $C_i^+ = \{v_{i+1,0}, v_{i+1,1}, v_{i+1,2}\}$, and w_i, and $C_i^+ \subset Q_i$ is identified with $C_{i+1}^- \subset Q_{i+1}$ for $0 \le i \le 4$ (indices modulo 5).

The graph $\tilde{H}$ is a counterexample to Conjecture 1.1:

Theorem 2.1 *The graph $\tilde{H}$ triangulates the orientable surface of genus 6 and the nonorientable surface of genus 12. Its chromatic number is 8 and it does not contain a subdivision of K_8.*

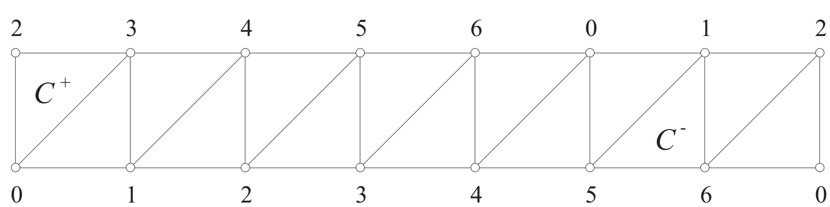

Figure 1: K_7 on the torus

Proof. The graph K_7 triangulates the torus. See Figure 1, where the edge on the left is identified with the rightmost one, and the two cycles of the resulting cylinder are then identified as shown by the labels. Consider 5 such triangulations using $Q_0, \ldots, Q_4$. We may assume that the triangles C_i^+ and C_i^- of each Q_i are facial; suppose that they correspond to the triangles that are labeled 023 and 156 in Figure 1. By identifying C_i^+

with C_{i+1}^- for $i = 0, 1, 2, 3$, we obtain the connected sum of five tori. Finally, by identifying C_4^+ with C_0^-, the orientable surface of genus 6 is obtained. Clearly, the resulting graph is isomorphic to $\tilde{H}$ and triangulates the surface. Instead of the orientable surface we can get a nonorientable surface of the same Euler characteristic by taking for C_0^- in Q_0 the triangle 165 instead of 156.

Let H be the subgraph of $\tilde{H}$ obtained by removing vertices $w_0, \ldots, w_4$. Clearly, no three vertices of H are independent. Thus, $\chi(H) \geq \lceil |V(H)|/2 \rceil = 8$. Also, it is easy to find an 8-coloring of H, and any such coloring extends to $\tilde{H}$ since the degrees of the removed vertices w_i are equal to 6. Consequently, $\chi(\tilde{H}) = \chi(H) = 8$.

Let us now prove that $\tilde{H}$ does not contain a subdivision of K_8. Assume, by reductio ad absurdum, that $L \subseteq \tilde{H}$ is such a subgraph. Vertices of degree 7 in L are *branch vertices*; vertices of degree 2 subdivide edges of K_8. Since $\deg_{\tilde{H}}(w_i) = 6$, no w_i is a branch vertex. Since the neighbors of w_i form a complete subgraph, we may also assume that w_i does not subdivide an edge in L for $i = 0, \ldots, 4$. In particular, we conclude that $L \subseteq H$.

Let $B \subseteq V(L)$ be the set of branch vertices and let s be the number of vertices of degree 2 in L. Clearly, $|B| = 8$ and $s \leq |V(H) \setminus B| = 7$. Let $B_i = B \cap C_i^+$ and $b_i = |B_i|$ for $0 \leq i \leq 4$. We may assume that $b_0 = \max_i b_i$ and that, subject to this assumption, $b_0 + b_1$ is maximum. Then it is easy to see that $(b_0, b_1) \in \{(3, 3), (3, 2), (3, 1), (2, 2)\}$. We let $b = b_0 + b_1$.

There are $8 - b$ branch vertices that need to be joined to b branch vertices in $B_0 \cup B_1$. Each such vertex $x \in B_2 \cup B_3 \cup B_4$ is linked to all vertices in $B_0 \cup B_1$; some of the subdivided edges must contain two or three vertices of degree 2. It is easy to see that for every such x, at least four vertices of degree 2 are needed, with two possible exceptions in the cases when $b_0 = b_1 = 2$ or $b_0 = 3$, $b_1 = 1$. In any case, we conclude that $s \geq 8$, and this contradiction completes the proof. □

3 Paley graphs

Let $q = p^r$ be a prime power, where $p \equiv 1 \pmod 4$. The *Paley graph* $\mathbf{P}_q$ is the Cayley graph of the additive group of the finite field $GF(q)$ generated by all squares, i.e., $V(\mathbf{P}_q) = GF(q)$ and two distinct vertices x, y are adjacent if and only if $x - y = z^2$ for some $z \in GF(q)$. Since $p \equiv 1 \pmod 4$, $x - y$ is a square if and only if $y - x$ is a square, and this assures that $\mathbf{P}_q$ is a graph and not a digraph.

Paley graphs have a number of intriguing properties. First of all, they are highly symmetric. Secondly, they are self-complementary. On the other hand, they resemble random graphs a lot. One of their properties which makes them similar to random graphs is the following result of Thomason [9]; see also [1, p. 363].

Theorem 3.1 *Let A be a set of vertices of the Paley graph $\mathbf{P}_q$. Let $a = |A|$, and let $e(A)$ be the number of edges in the induced subgraph $\mathbf{P}_q(A)$. Then*

$$\left| e(A) - \frac{1}{2}\binom{a}{2} \right| \leq \frac{a(q-a)}{4\sqrt{q}}.$$

This theorem implies that the maximum clique in $\mathbf{P}_q$ has at most $\sqrt{q}$ vertices. Equality is attained if q is an even power of a prime. But if q is a prime, it seems that the maximum cliques in $\mathbf{P}_q$ have far smaller orders. It is believed that their orders are at most polylogarithmic, possibly $o(\log^2 q)$ or even smaller. See [1] for some discussion on this problem.

Let z be a generator of the multiplicative group $GF(q)^*$. It is known that this group is cyclic, so $z^0, z^2, z^4, \ldots, z^{q-3}$ are precisely all squares. For each vertex $x \in V(\mathbf{P}_q)$, its neighbors are $x+z^0, x+z^2, x+z^4, \ldots, x+z^{q-3}$. This (cyclic) sequence defines a local rotation around x, i.e., a clockwise cyclic order of edges incident with x. The collection of all such local rotations defines an orientable embedding Π_q of $\mathbf{P}_q$ which is known as the *Paley map*. It has the following properties (see [12]):

Theorem 3.2 *Suppose that $q = p^r$, where $p \equiv 1 \pmod 8$ is a prime. Then the Paley map Π_q is self-dual – the geometric dual graph $\mathbf{P}_q^*$ is isomorphic to $\mathbf{P}_q$. Its genus is equal to $(q^2-9q+8)/8$. It has q faces, each of which has length $(q-1)/2$ and is bounded by a cycle of $\mathbf{P}_q$.*

We refer to [12] for more details concerning this interesting map.

From now on we assume that $q \equiv 1 \pmod 8$. Let T_q be the triangulation of the orientable surface of genus $(q^2-9q+8)/8$ obtained from the Paley map Π_q by adding a new vertex into each face and joining it to all vertices on the boundary of that face. Let $V_q = V(\mathbf{P}_q)$ and let V_q^* be the added vertices. Then $|V_q| = |V_q^*| = q$.

Theorem 3.3 *Suppose that $q = p^r$, where $p \equiv 1 \pmod 8$ is a prime. The triangulation T_q contains no subdivision of the complete graph of order $\geq \lambda\sqrt{q}$, where $\lambda = \frac{1}{4}(2+\sqrt{198}) <$ 4.0178.*

Proof. Let $s = \sqrt{q}$ and $k = \lceil \lambda s \rceil$. Suppose that T_q contains a subdivision K of K_k. Let $B_0 \subseteq V(K)$ be the set of branch vertices, and let $b = |B_0 \cap V_q|$ and $b^* = |B_0 \cap V_q^*|$. Clearly, $b + b^* = k$.

Let us count the number of subdivided edges in K. Since no two vertices in V_q^* are adjacent, $B_0 \cap V_q^*$ gives rise to $\binom{b^*}{2}$ subdivided edges. Concerning the set $B_0 \cap V_q$, Theorem 3.1 shows that this set gives rise to at least $\frac{1}{2}\binom{b}{2} - \frac{b(q-b)}{4s}$ subdivided edges. Since the total number of vertices of T_q is equal to $2q$, we see that:

$$k + \binom{b^*}{2} + \frac{1}{2}\binom{b}{2} - \frac{b(q-b)}{4s} \leq 2q. \tag{1}$$

Let $b^* = \alpha^* s$ and $b = \alpha s$. Then (1) expands to the following condition

$$s(2\alpha^{*2} + \alpha^2 - \alpha - 8) + 2\alpha^* + 3\alpha + \alpha^2 \leq 0. \tag{2}$$

Let $\Lambda = \alpha^* + \alpha$. Then (2) implies that

$$2(\Lambda - \alpha)^2 + \alpha^2 - \alpha - 8 < 0. \tag{3}$$

By considering (3) as a quadratic inequality in α, its discriminant is $97 + 8\Lambda - 8\Lambda^2$. Therefore, (3) has no solutions if

$$8\Lambda^2 - 8\Lambda - 97 \geq 0. \tag{4}$$

Since $\Lambda s = k = \lceil \lambda s \rceil \geq \lambda s$, we have $\Lambda \geq \lambda = \frac{1}{4}(2 + \sqrt{198})$. This implies (4) and proves that (3) has no solution. This contradiction completes the proof. $\square$

Theorem 3.3 now implies:

Theorem 3.4 *Suppose that $q = p^r$, where $p \equiv 1 \pmod 8$ is a prime. If the chromatic number of $\mathbf{P}_q$ is at least $\lambda\sqrt{q}$, where $\lambda = \frac{1}{4}(2 + \sqrt{198})$, then the triangulation T_q fails to satisfy the Hajós Conjecture.*

In the above theorem, it suffices to assume that $\chi(T_q) \geq \lambda\sqrt{q}$. But since the vertex set V_q^* is independent in T_q, $\chi(T_q) \leq \chi(\mathbf{P}_q) + 1$. Therefore, we find it more pleasing to use the natural value $\chi(\mathbf{P}_q)$.

The independence numbers $\alpha(\mathbf{P}_q)$ of Paley graphs of prime order q less than 7000 were computed by Shearer [8]. Using these calculations and estimate the chromatic number by $\chi(\mathbf{P}_q) \geq \lceil q/\alpha(\mathbf{P}_q) \rceil$, one finds out that Theorem 3.4 can be applied for many values of q (and when q gets large, almost all of them are good). All such primes $q \leq 7000$ with $q \equiv 1 \pmod 8$ are collected in Table 1.

As stated after Theorem 3.1, it is believed that for every prime q, $\omega(\mathbf{P}_q) = o(\sqrt{q})$. If true, this yields infinitely many cases where Theorem 3.4 can be applied.

4 Conclusions

We have proved that there is a variety of reasons why a triangulation of some surface may fail to satisfy the Hajós Conjecture. In this sense, this note can be viewed as an echo to the stimulating work of Thomassen [11]. As noted in that paper, graphs embedded in a surface Σ with sufficiently large edge-width satisfy the Hajós Conjecture. Recall that the *edge-width* of a graph embedded in a nonplanar surface is the length of a shortest noncontractible cycle. In the above observation of Thomassen, the required width depends on Σ. However, the following strengthening may be true:

Conjecture 4.1 *There is an absolute constant w_0 such that every graph which can be embedded in some surface with edge-width at least w_0 satisfies the Hajós Conjecture.*

Kühn and Osthus [5] proved that graphs whose girth is at least 186 satisfy the Hajós conjecture. This excludes the most obvious possibility of counterexamples to Conjecture 4.1. (Note that the edge-width of a graph of girth g is at least g.)

We do not dare to estimate what the best possible value for w_0 may be. However, if we restrict ourselves to triangulations, no counterexamples of edge-width 4 are known.

q	$\alpha(\mathbf{P}_q)$	$\lceil q/\alpha(\mathbf{P}_q)\rceil$	$\lceil\lambda\sqrt{q}\rceil$
3697	15	247	245
4217	15	282	261
4441	15	297	268
4457	15	298	269
4649	17	274	274
4673	17	275	275
4721	17	278	277
4729	17	279	277
4793	16	300	279
4817	17	284	279
4937	17	291	283
4969	17	293	284
4993	17	294	284
5081	17	299	287
5153	17	304	289
5233	17	308	291
5281	17	311	292
5297	17	312	293
5393	17	318	296
5441	17	321	297
5449	17	321	297
5569	18	310	300
5657	18	315	303
5737	17	338	305
5801	17	342	307
5849	17	345	308
5857	17	345	308
5881	19	310	309
5897	17	347	309

q	$\alpha(\mathbf{P}_q)$	$\lceil q/\alpha(\mathbf{P}_q)\rceil$	$\lceil\lambda\sqrt{q}\rceil$
5953	17	351	310
6073	19	320	314
6089	17	359	314
6113	19	322	315
6121	18	341	315
6217	19	328	317
6257	19	330	318
6329	19	334	320
6337	17	373	320
6353	17	374	321
6361	18	354	321
6449	19	340	323
6473	19	341	324
6481	17	382	324
6529	17	385	325
6553	19	345	326
6569	19	346	326
6577	19	347	326
6673	19	352	329
6689	19	353	329
6737	18	375	330
6761	20	339	331
6793	19	358	332
6833	19	360	333
6841	17	403	333
6857	17	404	333
6961	19	367	336
6977	18	388	336

Table 1: Some good values of q

Problem 4.2 *Does every triangulation without noncontractible triangles satisfy the Hajós Conjecture?*

For a graph H and a positive integer t, let $\tilde{H}_t$ be the graph obtained from the union of H and the clique K_t by adding all edges between them. Clearly, the Hajós Conjecture holds for H if and only if it holds for $\tilde{H}_t$.

Let v and e ($\tilde{v}$ and $\tilde{e}$) be the number of vertices and edges of H (and $\tilde{H}_t$), respectively. Clearly, $\tilde{v} = v + t$ and $\tilde{e} = e + \binom{t}{2} + vt$. If $\tilde{H}_t$ triangulates some surface, then Euler's formula implies that $\tilde{e}$ is divisible by 3. This holds if and only if one of the following holds:

(A) $e \equiv 0 \pmod 3$ and $t \equiv 0 \pmod 3$,

(B) $v + e \equiv 0 \pmod 3$ and $t \equiv 1 \pmod 3$, or

(C) $e - v \equiv 2 \pmod 3$ and $t \equiv 2 \pmod 3$.

Conjecture 4.3 *For every graph H, there exists a constant t_0 such that for every $t \geq t_0$ for which one of* (A)–(C) *is satisfied, the graph $\tilde{H}_t$ triangulates some surface.*

Conjecture 4.3 would imply that for every counterexample H to the Hajós Conjecture for which either $e \equiv 0 \pmod 3$, $v + e \equiv 0 \pmod 3$, or $e - v \equiv 2 \pmod 3$, there are infinitely many triangulations of the form $\tilde{H}_t$ which violate the Hajós Conjecture.

Conjecture 4.3 would follow from a solution of a more general open problem which has been raised in [7, Problem 4.4.10]:

Problem 4.4 *Is there an $\varepsilon > 0$ such that every graph of order n, with e edges, where e is divisible by* 3*, and with minimum degree at least $(1 - \varepsilon)n$ triangulates some surface?*

A positive answer to Problem 4.4 would imply that almost one third of random graphs in $\mathcal{G}(n, p)$, where $p > 1-\varepsilon$, would be graphs of some triangulations. As shown by Erdős and Fajtlowicz [4], almost all of these graphs would also be counterexamples to Conjecture 1.1.

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