Distinguishing Cartesian Powers of Graphs

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Abstract

Given a graph G, a labeling $c : V(G) \to \{1, 2, \ldots, d\}$ is said to be *d-distinguishing* if the only element in $\mathrm{Aut}(G)$ that preserves the labels is the identity. The *distinguishing number* of G, denoted by $D(G)$, is the minimum d such that G has a d-distinguishing labeling. If $G \square H$ denotes the Cartesian product of G and H, let $G^{2} = G \square G$ and $G^{r} = G \square G^{r-1}$. A graph G is said to be *prime* with respect to the Cartesian product if whenever $G \cong G_1 \square G_2$, then either G_1 or G_2 is a singleton vertex. This paper proves that if G is a connected, prime graph, then $D(G^{r}) = 2$ whenever $r \geq 4$.

1 Introduction

Given a graph G, a labeling $c : V(G) \to \{1, 2, \ldots, d\}$ is d-distinguishing if the only element in $\mathrm{Aut}(G)$ that preserves the labels is the identity. The idea is that the labeling together with the structure of G uniquely identifies every vertex. Formally, c is said to be *d-distinguishing* if $\phi \in \mathrm{Aut}(G)$ and $c(\phi(x)) = c(x)$ for all $x \in V(G)$ implies that $\phi = id$. The *distinguishing number* of G, denoted by $D(G)$, is the minimum d such that G has a d-distinguishing labeling. It is a measure of the relative symmetry of G.

It is immediate that $D(K_n) = n$ and when $q > p, D(K_{p,q}) = q$. It is straightforward to see that $D(K_{n,n}) = n + 1$. The original paper on distinguishing [1] was inspired by a recreational puzzle [5]. The solution requires showing that if $n \geq 6$, then $D(C_n) = 2$. The attraction of this puzzle is the contrast with smaller cycles where $D(C_n) = 3$ when $3 \leq n \leq 5$.

The inspiration for this paper is the solution to the problem of distinguishing the generalized cubes. Let Q_r denote the r-dimensional hypercube: $V(Q_r) = \{\mathbf{x} = (x_1, \ldots, x_r) : x_i \in \mathbb{Z}_2\}$ and $\mathbf{xy} \in E(Q_r)$ if $\mathbf{x}$ and $\mathbf{y}$ differ in exactly one coordinate. Note that $Q_2 = C_4$, Q_3 is the standard cube, and $D(Q_2) = D(Q_3) = 3$.

The Cartesian (or box) product of two graphs G and H, denoted by $G\Box H$, is the graph whose vertex set $V(G\Box H) = \{(u,v) : u \in V(G), v \in V(H)\}$. The vertex (u,v) is adjacent to the vertex (w,z) if either $u = w$ and $vz \in E(H)$ or $v = z$ and $uw \in E(G)$. The *box* notation illustrates the Cartesian product of two edges. Here we let G^2 denote $G\Box G$ and recursively let $G^r = G\Box G^{r-1}$. The connection between hypercubes and Cartesian products is that $Q_r = K_2^r$. For more on Cartesian products see [4].

Recently Bogstead and Cowen showed that if $r \geq 4$, then $D(Q_r) = 2$ [2]. Their proof idea is elegant: find H, an induced subgraph of G, such that any nontrivial automorphism of G maps some vertex in H to a vertex not in H. In such a circumstance the natural labeling $\{c(x) = 2$ if $x \in V(H)$ and $c(x) = 1$ otherwise$\}$ is 2-distinguishing. Using this technique it is straightforward to prove that $D(K_3^3) = D(P_3^2) = 2$, and it is natural to think that larger powers of these graphs will also be 2-distinguishable. All of this suggests the following conjecture.

Conjecture 1. If G is connected, then there exists $R = R(G)$ such that if $r \geq R$, then $D(G^r) = 2$.

The connectivity is necessary since if G is two independent vertices, then $D(G^r) = 2^r$.

This purpose of this note is to prove Theorem 2, a significant strengthening of the above conjecture for a slightly smaller class of graphs. In its full generality Conjecture 1 remains open.

2 Cartesian Products

A graph H is said to be *prime* with respect to the Cartesian product if whenever $H \cong H_1\Box H_2$, then either H_1 or H_2 is a singleton vertex. It is well known that if G is connected, then G has a unique prime factorization *i.e.* $G \cong H_1\Box H_2\Box\cdots\Box H_t$ such that for $1 \leq i \leq t$, H_i is prime. About thirty-five years ago Imrich and Miller independently showed the following theorem.

Theorem 1. [4] If G is connected and $G = H_1\Box H_2\Box\cdots\Box H_r$ is its prime decomposition, then every automorphism of G is generated by the automorphisms of the factors and the transpositions of isomorphic factors.

Corollary 1.1. If G is a connected prime graph with $|V(G)| = n$, then $\mathrm{Aut}(G^r) \leq \mathrm{Aut}(K_n^r)$

Proof. Since every automorphism of G is an automorphism of K_n, it follows that every automorphism of G^r is an automorphism of K_n^r. □

Corollary 1.2. If G is a connected prime graph with $|V(G)| = n$, then $D(G^r) \leq D(K_n^r)$.

Proof. Any labeling that destroys every automorphism of K_n^r must also destroy every automorphism of G^r. □

We now state our main result, though its proof will be postponed until the end of the next section.

Theorem 2. If G is a connected graph that is prime with respect to the Cartesian product, then $D(G^r) = 2$ whenever $r \geq 4$. Futhermore, if in addition, $|V(G)| \geq 5$, then $D(G^r) = 2$ whenever $r \geq 3$.

It is well known that almost all graphs are connected. Graham [3] has shown that almost all graphs are irreducible with respect to the Θ^* equivalence class; see [4]. Since every such irreducible graph is prime, almost all graphs satisfy the hypotheses of Theorem 2.

It seems that it should be possible to prove Theorem 2 using the Bogstead Cowen strategy. Whether there is such a proof remains open.

3 The Motion Lemma and Its Consequences

For $\sigma \in \mathrm{Aut}(G)$ let $m(\sigma) = |\{x \in V(G) : \sigma(x) \neq x\}|$ and let $m(G) = \min\{m(\sigma) : \sigma \neq id\}$. Call $m(\sigma)$ the *motion* of σ and $m(G)$ the *motion* of G. Using an appealing probabilistic argument Russell and Sundaram showed that if the motion of G is large, then the distinguishing number of G is small. Specifically they proved the *motion lemma*, Theorem 3.

Theorem 3. [6] If $d^{\frac{m(G)}{2}} > |\mathrm{Aut}(G)|$, then $D(G) \leq d$.

To apply the motion lemma we need determine $|\mathrm{Aut}(K_n^r)|$ and $m(K_n^r)$.

Theorem 4. $|\mathrm{Aut}(K_n^r)| = r!(n!)^r$.

Proof. K_n^r is vertex transitive and has n^r vertices. Each vertex, say x, is contained in exactly r cliques of size n and the vertices in these cliques are disjoint except for x. An automorphism might take x to any of the n^r vertices. Once the image of x is chosen, then a clique that contains x can be mapped to a clique that contains the image of x in any of $r(n-1)!$ ways. A second clique containing x can be mapped in any of $(r-1)(n-1)!$ ways. The j^{th} clique containing x can be mapped in any of $(r-j+1)(n-1)!$ ways. Once all cliques containing x are mapped, the entire automorphism is fixed. Alternatively, one can recognize $\mathrm{Aut}(K_n^r)$ as an appropriate wreath product and arrive at the count that way. $\square$

Theorem 5. If $n \geq 3$, then $m(K_n^r) = 2n^{r-1}$.

Proof. For every $x_2, \ldots, x_r$, let σ_0 be the automorphism of K_n^r in which $\sigma_0(1, x_2, \ldots, x_r) = (2, x_2, \ldots, x_r)$; $\sigma_0(2, x_2, \ldots, x_r) = (1, x_2, \ldots, x_r)$; and σ_0 fixes everything else. Clearly $m(\sigma_0) = 2n^{r-1}$. It remains to show that no non-trivial automorphism has smaller motion.

The proof that $m(K_n^r) \geq 2n^{r-1}$ will use a combination of induction and contradiction. The base case holds since when $r = 1$, any non-identity automorphism must move at least two vertices.

Let $F_{j_1,j_2,\ldots,j_t} = \{(x_1,\ldots,x_r) \in V(K_n^r) : x_1 = j_1, x_2 = j_2, \ldots, x_t = j_t\}$. The notation is chosen to emphasize that we are looking at vertices in K_n^r whose first coordinates are fixed. Let $L_k = \{(x_1,\ldots x_r) \in V(K_n^r) : x_r = k\}$. The notation is chosen to emphasize that we are looking at vertices in K_n^r whose last coordinate is fixed. Note that $|F_{j_1,j_2,\ldots,j_t}| = n^{r-t}$ and that $|L_k| = n^{r-1}$.

If $\sigma \in \mathrm{Aut}(K_n^r)$ is such that $0 < m(\sigma) < 2n^{r-1}$, then σ fixes more than $(n-2)n^{r-1}$ vertices. By the pigeonhole principle and appropriate reindexing there exists $j_1, j_2, \ldots, j_{r-1}$ such that σ fixes more than $(n-2)n^{r-2}$ vertices in F_{j_1}; σ fixes more than $(n-2)n^{r-3}$ vertices in F_{j_1,j_2}; σ fixes more than $(n-2)n^{r-s-1}$ vertices in $F_{j_1,j_2,\ldots,j_s}$; and σ fixes more than $n-2$ vertices in $F_{j_1,\ldots,j_{r-1}}$. Alternatively σ moves at most one vertex in this clique. Since $n \geq 3$, σ fixes the entire clique $F_{j_1,\ldots,j_{r-1}}$.

For $1 \leq k \leq n$, $L_k \cap F_{j_1,\ldots,j_{r-1}} = \{(j_1, j_2, \ldots, j_{r-1}, k)\}$. This vertex is fixed by σ. Now any vertex in K_n^r that is adjacent to $(j_1, j_2, \ldots, j_{r-1}, k)$ is either in $F_{j_1,\ldots,j_{r-1}}$ or in L_k. In the former case it is fixed by σ. In the latter case in order to preserve adjacency, it must be mapped to a vertex in L_k. Now all the vertices in L_k that are at distance two from $(j_1, j_2, \ldots j_{r-1}k$ must also be mapped to L_k. Continuing we see that σ maps L_k to itself.

Next, for the moment suppose that for a particular value of k, L_k is fixed by σ. Since every vertex in $K_n^r - L_k$ is adjacent to exactly one vertex in L_k, σ must map $L_1, L_2, \ldots L_n$ onto $L_1, L_2, \ldots, L_n$. Since σ is the identity on $F_{j_1,\ldots,j_{r-1}}$, σ is the identity on all of K_n^r.

Thus we may assume that for every k with $1 \leq k \leq n$, σ maps L_k to L_k moving some of the vertices in L_k. Since $\sigma|_{L_k}$ is an automorphism on K_n^{r-1} we can inductively assume that σ moves at least $2n^{r-2}$ vertices. Since this is true for each k, $m(\sigma) \geq 2n^{r-1}$. $\square$

We now turn to the proof of Theorem 2.

Proof. First we note that when $r > 1$, G^r is not rigid. Thus $D(K_n^r) > 1$. If $n = 2$, then Theorem 2 is just the result of Bogstead and Cowen. When $n \geq 3$ we can substitute the results of Theorems 3 and 4 into the Motion Lemma. Thus if $r!(n!)^r < 2^{n^{(r-1)}}$, then $D(K_n^r) \leq 2$.

Case (i): Suppose $n \geq r \geq 4$. It is straightforward to check the following inequalities. The logarithms are base 2.

$$\log(r!) + r\log(n!) < n\log(n) + n^2\log(n) < n^3 \leq n^{r-1}.$$

Exponentiating the extremes gives $r!(n!)^r < 2^{n^{(r-1)}}$.

Case (ii): Suppose $r > n \geq 3$ and $r \geq 5$. It is straightforward to check the following inequalities. The logarithms are base 2.

$$\log(r!) + r\log(n!) < r\log(r) + r^2\log(r) < 3^{r-1} \leq n^{r-1}.$$

Again exponentiating the extremes gives $r!(n!)^r < 2^{n^{(r-1)}}$.

Case (iii): Suppose $r = 4$ and $n = 3$. A direct calculation shows that $r!(n!)^r < 2^{n^{(r-1)}}$.

Finally it is straightforward to check that if $r = 3$ and $n \geq 5, 6(n!)^3 < 2^{n^2}$. $\square$

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