A Combinatorial Proof of a Symmetric q-Pfaff-Saalschütz Identity

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Abstract

We give a bijective proof of a symmetric q-identity on ${}_4\phi_3$ series, which is a symmetric generalization of the famous q-Pfaff-Saalschütz identity. An elementary proof of this identity is also given.

1 Introduction

Throughout this paper we regard q as an indeterminate, and we follow the notation and terminology in [5]. The *q-shifted factorials* are defined by

$$(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n,$$

where $(a;q)_0 = 1$ and $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ for $n \geq 1$.

In 1990, the second author [11] obtained a symmetric extension of a formula due to Ramanujan-Bailey, of which the analytical proof led to the following q-identity:

$$\begin{aligned} &\frac{(xz, yz; q)_m (z;q)_n}{(q, xyz; q)_m (q;q)_n} {}_4\phi_3 \left[\begin{matrix} q^{-n}, \ x, \ y, \ vq^m \\ v, \ xyzq^m, \ q^{1-n}/z, \end{matrix} ; q, q \right] \\ &\quad = \frac{(xz, yz; q)_n (z;q)_m}{(q, xyz; q)_n (q;q)_m} {}_4\phi_3 \left[\begin{matrix} q^{-m}, \ x, \ y, \ vq^n \\ v, \ xyzq^n, \ q^{1-m}/z, \end{matrix} ; q, q \right], \end{aligned} \tag{1.1}$$

where $m,n\in\mathbb{N}$. Indeed, applying Sears' transformation [5, p. 360, (III.15)] with $a=x, b=y, c=vq^m, d=v, e=xyzq^m$, and $f=q^{1-n}/z$ to the left-hand side of (1.1) yields the following identity

$$\begin{aligned}
&\frac{(xz,yz;q)_m(z;q)_n}{(q,xyz;q)_m(q;q)_n}{}_4\phi_3\left[\begin{matrix} q^{-n},\ x,\ y,\ vq^m \\ v,\ xyzq^m,\ q^{1-n}/z, \end{matrix} ;q,q\right] \\
&\quad=\frac{(xz;q)_m(xz;q)_n(yz;q)_{m+n}}{(xyz;q)_{m+n}(q;q)_m(q;q)_n}{}_4\phi_3\left[\begin{matrix} q^{-n},\ x,\ v/y,\ q^{-m} \\ v,\ xz,\ q^{1-m-n}/yz, \end{matrix} ;q,q\right]. \qquad (1.2)
\end{aligned}$$

It follows that the left-hand side of (1.1) is symmetric in m and n, which is exactly what (1.1) means.

In order to give a combinatorial proof of (1.1), we first rewrite (1.1) as

$$\begin{aligned}
&\frac{(xz,yz;q)_m}{(q,xyz;q)_m}\sum_{k=0}^{n}\frac{(x,y,vq^m;q)_k(z;q)_{n-k}}{(q,v,xyzq^m;q)_k(q;q)_{n-k}}z^k \\
&\quad=\frac{(xz,yz;q)_n}{(q,xyz;q)_n}\sum_{k=0}^{m}\frac{(x,y,vq^n;q)_k(z;q)_{m-k}}{(q,v,xyzq^n;q)_k(q;q)_{m-k}}z^k. \qquad (1.3)
\end{aligned}$$

Letting $x=q^{-(a-r)}$, $y=q^{-(b-r)}$, $z=q^{a+b+1}$, $v=q^{-(e-r)}$, $n=c-r$, and $m=d-r$, and using the formulas

$$(q^{-N};q)_k=(-1)^kq^{-kN+\binom{k}{2}}\frac{(q;q)_N}{(q;q)_{N-k}},\qquad (q^{N+1};q)_k=\frac{(q;q)_{N+k}}{(q;q)_N},\quad N\in\mathbb{N},$$

the left-hand side of (1.3) becomes

$$\begin{aligned}
&\frac{(q^{b+r+1},q^{a+r+1};q)_{d-r}}{(q,q^{2r+1};q)_{d-r}}\sum_{k=0}^{c-r}\frac{(q^{-(a-r)},q^{-(b-r)},q^{-(e-d)};q)_k(q^{a+b+1};q)_{c-r-k}}{(q,q^{-(e-r)},q^{d+r+1};q)_k(q;q)_{c-r-k}}q^{(a+b+1)k} \\
&=\frac{(q;q)_{b+d}(q;q)_{a+d}(q;q)_{2r}(q;q)_{a-r}(q;q)_{b-r}(q;q)_{e-d}}{(q;q)_{b+r}(q;q)_{a+r}(q;q)_{d-r}(q;q)_{a+b}(q;q)_{e-r}} \\
&\quad\times\sum_{k=0}^{c-r}q^{k(k+d+r)}\frac{(q;q)_{a+b+c-r-k}(q;q)_{e-r-k}}{(q;q)_k(q;q)_{a-r-k}(q;q)_{b-r-k}(q;q)_{e-d-k}(q;q)_{d+r+k}(q;q)_{c-r-k}},
\end{aligned}$$

where a,b,c,d,e are nonnegative integers and r is an integer such that $r\le\min\{a,b,c,d\}$ and $e\ge\max\{c,d\}$.

Exchanging c and d, we obtain a similar expression for the right-hand side of (1.3). Hence, after simplification, we see that (1.3) is equivalent to

$$\begin{aligned}
&\sum_{k=0}^{c-r}\frac{q^{k(k+d+r)}(q;q)_{b+d}(q;q)_{a+d}(q;q)_{e-d}(q;q)_{a+b+c-r-k}(q;q)_{e-r-k}}{(q;q)_{d-r}(q;q)_k(q;q)_{a-r-k}(q;q)_{b-r-k}(q;q)_{e-d-k}(q;q)_{d+r+k}(q;q)_{c-r-k}} \\
&=\sum_{k=0}^{d-r}\frac{q^{k(k+c+r)}(q;q)_{b+c}(q;q)_{a+c}(q;q)_{e-c}(q;q)_{a+b+d-r-k}(q;q)_{e-r-k}}{(q;q)_{c-r}(q;q)_k(q;q)_{a-r-k}(q;q)_{b-r-k}(q;q)_{e-c-k}(q;q)_{c+r+k}(q;q)_{d-r-k}}. \qquad (1.4)
\end{aligned}$$

Finally, shifting k to $k-r$ and using the q-binomial coefficient

$$\begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_M}{(q;q)_N (q;q)_{M-N}}, & \text{if } 0 \le N \le M, \\ 0, & \text{otherwise,} \end{cases}$$

we can rewrite (1.4) in the following form:

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} q^{(k-r)(k+d)} \begin{bmatrix} a+b+c-k \\ a-k \end{bmatrix} \begin{bmatrix} b+d \\ b-k \end{bmatrix} \begin{bmatrix} c-r \\ c-k \end{bmatrix} \begin{bmatrix} e-k \\ d-r \end{bmatrix} \\ &\quad = \sum_{k\in\mathbb{Z}} q^{(k-r)(k+c)} \begin{bmatrix} a+b+d-k \\ b-k \end{bmatrix} \begin{bmatrix} a+c \\ a-k \end{bmatrix} \begin{bmatrix} e-c \\ k-r \end{bmatrix} \begin{bmatrix} e-k \\ d-k \end{bmatrix}. \end{aligned} \tag{1.5}$$

Note that setting $d=r$ and letting $e \longrightarrow +\infty$ in (1.5) we recover the q-Pfaff-Saalschütz identity:

$$\sum_{k\in\mathbb{Z}} \frac{q^{k^2-r^2}[a+b+c-k]!}{[a-k]![b-k]![c-k]![k-r]![k+r]!} = \begin{bmatrix} a+b \\ a+r \end{bmatrix} \begin{bmatrix} a+c \\ c+r \end{bmatrix} \begin{bmatrix} b+c \\ b+r \end{bmatrix}, \tag{1.6}$$

where $[n]! = (q;q)_n/(1-q)^n$ for $n \ge 0$ and $1/[n]! = 0$ for $n < 0$. In the 1980's several authors [3, 6, 12] published combinatorial proofs of the q-Pfaff-Saalschütz identity. The main object of this paper is to provide a bijective proof of (1.5) by generalizing Zeilberger's combinatorial proof of the q-Pfaff-Saalschütz identity (1.6).

On the other hand, setting $x=0$ and letting $v \longrightarrow \infty$, identity (1.3) reduces to:

$$\frac{(yz;q)_m}{(q;q)_m} \sum_{k=0}^{n} \frac{(y;q)_k (z;q)_{n-k}}{(q;q)_k (q;q)_{n-k}} q^{mk} z^k = \frac{(yz;q)_n}{(q;q)_n} \sum_{k=0}^{m} \frac{(y;q)_k (z;q)_{m-k}}{(q;q)_k (q;q)_{m-k}} q^{nk} z^k, \tag{1.7}$$

while (1.2) reduces to

$${}_2\phi_1 \left[\begin{matrix} q^{-n}, \ y \\ q^{1-n}/z, \end{matrix} ; q, q^{m+1} \right] = \frac{(yz;q)_{m+n}}{(yz;q)_m (z;q)_n} {}_2\phi_1 \left[\begin{matrix} q^{-m}, \ q^{-n} \\ q^{1-m-n}/yz, \end{matrix} ; q, q/y \right],$$

or

$$\sum_{k=0}^{n} \frac{(y;q)_k (z;q)_{n-k}}{(q;q)_k (q;q)_{n-k}} q^{mk} z^k = \frac{(q;q)_m}{(yz;q)_m} \sum_{k=0}^{\min\{m,n\}} \frac{(yz;q)_{m+n-k} q^{\binom{k}{2}} (-z)^k}{(q;q)_k (q;q)_{m-k} (q;q)_{n-k}}. \tag{1.8}$$

The first author [7] has recently proved the $y = q/z$ case of (1.7) by using combinatorics of partition theory. Hence it is natural to ask for a combinatorial proof of (1.7) by extending the argument of [7]. Note that (1.8) shows that the left-hand side of (1.7) is symmetric in m and n, which establishes (1.7).

This paper is organized as follows: in Section 2 we give the bijective proof of (1.5) in the framework of words, and in Section 3 the combinatorial proof of (1.8) using partitions of integers. Finally, in Section 4, we give a short proof of (1.3) from scratch, in the same vein as the first author's approach to some other well-known q-identities [8].

2 A Bijective Proof of Equation (1.5)

Let $M(1^{n_1},\ldots,m^{n_m})$ denote the set of rearrangements of the word $1^{n_1}2^{n_2}\ldots m^{n_m}$. An *inversion* in a word $w=w_1w_2\cdots w_n$ on the alphabet $\{1,\ldots,m\}$ is a pair of indices (i,j) such that $i<j$ and $w_i>w_j$. The number of inversions of w is denoted by $\mathrm{inv}(w)$. For instance, for $w=1\,3\,1\,2\,2\,3\,2\,1\,1\in M(1^4,2^3,3^2)$, we have $\mathrm{inv}(w)=15$. It is folklore [2, Theorem 3.6] that

$$\sum_{w\in M(1^{n_1},\ldots,m^{n_m})} q^{\mathrm{inv}(w)}=\frac{[n_1+\cdots+n_m]!}{[n_1]!\cdots[n_m]!}. \tag{2.1}$$

On the other hand, for any word w on the alphabet of two letters $\{a,b\}$ with $a<b$, we define $\imath_{ab}(w)$ to be the word obtained from w by reversing the order of letters of w and then interchanging the letters a's and b's. For instance, if $a=1$ and $b=2$, then $\imath_{12}(11222221)=21111122$. It is easy to see that $\imath_{ab}$ is an involution such that $\mathrm{inv}(w)=\mathrm{inv}(\imath_{ab}(w))$.

If w_1 and w_2 are two words, we denote by w_1w_2 their concatenation.

Lemma 2.1 *Let $b,c,d,e\in\mathbb{N}$ and $r\in\mathbb{Z}$ such that $r\le\min\{b,c,d,e,d+e-c\}$. For any $k\in\mathbb{Z}$, define the sets*

$$A_k=M(2^{b-k},3^{k+d})\times M(2^{k-r},4^{c-k})\times M(3^{e-k},4^{d-r}),$$
$$B_k=M(2^{b-k},3^{k+c})\times M(2^{k-r},4^{e+d-c-k})\times M(3^{d-k},4^{e-r}),$$

where $M(a^i,b^j)=\emptyset$ if $i<0$ or $j<0$ by convention. Set $A=\cup_kA_k$ and $B=\cup_kB_k$. For each triple $w=(w_1,w_2,w_3)\in A\cup B$, define the statistic $\mathrm{inv}(w)=\mathrm{inv}(w_1w_2)+\mathrm{inv}(w_3)$. Then there is a bijection $\theta\colon A\longrightarrow B$ such that $\mathrm{inv}(w)=\mathrm{inv}(\theta(w))$.

Proof. Start with a triple $(w_1,w_2,w_3)\in A_k$. Replacing all the 4's in w_2 by the leftmost $c-k$ letters in w_3 one by one, we obtain a word w_2'. Denote by w_2'' the word obtained from w_2' by deleting all the 4's. Let v_3' be the word obtained from w_2' by replacing every 2 by 3. Note that both w_1 and w_2'' are words on $\{2,3\}$. Let v_1 be the word corresponding to the leftmost $b+c$ letters in w_1w_2'', and let v_1' be the word such that $v_1v_1'=w_1w_2''$.

Let w_3' be the word obtained from w_3 by deleting the leftmost $c-k$ letters. It is easy to see that the number of 4's in w_3' is exactly equal to the length of v_1'. Let $w_3''=\imath_{34}(w_3')$. Replacing all the letters 3's in w_3'' by those in v_1' one by one, we obtain a word w_3'''. Let v_2 be the word obtained from w_3''' by replacing every 3 by 4, and let v_2' be the word obtained from w_3''' by deleting all the 2's. Finally, let $v_3=v_2'\imath_{34}(v_3')$. Clearly $v=(v_1,v_2,v_3)\in B$.

Conversely, if $v=(v_1,v_2,v_3)\in B_k$, then the above procedure with $c-k$ changed to $e+d-c-k$ and $b+c$ to $b+d$ also defines a mapping θ from B to A such that θ^2 is the identity mapping. Thus, the mapping $\theta\colon w\mapsto v$ is a bijection from A to B.

It remains to show that $\mathrm{inv}(w)=\mathrm{inv}(v)$. Let $|u|_i$ denote the number of occurrences of i in the word u. Note that, in our construction from w to v, we have the following obvious relations:

$$
\begin{aligned}
\operatorname{inv}(w_2) &= \operatorname{inv}(w_2') = \operatorname{inv}(w_2'') + \operatorname{inv}(v_3'),\\
\operatorname{inv}(w_3''') &= \operatorname{inv}(w_3'') + \operatorname{inv}(v_1') = \operatorname{inv}(w_3') + \operatorname{inv}(v_1'),\\
\operatorname{inv}(w_3''') &= \operatorname{inv}(v_2) + \operatorname{inv}(v_2'),\\
\operatorname{inv}(v_3) &= \operatorname{inv}(v_2') + \operatorname{inv}(v_3') + |v_2'|_4 \cdot |v_3'|_4,\\
\operatorname{inv}(v_1) + \operatorname{inv}(v_1') + |v_1|_3 \cdot |v_1'|_2 &= \operatorname{inv}(w_1) + \operatorname{inv}(w_2'') + |w_1|_3 \cdot |w_2''|_2,\\
&= \operatorname{inv}(w_1) + \operatorname{inv}(w_2'') + |w_1|_3 \cdot |w_2|_2,\\
|v_2|_2 &= |v_1'|_2,\\
|v_2'|_4 &= |w_3'''|_4 = |w_3''|_4 = |w_3'|_3,\\
|v_3'|_4 &= |w_2'|_4,\\
\operatorname{inv}(w_3) &= \operatorname{inv}(w_3') + |w_3'|_3 \cdot |w_2'|_4.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\operatorname{inv}(v_1 v_2) + \operatorname{inv}(v_3)\\
&\quad = \operatorname{inv}(v_1) + \operatorname{inv}(v_2) + |v_1|_3 \cdot |v_2|_2 + \operatorname{inv}(v_2') + \operatorname{inv}(v_3') + |v_2'|_4 \cdot |v_3'|_4\\
&\quad = \operatorname{inv}(v_1) + \operatorname{inv}(v_1') + \operatorname{inv}(w_3') + |v_1|_3 \cdot |v_2|_2 + \operatorname{inv}(v_3') + |w_3'|_3 \cdot |w_2'|_4\\
&\quad = \operatorname{inv}(w_1) + \operatorname{inv}(w_2'') + |w_1|_3 \cdot |w_2|_2 + \operatorname{inv}(w_3') + \operatorname{inv}(v_3') + |w_3'|_3 \cdot |w_2'|_4\\
&\quad = \operatorname{inv}(w_1) + \operatorname{inv}(w_2) + |w_1|_3 \cdot |w_2|_2 + \operatorname{inv}(w_3') + |w_3'|_3 \cdot |w_2'|_4\\
&\quad = \operatorname{inv}(w_1 w_2) + \operatorname{inv}(w_3).
\end{aligned}
$$

This completes the proof. ∎

Example 2.1 Let $b = 7$, $c = 8$, $d = 9$, $e = 10$, $r = 0$ and $k = 4$. If

$$w = (w_1, w_2, w_3) = (3323333233323333,\ 42244242,\ 334344344434344),$$

then, $w_2' = 32234232$, i.e., $w_2'' = 3223232$ and $v_3' = 33334333$. So, $v_1 = 332333323332333$ and $v_1' = 33223232$.

On the other hand, $w_3' = 44344434344$, i.e., $w_3'' = 33434333433$. Hence, $w_3''' = 33424232432$, i.e., $v_2 = 44424242442$ and $v_2' = 3344343$. Thus, $v_3 = v_2' \imath_{34}(v_3') = 334434344434444$. Namely,

$$v = (v_1, v_2, v_3) = (332333323332333,\ 44424242442,\ 334434344434444).$$

It is easy to see that $\operatorname{inv}(w) = \operatorname{inv}(v) = 95$.

Proof of (1.5). Let $M = \cup_k M_k$ and $N = \cup_k N_k$, where

$$
\begin{aligned}
M_k &= M(1^{b+c}, 2^{a-k}) \times M(1^{b-k}, 3^{k+d}) \times M(2^{k-r}, 4^{c-k}) \times M(3^{e-k}, 4^{d-r}),\\
N_k &= M(1^{a+d}, 2^{b-k}) \times M(1^{a-k}, 3^{k+c}) \times M(2^{k-r}, 4^{e+d-c-k}) \times M(3^{d-k}, 4^{e-r}).
\end{aligned}
$$

For each element $w=(w_1,w_2,w_3,w_4)$ of $M\cup N$ define

$$\mathrm{inv}(w)=\mathrm{inv}(w_1)+\mathrm{inv}(w_2w_3)+\mathrm{inv}(w_4).$$

Then, in view of (2.1), we have

$$\begin{aligned}
\sum_{w\in M_k} q^{\mathrm{inv}(w)} &= q^{(k-r)(k+d)} \begin{bmatrix} a+b+c-k \\ a-k \end{bmatrix} \begin{bmatrix} b+d \\ b-k \end{bmatrix} \begin{bmatrix} c-r \\ c-k \end{bmatrix} \begin{bmatrix} e+d-r-k \\ d-r \end{bmatrix}, \\
\sum_{w\in N_k} q^{\mathrm{inv}(w)} &= q^{(k-r)(k+c)} \begin{bmatrix} a+b+d-k \\ b-k \end{bmatrix} \begin{bmatrix} a+c \\ a-k \end{bmatrix} \begin{bmatrix} e+d-r-c \\ k-r \end{bmatrix} \begin{bmatrix} e+d-r-k \\ d-k \end{bmatrix}.
\end{aligned}$$

Hence, replacing e by $e+d-r$, identity (1.5) can be rephrased as follows:

$$\sum_{w\in M} q^{\mathrm{inv}(w)} = \sum_{w\in N} q^{\mathrm{inv}(w)}.$$

We now give a bijection $\eta\colon M \longrightarrow N$ to interpret the above identity. Start with a quadruple $w=(w_1,w_2,w_3,w_4)$ in M_k. Replacing all the $b-k$ 1's in w_2 by the rightmost $b-k$ letters in w_1, we obtain a word w_2'. Denote by w_2'' the word obtained from w_2' by deleting all the 1's. Let v_1' be the word obtained from w_2' by replacing every 3 by 2. Note that w_2'' is a word on $\{2,3\}$. Applying Lemma 2.1, we obtain $(v_3',v_3,v_4)=\theta(w_2'',w_3,w_4)$.

Let w_1' be the subword of w_1 corresponding to the leftmost $a+c$ letters. It is easy to see that the number of 1's in w_1' is exactly equal to the length of v_3'. Let $w_1''=\imath_{12}(w_1')$. Replacing all the 2's in w_1'' by the word v_3', we obtain a word w_1'''. Let v_2 be the word obtained from w_1''' by replacing every 2 by 1, and let v_2' be the subword of w_1''' by deleting all the 3's. Finally, let $v_1=\imath_{12}(v_1')v_2'$. Suppose v_3 has $d-k'$ 3's. Then it is easy to see that $v=(v_1,v_2,v_3,v_4)\in N_{k'}$.

Conversely, if $v=(v_1,v_2,v_3,v_4)\in N_k$, then the above procedure with $b-k$ changed to $a-k$, and $a+c$ changed to $b+d$ also defines a mapping from N to M, also denoted by η. It is easy to check that η^2 is the identity mapping. Namely, $\eta\colon w\mapsto v$ is a bijection from M to N. Moreover, an argument similar to that in the proof of Lemma 2.1 shows that $\mathrm{inv}(w)=\mathrm{inv}(v)$. This completes the proof. ∎

3 A Combinatorial Proof of Equation (1.8)

Replacing q by q^2, y by $-yq$, and z by $-zq$, we can rewrite (1.8) as

$$\sum_{k=0}^{n}(-1)^k\frac{(-yq;q^2)_k(-zq;q^2)_{n-k}}{(q^2;q^2)_k(q^2;q^2)_{n-k}}q^{(2m+1)k}z^k = \sum_{k=0}^{\min\{m,n\}} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} \frac{(yzq^{2m+2};q^2)_{n-k}q^{k^2}z^k}{(q^2;q^2)_{n-k}}. \quad (3.1)$$

A *partition* λ is a finite sequence of nonnegative integers $(\lambda_1,\lambda_2,\ldots,\lambda_m)$ such that $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_m\geq 0$. Each $\lambda_i>0$ is called a part of λ. The numbers of parts, odd parts, and even parts of λ are denoted by $\ell(\lambda)$, $\mathrm{odd}(\lambda)$, and $\mathrm{even}(\lambda)$, respectively. Write

$|\lambda| = \sum_{i=1}^{m} \lambda_i$, called the *weight* of λ. The set of all partitions into even parts is denoted by $\mathcal{P}_{\text{even}}$. The set of all partitions into distinct odd (resp. even) parts is denoted by $\mathcal{D}_{\text{odd}}$ (resp. $\mathcal{D}_{\text{even}}$). Let $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) denote the set of partitions with no repeated odd (resp. even) parts. Given two partitions λ and μ, we define $\lambda \cup \mu$ to be the partition whose parts are those of λ and μ in decreasing order, and $\lambda + \mu$ to be the partition of which the i-th part is the sum of λ_i and μ_i. If t is a part of λ, then $\lambda \setminus t$ denotes the partition obtained from λ by deleting one part equal to t.

The following lemma is a combinatorial version of the q-binomial theorem, as shown in [7]. See also Chapman [4]. For the convenience of the reader, we sketch a proof here. Other models, as *overpartitions*, of the q-binomial theorem have been given by Joichi and Stanton [9] and Alladi [1]. See also Pak's survey [10].

Lemma 3.1 *There is an involution σ on $\mathcal{P}_1$ such that for each $\lambda \in \mathcal{P}_1$, we have*

$$|\sigma(\lambda)| = |\lambda|, \ \ell(\sigma(\lambda)) = \lceil \lambda_1/2 \rceil, \ \textit{and} \ \mathrm{odd}(\sigma(\lambda)) = \mathrm{odd}(\lambda).$$

Proof. Given a partition $\lambda \in \mathcal{P}_1$, we draw the 2-*modular diagram* of λ as follows: an even part $2k$ will give a row of k 2's, while an odd part $2k+1$ will give a row of k 2's followed by a 1. So each part λ_i corresponds to a row of length $\lceil \lambda_i/2 \rceil$, and the number of 1's in the 2-modular diagram is $\mathrm{odd}(\lambda)$. Since no odd part of λ is repeated, the 1's can only occur at the bottom of columns. We identify elements of $\mathcal{P}_1$ with their diagrams, and then define σ to be conjugation of diagrams. Clearly, the number of rows in the diagram is $\ell(\lambda)$, while the number of columns is $\lceil \lambda_1/2 \rceil$. Thus, σ has the required property and Lemma 3.1 is proved. ∎

Example 3.1 Let $\lambda = (10, 9, 7, 4, 4, 4, 3, 2, 2, 1)$. Then, λ gives the left 2-modular diagram below, while its conjugation $\sigma(\lambda)$ gives the right 2-modular diagram below:

$$\begin{array}{ccccc} 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 & \\ 2 & 2 & & & \\ 2 & 2 & & & \\ 2 & 2 & & & \\ 2 & 1 & & & \\ 2 & & & & \\ 2 & & & & \\ 1 & & & & \end{array} \quad \longrightarrow \quad \begin{array}{cccccccccc} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 1 & & & \\ 2 & 2 & 2 & & & & & & & \\ 2 & 2 & 1 & & & & & & & \\ 2 & 1 & & & & & & & & \end{array}$$

Namely, $\sigma(\lambda) = (19, 13, 6, 5, 3)$.

We derive immediately the following result.

Lemma 3.2 *We have*

$$\sum_{\substack{\mu\in\mathcal{P}_1\\ \ell(\mu)\le n}} q^{|\mu|}z^{\mathrm{odd}(\mu)} = \sum_{\substack{\lambda\in\mathcal{P}_1\\ \lambda_1\le 2n}} q^{|\lambda|}z^{\mathrm{odd}(\lambda)} = \frac{(-zq;q^2)_n}{(q^2;q^2)_n}. \tag{3.2}$$

We also need some other lemmas. Set

$$\mathcal{A}_{m,n} = \{(\lambda,\,\mu)\in\mathcal{P}_2\times\mathcal{P}_1\colon \ell(\lambda)+\ell(\mu)\le n \text{ and } \lambda_{\ell(\lambda)}\ge 2m+1\}.$$

Lemma 3.3 *For $m\ge 0$ and $n\ge 1$, we have*

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\frac{(-yq;q^2)_k(-zq;q^2)_{n-k}}{(q^2;q^2)_k(q^2;q^2)_{n-k}}q^{(2m+1)k}z^k\\ &\quad=\sum_{(\lambda,\,\mu)\in\mathcal{A}_{m,n}}(-1)^{\ell(\lambda)}q^{|\lambda|+|\mu|}y^{\mathrm{even}(\lambda)}z^{\mathrm{odd}(\mu)+\ell(\lambda)}.\end{aligned} \tag{3.3}$$

Proof. Let $\nu=(2m+1,\ldots,2m+1)$ be a partition with k parts. By Lemma 3.2,

$$\frac{(-yq;q^2)_k}{(q^2;q^2)_k}q^{(2m+1)k}z^k = q^{|\nu|}z^{\ell(\nu)}\sum_{\substack{\tau\in\mathcal{P}_1\\ \ell(\tau)\le k}} q^{|\tau|}y^{\mathrm{odd}(\tau)} = \sum_{\substack{\lambda\in\mathcal{P}_2\\ \ell(\lambda)=k\\ \lambda_{\ell(\lambda)}\ge 2m+1}} q^{|\lambda|}y^{\mathrm{even}(\lambda)}z^{\ell(\lambda)},$$

where $\lambda=\tau+\nu$. Also,

$$\frac{(-zq;q^2)_{n-k}}{(q^2;q^2)_{n-k}} = \sum_{\substack{\mu\in\mathcal{P}_1\\ \ell(\mu)\le n-k}} q^{|\mu|}z^{\mathrm{odd}(\mu)}.$$

Multiplying the above two identities and summing over k, we get the desired identity. ∎

Let $\mathcal{B}_{m,n}$ be the subset of $\mathcal{A}_{m,n}$ consisting of the pairs (λ,μ) such that λ_i is odd for some i, or μ_j is odd for some j and $\mu_j\ge 2m+1$.

Lemma 3.4 *For $m\ge 0$ and $n\ge 1$, we have*

$$\sum_{(\lambda,\,\mu)\in\mathcal{B}_{m,n}}(-1)^{\ell(\lambda)}q^{|\lambda|+|\mu|}y^{\mathrm{even}(\lambda)}z^{\mathrm{odd}(\mu)+\ell(\lambda)}=0. \tag{3.4}$$

Proof. We will construct a weight preserving and sign reversing involution ϕ on $\mathcal{B}_{m,n}$. For any $(\lambda,\,\mu)\in\mathcal{B}_{m,n}$, as no odd part of μ is repeated, let t be the largest odd part in $\lambda\cup\mu$. By the definition of $\mathcal{B}_{m,n}$, we see that $t\ge 2m+1$. Now define

$$\phi((\lambda,\,\mu))=\begin{cases}(\lambda\cup t,\,\mu\setminus t), & \text{if } t \text{ is a part of } \mu,\\ (\lambda\setminus t,\,\mu\cup t), & \text{if } t \text{ is not a part of } \mu.\end{cases}$$

It is straightforward to verify that ϕ is an involution on $\mathcal{B}_{m,n}$ which preserves $|\lambda|+|\mu|$, $\mathrm{even}(\lambda)$ and $\mathrm{odd}(\mu)+\ell(\lambda)$ and reverses the sign $(-1)^{\ell(\lambda)}$. ∎

Proof of (3.1). Note that $(\lambda,\,\mu)\in\mathcal{A}_{m,n}\setminus\mathcal{B}_{m,n}$ if and only if $\lambda\in\mathcal{D}_{\text{even}}$ and for any i if μ_i is odd then $\mu_i\le 2m-1$. Combining Lemmas 3.3 and 3.4, we see that the left-hand side of (3.1) is equal to

$$\begin{aligned}
&\sum_{(\lambda,\,\mu)\in\mathcal{A}_{m,n}\setminus\mathcal{B}_{m,n}}(-1)^{\ell(\lambda)}q^{|\lambda|+|\mu|}y^{\ell(\lambda)}z^{\mathrm{odd}(\mu)+\ell(\lambda)}\\
&\quad=\sum_{k=0}^{\min\{m,n\}}z^k\sum_{\substack{\eta\in\mathcal{D}_{\text{odd}}\\ \ell(\eta)=k\\ \eta_1\le 2m-1}}q^{|\eta|}\sum_{\substack{\tau\in\mathcal{D}_{\text{odd}}\\ \nu\in\mathcal{P}_{\text{even}}\\ \ell(\tau)+\ell(\nu)\le n-k}}q^{|\tau|+|\nu|}(-yzq^{2m+1})^{\ell(\tau)},
\end{aligned}\tag{3.5}$$

where $k=\mathrm{odd}(\mu)$, $\mu=\eta\cup\nu$, and $\tau_i=\lambda_i-(2m+1)$.

Now, setting $\pi_i=\eta_i-(2i-1),\ 1\le i\le k$, and using the result (see [2, Theorem 3.1])

$$\sum_{\substack{\ell(\alpha)\le k\\ \alpha_1\le m-k}}q^{|\alpha|}=\begin{bmatrix}m\\k\end{bmatrix},$$

we have

$$\sum_{\substack{\eta\in\mathcal{D}_{\text{odd}}\\ \ell(\eta)=k\\ \eta_1\le 2m-1}}q^{|\eta|}=q^{k^2}\sum_{\substack{\pi\in\mathcal{P}_{\text{even}}\\ \ell(\pi)\le k\\ \pi_1\le 2m-2k}}q^{|\pi|}=q^{k^2}\begin{bmatrix}m\\k\end{bmatrix}_{q^2}.\tag{3.6}$$

Also, replacing z by $-yzq^{2m+1}$ and n by $n-k$ in (3.2) yields

$$\sum_{\substack{\tau\in\mathcal{D}_{\text{odd}}\\ \nu\in\mathcal{P}_{\text{even}}\\ \ell(\tau)+\ell(\mu)\le n-k}}q^{|\tau|+|\nu|}(-yzq^{2m+1})^{\ell(\tau)}=\frac{(yzq^{2m+2};q^2)_{n-k}}{(q^2;q^2)_{n-k}}.\tag{3.7}$$

Finally, combining (3.5), (3.6) and (3.7) completes the proof. ∎

4 An Elementary Proof of Equation (1.3)

Lemma 4.1 *For $m,n\in\mathbb{N}$, we have*

$$\frac{(xq,yq;q)_m}{(q,xyq;q)_m}\sum_{k=0}^{n}\frac{(x,y,vq^m;q)_k}{(q,v,xyq^{m+1};q)_k}q^k=\frac{(xq,yq;q)_n}{(q,xyq;q)_n}\sum_{k=0}^{m}\frac{(x,y,vq^n;q)_k}{(q,v,xyq^{n+1};q)_k}q^k.\tag{4.1}$$

Proof. For $k\ge0$, let $B(-1,k)=0$ and

$$B(r,k)=\frac{(xq,yq;q)_r}{(q,xyq;q)_r}\frac{(x,y,vq^r;q)_k}{(q,v,xyq^{r+1};q)_k}q^k,\quad r\ge0.$$

For $r, k \geq 0$, set

$$A(r,k) := B(r,k) - B(r-1,k).$$

Then (4.1) may be written as

$$\sum_{k=0}^{n} \sum_{r=0}^{m} A(r,k) = \sum_{k=0}^{m} \sum_{r=0}^{n} A(r,k).$$

Since $A(r,k) = A(k,r)$, the above identity is then obvious. ∎

Proof of (1.3). Since both sides of (1.3) are rational fractions of z, it suffices to show that (1.3) holds for all $z = q^c$ ($c \geq 1$). We proceed by induction on c. The $z = q$ case of (1.3) is equivalent to (4.1) and has been proved. Suppose (1.3) holds for $z = q^c$. Denote the left-hand side of (1.3) by $S(m,n,x,z)$ for nonnegative integers m and n. Then (1.3) means nothing else that $S(m,n,x,z)$ is symmetric in m and n. Multiplying both sides of

$$\frac{1 - xyz^2q^{m+n}}{1 - xzq^n}(1 - xq^k) + \frac{x(1 - yzq^m)}{1 - xzq^n}(q^k - zq^n) = 1 - xyzq^{m+k} \tag{4.2}$$

by

$$\frac{1}{1 - xyzq^{m+k}} \frac{(xz, yz; q)_m}{(q, xyz; q)_m} \frac{(x, y, vq^m; q)_k (z; q)_{n-k}}{(q, v, xyzq^m; q)_k (q; q)_{n-k}} z^k,$$

and summing over k from 0 to n, we obtain

$$a\, S(m,n,xq,z) + b\, S(m,n,x,zq) = S(m,n,x,z), \tag{4.3}$$

where the coefficients a and b are two symmetric expressions in m and n:

$$a = \frac{(1 - xyz^2q^{m+n})(1 - x)(1 - xz)}{(1 - xyz)(1 - xzq^m)(1 - xzq^n)}, \quad b = \frac{x(1 - z)(1 - xz)(1 - yz)}{(1 - xyz)(1 - xzq^m)(1 - xzq^n)}.$$

Since both $S(m,n,xq,z)$ and $S(m,n,x,z)$ are symmetric in m and n by induction hypothesis, we deduce that $S(m,n,x,zq)$ is symmetric in m and n, i.e., (1.3) holds for zq. This completes the proof. ∎

Remark. Equation (4.2) can be obtained from the $n = 1$ case of the q-Pfaff-Saalschütz identity [5, (1.7.2)]. Krattenthaler has indicated how to derive contiguous relations as (4.3) from special cases of terminating basic hypergeometric summation or transformation formulas (see "A systematic list of two- and three-term contiguous relations for basic hypergeometric series," available at `http://euler.univ-lyon1.fr/home/kratt/papers.html`).

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