A Sperner-Type Theorem for Set-Partition Systems

Karen Meagher

Department of Mathematics and Statistics
University of Ottawa, Ottawa, Ontario, Canada
kmeagher@site.uottawa.ca

Lucia Moura

School of Information Technology and Engineering
University of Ottawa, Ottawa, Ontario, Canada
lucia@site.uottawa.ca

Brett Stevens

School of Mathematics and Statistics
Carleton University, Ottawa, Ontario, Canada
brett@math.carleton.ca

Submitted: Aug 15, 2005; Accepted: Oct 26, 2005; Published: Oct 31, 2005
Mathematics Subject Classifications: 05D05

Abstract

A *Sperner partition system* is a system of set partitions such that any two set partitions P and Q in the system have the property that for all classes A of P and all classes B of Q, $A \not\subseteq B$ and $B \not\subseteq A$. A *k-partition* is a set partition with k classes and a k-partition is said to be *uniform* if every class has the same cardinality $c = n/k$. In this paper, we prove a higher order generalization of Sperner's Theorem. In particular, we show that if k divides n the largest Sperner k-partition system on an n-set has cardinality $\binom{n-1}{n/k-1}$ and is a uniform partition system. We give a bound on the cardinality of a Sperner k-partition system of an n-set for any k and n.

1 Introduction

In this paper, we prove a Sperner-type theorem for systems of set partitions and related results. These theorems are stated after some notation and background results are introduced.

For i, j positive integers with $i \leq j$, let $[i, j]$ denote the set $\{i, i+1, \ldots, j\}$. For k, n positive integers, set $\binom{[n]}{k} = \{A \subseteq [1, n] : |A| = k\}$. This set is also known as a *complete k-uniform hypergraph*. A system $\mathcal{A}$ of subsets of $[1, n]$ is said to be *k-uniform* if $\mathcal{A} \subseteq \binom{[n]}{k}$.

Two subsets A, B are *incomparable* if $A \not\subseteq B$ and $B \not\subseteq A$. A set system on an n-set $\mathcal{A}$ is said to be a *Sperner set system*, if any two distinct sets in $\mathcal{A}$ are incomparable.

Sperner's Theorem is concerned with the maximal cardinality of Sperner set systems as well as with the structure of such maximal systems.

Theorem (Sperner's Theorem [14]). *A Sperner set system $\mathcal{A}$ of subsets of $[1,n]$ consists of at most $\binom{n}{\lfloor n/2 \rfloor}$ sets. Moreover, a Sperner set system meets this bound if and only if $\mathcal{A} = \binom{[n]}{\lfloor n/2 \rfloor}$ or $\mathcal{A} = \binom{[n]}{\lceil n/2 \rceil}$.*

A sharper version of Sperner's theorem is the *LYM Inequality* named after Lubell [11], Meshalkin [13] and Yamamoto [15], who each independently established the result.

Theorem (LYM Inequality). *Let n be a positive integer and $\mathcal{A}$ be a Sperner set system on an n-set. Let p_i denote the number of subsets in $\mathcal{A}$ of size i, then*

$$\sum_{i=1}^{n} \frac{p_i}{\binom{n}{i}} \leq 1. \tag{1}$$

A *set partition* of $[1,n]$ is a set of disjoint non-empty subsets (called classes) of $[1,n]$ whose union is $[1,n]$. Throughout this paper, we refer to set partitions as simply *partitions*. A partition P is called a *k-partition* if it contains k classes, that is $|P| = k$. Denote by $\mathcal{P}_k^n$ the set of all k-partitions of $[1,n]$. If $n = ck$, a partition $P \in \mathcal{P}_k^n$ is said to be *c-uniform* if every class of P has the same cardinality, that is $|A| = c$, for all $A \in P$. If k does not divide n, a partition $P \in \mathcal{P}_k^n$ is said to be *almost uniform* if every class $A \in P$ has $|A| = \lfloor n/k \rfloor$ or $|A| = \lceil n/k \rceil$.

A partition system $\mathcal{P} \subseteq \mathcal{P}_k^n$ is a *Sperner partition system* if all distinct $P, Q \in \mathcal{P}$, with $P = \{P_1, \ldots, P_k\}$ and $Q = \{Q_1, \ldots, Q_k\}$, the following holds

$$P_i \not\subseteq Q_j, \text{ and } Q_i \not\subseteq P_j, \text{ for all } i, j \in \{1, \ldots, k\}.$$

Let $\mathrm{SP}(n,k)$ denote the size of the largest Sperner partition system in $\mathcal{P}_k^n$.

If $\mathcal{P}$ is a partition system, then $\mathcal{P}$ is a Sperner partition system if and only if all the partitions in $\mathcal{P}$ are disjoint (no two partitions have a common class) and the union of all the classes of all the partitions in $\mathcal{P}$ is a Sperner set system. In design theory, a collection of disjoint subsets of an n-set whose union is the n-set is called a *resolution class*. Any set system that can be partitioned into resolution classes is called *resolvable*. In this sense, Sperner partition systems can be considered resolvable Sperner set systems.

There have been extensions of Sperner's Theorem to systems of families of sets [5] and to systems of subsets of a set X with a 2-partition $X = X_1 \cup X_2$ such that no two subsets A, B in the system satisfy both $A \cap X_i = B \cap X_i$ and $A \cap \overline{X_i} \subseteq B \cap \overline{X_i}$ where $i \in \{1,2\}$ [6, 8, 9]. Our notion of a Sperner partition system is quite different; our result extends Sperner's Theorem from sets to set-partitions. A related extension of the Erdős-Ko-Rado Theorem to set partitions is found in [12].

Bollobás [2] gives a generalization of the LYM Inequality to two families of sets. For positive integers n, m let $\mathcal{A} = \{A_i, B_i : i = 1, \ldots, m\}$ be a set system of subsets from $[1,n]$

with the property that $A_i \cap B_i \neq \emptyset$ and $A_i \not\subseteq A_j \cup B_j$ for $i \neq j$. Then $\sum_{i=1}^{m} \binom{n-|B_i|}{|A_i|} \leq 1$. This result implies both Sperner's Theorem and the LYM Inequality but does not generalize to three families of sets.

Another generalization of Sperner's Theorem that involves partition systems looks at the poset of partitions ordered by refinement [3, 4]. This has no direct relationship to our result as a system of k-partitions can not contain both a partition and any of its refinements.

Our first result is the exact size of the largest Sperner partition system in $\mathcal{P}_k^n$ when k divides n.

Theorem 1. *Let n, k, c be integers with $n = kc$. Then $\mathrm{SP}(n,k) = \binom{ck-1}{c-1}$. Moreover, a Sperner partition system meets this bound only if it is a c-uniform partition system.*

Our second result is a bound on the size of $\mathrm{SP}(n,k)$ for general n and k.

Theorem 2. *Let n, k, c and r be integers with $n = ck + r$ and $0 \leq r < k$. Then*

$$\mathrm{SP}(n,k) \leq \frac{1}{(k-r) + \frac{r(c+1)}{n-c}} \binom{n}{c}.$$

In Section 2, we prove the bound on the size of a Sperner partition system stated in Theorem 2. In Section 3, we prove Theorem 1.

2 A Bound on the Cardinality of Sperner Partition Systems in $\mathcal{P}_k^n$

Theorem 2 states that for integers n, k, c, r with $n = ck + r$ and $0 \leq r < k$, we have the following bound on the cardinality of a Sperner partition system in $\mathcal{P}_k^n$:

$$\mathrm{SP}(n,k) \leq \frac{\binom{n}{c}}{(k-r) + \frac{r(c+1)}{n-c}}.$$

Proof of Theorem 2. If $k = 1$ then $n = c$ and $r = 0$. Further, $\mathcal{P}_k^n$ has only one partition, namely $\{\{1, \ldots, n\}\}$ and the theorem holds trivially. So we assume that $k \geq 2$.

Let $\mathcal{P} \subseteq \mathcal{P}_k^n$ be a Sperner partition system. Let $\mathcal{A}$ be the Sperner set system formed by taking all classes from all the partitions in $\mathcal{P}$. Thus $|\mathcal{A}| = k|\mathcal{P}|$.

Let p_i, $i \in \{1, \ldots, n\}$ be the number of sets in $\mathcal{A}$ with size i. By the LYM Inequality

$$\sum_{i=1}^{n} \frac{p_i}{\binom{n}{i}} \leq 1.$$

Following the notation from [7], define the function $f(i) = \binom{n}{i}^{-1}$. With this, we get

$$\sum_{i=1}^{n} \frac{p_i}{|\mathcal{A}|} f(i) \leq \frac{1}{|\mathcal{A}|}. \tag{2}$$

In [7], it is shown that the function $f(i)$, for $i = 1, \ldots, n$, can be extended to a convex function on the real numbers by

$$f(i+u) = (1-u)f(i) + uf(i+1), \text{ for } 0 \leq u \leq 1.$$

Since the set system $\mathcal{A}$ is formed from a k-partition system on an n-set,

$$\sum_{i=1}^{n} ip_i = \sum_{A \in \mathcal{A}} |A| = n|\mathcal{P}| = n\frac{|\mathcal{A}|}{k}.$$

Using the above equality, the fact that $f(i)$ is a convex function with $\sum_{i=1}^{n} \frac{p_i}{|\mathcal{A}|} = 1$, and Equation (2), we get

$$f\left(\frac{n}{k}\right) = f\left(\sum_{i=1}^{n} i\frac{p_i}{|\mathcal{A}|}\right) \leq \sum_{i=1}^{n} f(i)\frac{p_i}{|\mathcal{A}|} \leq \frac{1}{|\mathcal{A}|}.$$

From the definition of $f(i)$ we get

$$f\left(\frac{n}{k}\right) = f\left(\frac{ck+r}{k}\right) = f\left(c + \frac{r}{k}\right) = \left(1 - \frac{r}{k}\right)\binom{n}{c}^{-1} + \frac{r}{k}\binom{n}{c+1}^{-1}.$$

Thus,

$$|\mathcal{A}| \leq \frac{1}{(1-\frac{r}{k})\binom{n}{c}^{-1} + \frac{r}{k}\binom{n}{c+1}^{-1}} = \binom{n}{c}\frac{k}{(k-r) + \frac{r(c+1)}{n-c}}$$

and

$$|\mathcal{P}| \leq \frac{\binom{n}{c}}{(k-r) + \frac{r(c+1)}{n-c}}.$$

□

We would like to know the exact cardinality and structure of the largest Sperner partition system. We conjecture that the largest Sperner partition system is an almost-uniform partition system.

Conjecture. *Let n, k be positive integers. The largest Sperner partition system in $\mathcal{P}_k^n$ is an almost-uniform partition system.*

For the case where $n = ck$, where c and k are integers, an almost-uniform partition system is a uniform partition system. In the next section, we prove this conjecture for this case. Specifically, we show that the largest Sperner partition system in $\mathcal{P}_k^{ck}$ is a uniform partition system.

3 Sperner's Theorem for Partition Systems in $\mathcal{P}_k^{ck}$

When $n = ck$, a 1-factor of the complete uniform hypergraph $\binom{[n]}{c}$ is equivalent to a uniform k-partition of an n-set, and a 1-factorization of $\binom{[n]}{c}$ corresponds to a Sperner partition system. If c divides n, the hypergraph $\binom{[n]}{c}$ has a 1-factorization with $\binom{n-1}{c-1}$ factors. Rephrasing this result using the above equivalence, we get the following result.

Theorem (Baranyai [1]). *Let n, k, c be positive integers with $n = ck$, then there exists a Sperner partition system in $\mathcal{P}_k^n$ of cardinality $\binom{n-1}{c-1}$.*

The proof that this is the largest Sperner partition system uses a result by Kleitman and Milner on Sperner set systems. For a set system $\mathcal{A}$, define the *volume* of $\mathcal{A}$ to be $t(\mathcal{A}) = \sum_{A\in\mathcal{A}} |A|$.

Theorem (Kleitman and Milner [10]). *Let $\mathcal{A}$ be a Sperner set system on an n-set, with $|\mathcal{A}| \geq \binom{n}{c}$ and $c \leq \frac{n}{2}$, then*

$$\frac{t(\mathcal{A})}{|\mathcal{A}|} \geq c.$$

This inequality is strict in all cases except when $\mathcal{A} = \binom{[n]}{c}$.

Proof of Theorem 1. If $k = 1$ then $\mathcal{P}_k^n$ has only one partition, namely $\{\{1, \ldots, n\}\}$, and the theorem holds trivially. So we assume that $k \geq 2$.

Let n, k and c be positive integers with $n = kc$. From Theorem 2, with $r = 0$ and Baranyai's Theorem, $\mathrm{SP}(n, k) = \frac{1}{k}\binom{n}{c}$.

It remains to show that if a Sperner partition system meets this bound then it is c-uniform. Assume that $|\mathcal{P}| = \frac{1}{k}\binom{n}{c}$. Let $\mathcal{A}$ be the Sperner set system formed by taking all classes from all the partitions in $\mathcal{P}$, then $|\mathcal{A}| = \binom{n}{c}$ and $c \leq \frac{n}{2}$.

Since $\frac{t(\mathcal{A})}{|\mathcal{A}|} = \frac{\sum_{A\in\mathcal{A}}|A|}{|\mathcal{A}|} = c$, from Kleitman and Milner's theorem, it follows that $\mathcal{A} = \binom{[n]}{c}$ and $\mathcal{P}$ is a uniform partition system. $\square$

Theorem 1 is a natural extension of Sperner's Theorem. Sperner's Theorem says that the Sperner set system with maximum cardinality on an n-set is the collection of all $\lfloor \frac{n}{2} \rfloor$-sets. Theorem 1 says that for integers n, k such that k divides n, the Sperner k-partition system on an n-set with the largest cardinality is the collection of all $(\frac{n}{k})$-sets arranged in resolution classes.

The following corollary follows from Theorem 1 and the fact that $\mathrm{SP}(n, k)$ is a non-decreasing function for fixed k.

Corollary 3. *Let n, k, c be positive integers with $n = ck + r$ where $0 \leq r < k$. Then*

$$\frac{1}{k}\binom{ck}{c} \leq \mathrm{SP}(n,k) \leq \frac{1}{k}\binom{(c+1)k}{c+1}.$$

From Corollary 3, we can calculate the asymptotic growth of $\mathrm{SP}(n, k)$ for the general case:

$$\lim_{n\to\infty} \frac{\log \mathrm{SP}(n,k)}{n} = \log k - \frac{k-1}{k}\log(k-1).$$

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