# Pan-factorial Property in Regular Graphs*

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**Abstract**

Among other results, we show that if for any given edge $e$ of an $r$-regular graph $G$ of even order, $G$ has a 1-factor containing $e$, then $G$ has a $k$-factor containing $e$ and another one avoiding $e$ for all $k$, $1 \le k \le r-1$.

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For a function $f : V(G) \to \{0, 1, 2, 3, \ldots\}$, a spanning subgraph $F$ of $G$ with $\deg_F(x) = f(x)$ for all $x \in V(G)$ is called an *$f$-factor* of $G$, where $\deg_F(x)$ denotes the degree of $x$ in $F$. If $f(x) = k$ for all vertices $x \in V(G)$, then an $f$-factor is also called a *$k$-regular factor* or a *$k$-factor*. An *$[a, b]$-factor* is a spanning subgraph $F$ of $G$ such that $a \le \deg_F(x) \le b$ for all $x \in V(G)$.

A graph $G$ is *pan-factorial* if $G$ contains all $k$-factors for $1 \le k \le \delta(G)$. In this note, we investigate the pan-factor property in regular graphs. Moreover, we proved that the existence of 1-factor containing any given edge implies the existence of $k$-factors containing or avoiding any given edge.

The first of our main results is the following.

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**Theorem 1** *Let $G$ be a connected $r$-regular graph of even order. If for every edge $e$ of $G$, $G$ has a 1-factor containing $e$, then $G$ has a $k$-factor containing $e$ and another $k$-factor avoiding $e$ for all integers $k$, $1 \leq k \leq r-1$.*

The next theorem is also one of our main results.

**Theorem 2** *Let $G$ be a connected graph of even order, $e$ be an edge of $G$, and $a, b, c$ be odd integers such that $1 \leq a < c < b$. If $G$ has both an $a$-factor and a $b$-factor containing $e$, then $G$ has a $c$-factor containing $e$. Similarly, if $G$ has both an $a$-factor and a $b$-factor avoiding $e$, then $G$ has a $c$-factor avoiding $e$.*

The above theorem shows that there exists a kind of continuity relation among regular factors, which is an improvement of the following theorem obtained by Katerinis [1].

**Theorem 3** (*Katerinis* [1]) *Let $G$ be a connected graph of even order, and $a, b$ and $c$ be odd integers such that $1 \leq a < c < b$. If $G$ has both an $a$-factor and a $b$-factor, then $G$ has a $c$-factor.*

We need a few known results as lemmas for the proof of our theorems. Firstly, we quote Petersen's classic decomposition theorem about regular graphs of even degree.

**Lemma 1** (*Petersen* [2]) *Every $2r$-regular graph can be decomposed into $r$ disjoint 2-factors.*

For the introduction of Tutte's $f$-factors theorem, we require the following notation.
For a graph $G$ and $S, T \subseteq V(G)$ with $S \cap T = \emptyset$, define

$$\delta_G(S,T) = \sum_{x \in S} f(x) + \sum_{x \in T} (d_{G-S}(x) - f(x)) - h_G(S,T),$$

where $h_G(S,T)$ is the number of components $C$ of $G - (S \cup T)$ such that $\sum_{x \in V(C)} f(x) + e_G(V(C), T) \equiv 1 \pmod 2$ and such a component $C$ is called an *$f$-odd component* of $G - (S \cup T)$.

**Lemma 2** (*Tutte's $f$-factor Theorem* [3]) *Let $G$ be a graph and $f : V(G) \rightarrow \{0, 1, 2, 3, \ldots\}$ be a function. Then*
*(a) $G$ has an $f$-factor if and only if $\delta_G(S,T) \geq 0$ for all $S$, $T \subseteq V(G)$ with $S \cap T = \emptyset$;*
*(b) $\delta_G(S,T) \equiv \sum_{x \in V(G)} f(x) \pmod 2$ for all $S$, $T \subseteq V(G)$ with $S \cap T = \emptyset$.*

**Lemma 3** *Let $G$ be a connected graph. If for any edge $e$ there exists a 1-factor containing $e$, then there exists another 1-factor avoiding $e$.*

*Proof.* For any edge $e \in E(G)$, we will show that there exists a 1-factor avoiding $e$. Choose an edge $e'$ incident to the given edge $e$, then there exists a 1-factor $F$ containing $e'$ and thus $F$ is the 1-factor avoiding $e$. $\square$

Now we are ready to show the main results. We start with the proof of Theorem 2 and then derive the proof of Theorem 1 from it.

*Proof of Theorem 2.* Let $e$ be an edge of $G$. Assume that $G$ has both $a$-factor and $b$-factor avoiding $e$. By applying Theorem 3 to $G-e$, we see that $G-e$ has a $c$-factor, which implies that $G$ has a $c$-factor avoiding $e$.

We now prove that if $G$ has both $a$-factor and $b$-factor containing $e$, then $G$ has a $c$-factor containing $e$.

We define a new graph $G^*$ by inserting a new vertex $w$ on the edge $e$, and define an integer-value function $f_k : V(G^*) \to \{k, 2\}$ such that

$$f_k(x) = \begin{cases} k & \text{if } x \in V(G); \\ 2 & \text{if } x = w. \end{cases}$$

Then $G$ has a $k$-factor containing $e$ if and only if $G^*$ has a $f_k$-factor. It is obvious that $\sum_{x \in V(G^*)} f_k(x) = k|V(G)| + 2 \equiv 0 \pmod 2$ since $G$ is of even order.

Assume that $G^*$ has no $f_c$-factor. Then, by Tutte's $f$-factor Theorem, there exist two disjoint subsets $S, T \subseteq V(G^*)$ such that

$$\delta(S,T;f_c) = \sum_{x \in S} f_c(x) + \sum_{x \in T} (\deg_{G^*-S}(x) - f_c(x)) - h(S,T;f_c) \leq -2. \tag{1}$$

On the other hand, since $G^*$ has both $f_a$-factor and $f_b$-factor, we have

$$\delta(S,T;f_a) = \sum_{x \in S} f_a(x) + \sum_{x \in T} (\deg_{G^*-S}(x) - f_a(x)) - h(S,T;f_a) \geq 0, \tag{2}$$

$$\delta(S,T;f_b) = \sum_{x \in S} f_b(x) + \sum_{x \in T} (\deg_{G^*-S}(x) - f_b(x)) - h(S,T;f_b) \geq 0. \tag{3}$$

Now depending on the location of $w$, we consider three cases:

*Case 1.* $w \notin S \cup T$.

(1), (2) and (3) can be rewritten as

$$c|S| + \sum_{x \in T} \deg_{G^*-S}(x) - c|T| - h(S,T;f_c) \quad \leq \quad -2, \tag{4}$$

$$a|S| + \sum_{x \in T} \deg_{G^*-S}(x) - a|T| - h(S,T;f_a) \quad \geq \quad 0, \tag{5}$$

$$b|S| + \sum_{x \in T} \deg_{G^*-S}(x) - b|T| - h(S,T;f_b) \quad \geq \quad 0. \tag{6}$$

Subtracting (5) from (4), we have

$$(c-a)(|S| - |T|) + h(S,T;f_a) - h(S,T;f_c) \leq -2. \tag{7}$$

Similarly, from (6) and (4), we have

$$(c-b)(|S| - |T|) + h(S,T;f_b) - h(S,T;f_c) \leq -2. \tag{8}$$

Recall that $h(S,T;f_k)$ is the number of $f_k$-odd components $C$ of $G^*-(S\cup T)$, which satisfies $\sum_{x\in V(C)} f_k(x)+e_{G^*}(C,T)\equiv 1 \pmod 2$. Since all $a$, $b$ and $c$ are odd integers, it follows that if $w\notin V(C)$, then

$$\begin{aligned}
& \sum_{x\in V(C)} f_a(x)+e_{G^*}(C,T) = a|C|+e_{G^*}(C,T) \\
\equiv\ & b|C|+e_{G^*}(C,T) = \sum_{x\in V(C)} f_b(x)+e_{G^*}(C,T) \pmod 2 \\
\equiv\ & c|C|+e_{G^*}(C,T) = \sum_{x\in V(C)} f_c(x)+e_{G^*}(C,T) \pmod 2.
\end{aligned}$$

Therefore we obtain

$$h(S,T;f_c)-h(S,T;f_a)\le 1 \qquad \text{and} \qquad h(S,T;f_c)-h(S,T;f_b)\le 1.$$

If $|S|\ge|T|$, then (7) implies

$$-1\le (c-a)(|S|-|T|)+h(S,T;f_a)-h(S,T;f_c)\le -2,$$

a contradiction. If $|S|<|T|$, then (8) implies

$$-1\le (c-b)(|S|-|T|)+h(S,T;f_b)-h(S,T;f_c)\le -2,$$

a contradiction again.

*Case 2.* $w\in S$.

In this case, (1), (2) and (3) become

$$\begin{aligned}
2+c(|S|-1)+\sum_{x\in T}\deg_{G^*-S}(x)-c|T|-h(S,T;f_c) &\le -2 \\
2+a(|S|-1)+\sum_{x\in T}\deg_{G^*-S}(x)-a|T|-h(S,T;f_a) &\ge 0 \\
2+b(|S|-1)+\sum_{x\in T}\deg_{G^*-S}(x)-b|T|-h(S,T;f_b) &\ge 0.
\end{aligned}$$

It is clear that $h(S,T;f_c)=h(S,T;f_a)=h(S,T;f_b)$. If $|S|\ge|T|+1$, we have $0\le (c-a)(|S|-1-|T|)\le -2$, a contradiction; if $|S|<|T|+1$, then $0\le(c-b)(|S|-1-|T|)\le -2$, a contradiction as well.

*Case 3.* $w\in T$.

In this case, (1), (2) and (3) become

$$\begin{aligned}
c|S|+\sum_{x\in T}\deg_{G^*-S}(x)-2-c(|T|-1)-h(S,T;f_c) &\le -2 \\
a|S|+\sum_{x\in T}\deg_{G^*-S}(x)-2-a(|T|-1)-h(S,T;f_a) &\ge 0 \\
b|S|+\sum_{x\in T}\deg_{G^*-S}(x)-2-b(|T|-1)-h(S,T;f_b) &\ge 0.
\end{aligned}$$

Discussing similarly as in Case 2, we yield contradictions. Consequently the theorem is proved. $\square$

With help of Theorem 2 and Petersen's Theorem (Lemma 1), we can provide a clean proof for Theorem 1.

*Proof of Theorem 1.* For any edge $e$ of $G$, let $F_1$ be a 1-factor containing $e$. From Lemma 3, there exists another 1-factor $F_2$ avoiding $e$. According to the parity of $r$ we consider two cases.

*Case 1.* $r$ is odd.

Since $G - F_1$ is an even regular graph, by Lemma 1, $G - F_1$ can be decomposed into 2-factors $T_1, T_2, \ldots, T_m$, where $m = (r-1)/2$. For an integer $k$ $(1 \leq k \leq m-1)$, $F_1 \cup T_1 \cup \cdots \cup T_k$ is a $(2k+1)$-factor containing $e$. In the mean time, $T_1 \cup \cdots \cup T_k$ is a $2k$-factor avoiding $e$. Moreover, $G - F_1$ is a $2m$-factor avoiding $e$.

Similarly, $G - F_2$ has disjoint 2-factors $T_1, T_2, \ldots, T_m$. Without loss of generality, we may assume $e \in T_1$. Then $F_2 \cup T_2 \cup \cdots \cup T_{k+1}$ is a $(2k+1)$-factor avoiding $e$, and $T_1 \cup \cdots \cup T_k$ is a $2k$-factor containing $e$. Furthermore, $G - F_2$ is a $2m$-factor containing $e$. Therefore the theorem holds in this case.

*Case 2.* $r$ is even.

For even $k$, similar to Case 1, $G$ can be decomposed into 2-factors $T_1, T_2, \ldots, T_m$, where $m = r/2$. Without loss of generality, assume $e \in T_1$. Then $T_1$, $T_1 \cup T_2$, $\ldots$, $T_1 \cup \ldots \cup T_m$ are 2-factor, 4-factor, $\ldots$, $r$-factor containing $e$, respectively. Moreover, $T_2$, $T_2 \cup T_3$, $\ldots$, $T_2 \cup T_3 \cup \ldots \cup T_m$ are 2-factor, 4-factor, $\ldots$, $(r-2)$-factor avoiding $e$, respectively.

For odd $k$, it is clear that $G - F_2$ is a $(r-1)$-factor containing $e$ and $G - F_1$ is an $(r-1)$-factor avoiding $e$. By Theorem 2, the odd-factors $F_1$ and $G - F_2$ containing $e$, respectively, imply the existence of $k$-factors containing $e$, $1 \leq k \leq r-1$. Similarly, we obtain $k$-factors avoiding $e$, $1 \leq k \leq r-1$.

So the desired statement holds and consequently the theorem is proved. $\square$

Next we consider the existence of factors containing or avoiding a given edge in a regular graph of *odd* order and prove a similar but slightly weaker result than Theorem 1.

**Theorem 4** *Let $G$ be a connected $2r$-regular graph of odd order. For any given edge $e$ and any vertex $v \in V(G) - V(e)$, if $G - v$ has a 1-factor containing $e$, then $G - v$ has a $[k, k+1]$-factor containing or avoiding $e$ for $1 \leq k \leq 2r-2$.*

*Proof.* For any edge $e$ of $G$ and any vertex $u \in V(G) - V(e)$, let the neighbor vertices of $u$ be $x_1, x_2, \ldots, x_{2r}$. We construct a new graph $G^*$ by using two copies of $G - u$ and joining two sets of vertices $\{x_1, x_2, \ldots, x_{2r}\}$ by a matching $M$. Then the resulting graph $G^*$ is a $2m$-regular graph with $2(|V(G)| - 1)$ vertices. Since $G - u$ has a 1-factor containing $e$, so does $G^*$. By Theorem 1, $G^*$ has a $k$-factor containing $e$ and another $k$-factor avoiding $e$ for all $k$, $1 \leq k \leq 2r-1$. Deleting the matching $M$ from $G^*$, we obtain a $[k, k+1]$-factor containing or avoiding $e$ for $1 \leq k \leq 2r - 2$. $\square$

## References


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