A note on graphs without short even cycles

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Abstract

In this note, we show that any n-vertex graph without even cycles of length at most $2k$ has at most $\frac{1}{2}n^{1+1/k} + O(n)$ edges, and polarity graphs of generalized polygons show that this is asymptotically tight when $k \in \{2, 3, 5\}$.

1 Introduction

In this note, we study graphs without cycles of prescribed even lengths. For a finite or infinite set $\mathcal{C}$ of cycles, define $\mathrm{ex}(n, \mathcal{C})$ to be the maximum possible number of edges in an n-vertex graph which does not contain any of the cycles in $\mathcal{C}$. The asymptotic behaviour of the function $\mathrm{ex}(n, \mathcal{C})$ is particularly interesting when at least one of the cycles in $\mathcal{C}$ is of even length, and was initiated by Erdős [5]. In general, it is the lower bounds for $\mathrm{ex}(n, \mathcal{C})$ – that is, the construction of dense graphs without certain even cycles – which are hard to come by. The best known lower bounds are based on finite geometries, such as polarity graphs of generalized polygons [9], and the algebraic constructions given by Lazebnik, Ustimenko and Woldar [8] and Ramanujan graphs of Lubotsky, Phillips and Sarnak [11]; see also [10]. In the direction of upper bounds, the first major result is known as the even circuit theorem, due to Bondy and Simonovits [3], who proved that $\mathrm{ex}(n, \{C_{2k}\}) \leq 100kn^{1+\frac{1}{k}}$. A more extensive study of $\mathrm{ex}(n, \mathcal{C})$ was carried out by Erdős and Simonovits [6]. Our point of departure is the study of $\mathrm{ex}(n, \mathcal{C})$ when $\mathcal{C}$ consists only of the even cycles of length at most $2k$. The main result of this article is the following:

Theorem 1 *Let $k \geq 2$ be an integer. Then, for all n,*

$$ex(n, \{C_4, C_6, \ldots, C_{2k}\}) \;\; \leq \;\; \tfrac{1}{2}n^{1+\frac{1}{k}} + 2^{k^2}n.$$

Furthermore, when $k \in \{2, 3, 5\}$, the n-vertex polarity graphs of generalized $(k+1)$-gons in [9] have $\frac{1}{2}n^{1+1/k} + O(n)$ edges and no even cycles of length at most $2k$.

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For the statement about the number of edges in the polarity graphs, see [9], page 9. Theorem 1 extends the Moore bound (see [2]) up to an additive term, and a more recent result of Alon, Hoory, and Linial [1], who proved that an n-vertex graph without cycles of length at most $2k$ has at most $\frac{1}{2}(n^{1+1/k}+n)$ edges (see Proposition 6). In other words, we do not require that the odd cycles be forbidden, and the same bound still holds, but with a weaker additive linear term. Our result is also best possible in the following sense: if we forbid only the $2k$-cycle in our graphs, then the upper bounds in Theorem 1 no longer hold – it was shown recently, in [7], that $\mathrm{ex}(n,\{C_6\}) > 0.534n^{4/3}$ and $\mathrm{ex}(n,\{C_{10}\}) > 0.598n^{6/5}$ as n tends to infinity.

2 Local Structure

Let G be a graph with no even cycles of length less than or equal to $2k$. We write $P[u,v]$ to indicate that a path $P \subset G$ has end vertices u and v, and we order the vertices of P from u to v. Let $\prec$ denote this ordering along P. A *vine* on a path P is a graph consisting of the union of P together with paths $Q[u_i,v_i]$ which are internally disjoint from P for $i=1,2,\dots,r$, and where $u \preceq u_1 \prec v_1 \preceq u_2 \prec v_2 \preceq \cdots \preceq u_r \prec v_r \preceq v$. A uv-path of shortest length is called a *uv-geodesic*. A *θ-graph* consists of three internally disjoint paths with the same pair of endpoints.

Lemma 2 *Any θ-graph contains an even cycle.*

Proof. If P,Q and R are the internally disjoint paths in the θ-graph with the same pair of endpoints, then $|P\cup Q|+|Q\cup R|+|P\cup R| = 2|P|+2|Q|+2|R|$, which is even. Therefore one of the cycles $P\cup Q$, $Q\cup R$ or $P\cup R$ must have even length. ∎

Lemma 3 *Let P^* be a uv-geodesic of length at most k. Then the union H of all uv-paths of length at most k is a vine on P^* and P^* is the unique uv-geodesic.*

Proof. Suppose, for a contradiction, that H is not a vine on P^*. Let $x \prec v$ be a vertex of P^* at a maximum distance from u on P^* such that the union of all ux-paths in H is a vine on $P^*[u,x]$. By the maximality of x, there is a uv-path P of length at most k such that x has degree three in $P\cup P^*$. If P has minimum possible length, then $P[x,y]\cup P^*[x,y]$ is the only cycle in $P\cup P^*$ for some $y \succ x$ on P^*. By the maximality of x, the union of all uy-paths in H is not a vine. Therefore there must be a uv-path Q of length at most k such that $Q\cup P\cup P^*$ is not a vine on P^*. If Q has minimum possible length, then $P\cup Q$ and $P^*\cup Q$ each have exactly one cycle. It follows that there is a path $Q[w,z]\subset Q$ such that

$$Q[u,x] = P^*[u,x] \quad \text{and} \quad Q[x,w]\cup Q[z,v] \subset P[x,v]\cup P^*[x,v]$$

and $Q[w,z]$ is internally disjoint from $P\cup P^*$. Since $P\cup P^*\cup Q$ is not a vine, $w \in P[x,y]\cup P^*[x,y]$ and $w\neq y$. If $z\in P^*[y,v]$, then $P^*[x,z]\cup P[x,z]\cup Q[w,z]$ is a θ-graph (see Figure 1).

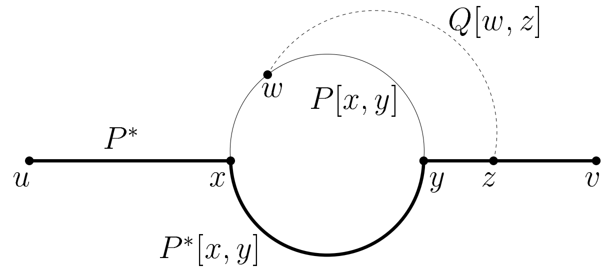

Figure 1 : A θ-graph in $Q \cup P \cup P^*$.

The cycles in this θ graph are $P[w,z] \cup Q[w,z] \subset P \cup Q$ and $P[x,y] \cup P^*[x,y] \subset P \cup P^*$ and $P^*[x,z] \cup Q[x,z] \subset P^* \cup Q$. Each of these cycles has length at most $2k$, since the paths P, Q and P^* each have length at most k. By Lemma 2, one of these cycles has even length, which is a contradiction. A similar argument works when $z \notin P^*[y,v]$. Therefore H is a vine on P^*.

To complete the proof, we must show that P^* is the unique uv-geodesic. By definition, H consists of the union of P^* and paths $P_i = P_i[u_i, v_i]$ for $i \in [r]$, and let $P_i^* = P^*[u_i, v_i]$. Since each cycle $P_i^* \cup P_i$ is of length at most $2k$, each cycle in the vine has odd length. Now suppose P is another uv-geodesic. Then $P_i \subset P$ for some i. Since $P_i \cup P_i^*$ is an odd cycle, we may assume $|P_i| < |P_i^*|$. By replacing P_i^* with P_i on P^*, we obtain a uv-path of length $|P^*| - |P_i^*| + |P_i| < |P^*|$, which contradicts the fact that P^* is a uv-geodesic. So P^* is the unique uv-geodesic. ∎

Henceforth, the paths in the vine on P^* will be denoted $P_i = P_i[u_i, v_i]$, and $P^*[u_i, v_i] = P_i^*$, for $i \in [r]$. Let $\mathcal{P}_k(u,v)$ denote the set of all uv-paths of length k, and define the map

$$f : \mathcal{P}_k(u,v) \to 2^{[r]} \quad \text{by} \quad f(P) = \{i \in [r] \mid P_i[u_i, v_i] \subset P\}.$$

Then $f(P)$ records the set of integers i for which the path $P \in \mathcal{P}_k(u,v)$ uses the path $P_i[u_i, v_i]$ in the vine on P^* instead of $P^*[u_i, v_i]$. Let $\mathcal{F}$ be the image of $\mathcal{P}_k(u,v)$ under f.

Lemma 4 *The map f is an injection, and the family $\mathcal{F}$ is an antichain of sets of size at most $k - |P^*|$ in the partially ordered set of all subsets of $[r]$.*

Proof. By Lemma 3, each $P \in \mathcal{P}_k(u,v)$ is the union of some (possibly none) of the paths P_i together with internally disjoint subpaths of P^*. Therefore the set $f(P)$ uniquely determines P, and f is an injection. If two sets in $\mathcal{F}$ are comparable, say $f(P) \subset f(Q)$, then $|Q| > |P|$ and $Q \notin \mathcal{P}_k(u,v)$, which is a contradiction. So $\mathcal{F}$ is an antichain. Finally, any path $P \in \mathcal{P}_k(u,v)$ has length at least $|P^*| + |f(P)|$, by Lemma 3, so all sets in $\mathcal{F}$ have size at most $k - |P^*|$. ∎

Theorem 5 *Let G be a graph containing no even cycles of length at most $2k$. Then*

$$|\mathcal{P}_k(u,v)| \;\le\; \max\left(\binom{r}{m} : r \le k \text{ and } m = \min\left\{\left\lfloor\frac{r}{2}\right\rfloor, k-r\right\}\right).$$

The equality is achieved when $r = |P^*|$ and the vine on P^* comprises $|P^*|$ triangles.

Proof. The family $\mathcal{F}$ is an antichain, by Lemma 4. By Sperner's Theorem and the LYM inequality [4], this means that $|\mathcal{F}| \le \binom{r}{m}$ where $m = \min\left\{\lfloor\frac{r}{2}\rfloor, k - |P^*|\right\}$. ∎

A *non-returning* walk of length r in G is a walk whose consecutive edges are distinct. Let $\mathcal{W}_r$ be the set of non-returning r-walks (for $r = 0$, $\mathcal{W}_0$ consists of single vertices). The final result required for the proof of Theorem 1 is the following lower bound on the number of non-returning walks, by Alon, Hoory and Linial [1], which gives the best known upper bound on $\mathrm{ex}(n, \{C_3, C_4, \dots, C_{2k}\})$:

Proposition 6 *Let G be an n-vertex graph of average degree $d \ge 2$. Then $|\mathcal{W}_r| \ge nd(d-1)^{r-1}$. Moreover, if G has average degree $d \ge 2$ and no cycles of length at most $2k$, then $d(d-1)^{k-1} \le n$.*

In [1], the number $\mathcal{W}_r/nd$ is denoted N_{r-1} and shown to be less than $(d-1)^{r-1}$. The second statement of the Proposition is an immediate consequence of the main theorem there.

3 Proof of Theorem 1

Let G be a counterexample to Theorem 1 with minimal number of vertices n and average degree d. Then $d > n^{\frac{1}{k}} + 2^{k^2}$, and G has minimum degree at least $\lfloor d/2 \rfloor + 1$, otherwise we remove a vertex of lower degree, keeping the average degree non-increasing, to obtain a smaller counterexample than G. We may also assume $n > 2^{k^2}$. Now let v be a vertex of G of maximum degree, Δ. Pick a breadth-first search tree T rooted at v, and let T_r be the set of vertices of G at distance at most r from v. Then no vertex of T_r is joined to two vertices in T_{r-1}, and the set of edges in $T_{r-1} \backslash T_{r-2}$ form a matching, for all $r \le k$. So every vertex of T has degree at least $\delta - 2$, where δ is the minimum degree in G, from which we deduce

$$1 + \Delta + \Delta(\delta-2) + \dots + \Delta(\delta-2)^{k-1} \;\le\; |V(T)| \;\le\; n.$$

Since $\delta > \lfloor d/2 \rfloor$ and $d > n^{\frac{1}{k}} + 4$, we find $\Delta < 2^{k-1} n^{\frac{1}{k}}$.

Now let $\mathcal{P}_r$ be the set of paths of length r in G, and let $\mathcal{Q}_r = \mathcal{W}_r - \mathcal{P}_r$ be the set of non-returning walks with r edges which are not paths. There are at least $\delta - k$ extensions of a given path of length r in G, for any $r < k$. Therefore

$$|\mathcal{P}_k| \ge (\delta-k)^{k-\ell} |\mathcal{P}_\ell| \quad \text{and} \quad |\mathcal{Q}_k| \le \Delta^{k-1} kn < k 2^{(k-1)^2} n^{\frac{2k-1}{k}}. \tag{1}$$

By Lemma 3, for any pair (u, v) of distinct vertices, joined by at least two paths of length k, there is a uv-geodesic of length $\ell < k$. By Theorem 5, $|\mathcal{P}_k(u, v)| < 2^k$, so the number of ordered pairs of vertices joined by exactly one k-path is at least

$$\begin{aligned}
|\mathcal{P}_k| - 2^k \sum_{\ell=1}^{k-1} |\mathcal{P}_\ell| &\geq |\mathcal{P}_k| \left(1 - \frac{2^k}{\delta - k - 1}\right) \\
&= (\,|\mathcal{W}_k| - |\mathcal{Q}_k|\,) \cdot \left(1 - \frac{2^k}{\delta - k - 1}\right) \\
&> \left(nd(d-1)^{k-1} - k2^{(k-1)^2} n^{\frac{2k-1}{k}}\right) \cdot \left(1 - \frac{2^k}{\delta - k - 1}\right).
\end{aligned}$$

In the last line, we used (1) and Proposition 6. There are $n(n-1)$ (ordered) pairs of distinct vertices which could be joined by a unique path of length k, so the expression above is less than n^2. Using $\delta - k - 1 \geq \frac{d}{4}$ and substituting $d = n^{\frac{1}{k}} + 2^{k^2}$ into the last line, we get

$$\begin{aligned}
n^2 &> \left(n(n^{\frac{1}{k}} + 2^{k^2})(n^{\frac{1}{k}} + 2^{k^2} - 1)^{k-1} - k2^{(k-1)^2} n^{\frac{2k-1}{k}}\right) \left(1 - \frac{2^{k+2}}{n^{\frac{1}{k}} + 2^{k^2}}\right) \\
&= \left(n^{\frac{2k-1}{k}} (n^{\frac{1}{k}} + 2^{k^2})(1 + n^{-\frac{1}{k}}(2^{k^2} - 1))^{k-1} - k2^{(k-1)^2} n^{\frac{2k-1}{k}}\right) \left(1 - \frac{2^{k+2}}{n^{\frac{1}{k}} + 2^{k^2}}\right) \\
&> \left(n^{\frac{2k-1}{k}} (n^{\frac{1}{k}} + 2^{k^2})(1 + n^{-\frac{1}{k}}(k-1)(2^{k^2} - 1)) - k2^{(k-1)^2} n^{\frac{2k-1}{k}}\right) \left(1 - \frac{2^{k+2}}{n^{\frac{1}{k}} + 2^{k^2}}\right) \\
&> n^2 \left(1 + \frac{2^{k^2}}{n^{\frac{1}{k}} + 2^{k^2}}\right) \left(1 - \frac{2^{k+2}}{n^{\frac{1}{k}} + 2^{k^2}}\right) > n^2
\end{aligned}$$

which gives a contradiction. We must thus have $d < n^{\frac{1}{k}} + 2^{k^2}$. ∎

4 Concluding Remarks

If G is d-regular, then picking a breadth first search tree as in the calculation of the maximum degree we obtain

$$1 + d + d(d-2) + \cdots + d(d-2)^{k-1} \leq n.$$

So in this case we have $d < n^{\frac{1}{k}} + 2$. The main points at which the large linear term is introduced in the proof of Theorem 1 is in the estimate of the maximum degree and the upper bound on $|\mathcal{Q}_k|$. We believe it should be possible to circumvent these bounds to obtain a linear term of the form cn, for some absolute constant c. Finally, we note that the analogous extremal problem when some of the short odd cycles are forbidden seems to be very difficult. For example, it is known that

$$\frac{1}{2\sqrt{2}} \leq \liminf_{n\to\infty} \frac{\mathrm{ex}(n,\{C_3,C_4\})}{n^{3/2}} \leq \limsup_{n\to\infty} \frac{\mathrm{ex}(n,\{C_3,C_4\})}{n^{3/2}} \leq \frac{1}{2},$$

but the asymptotic value of $\mathrm{ex}(n,\{C_3,C_4\})$ remains an open question (posed by Erdős).

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