Extending Arcs: An Elementary Proof

T. Alderson*

Department of Mathematical Sciences
University of New Brunswick, Saint John, N.B., Canada
talderso@unb.ca

Submitted: Jan 19, 2005; Accepted: Apr 7, 2005; Published: Apr 28, 2005
Mathematics Subject Classifications: 51E21,51E15

Abstract

In a finite projective plane π we consider two configuration conditions involving arcs in π and show via combinatorial means that they are equivalent. When the conditions hold we are able to obtain embeddability results for arcs, all proofs being elementary. In particular, when $\pi = PG(2, q)$ with q even we provide short proofs of some well known embeddability results.

1 Introduction

Let π be a projective plane of order n where n can be even or odd. An *arc* of size k or a k*-arc* in π is defined as a set of k points no three of which are collinear. If $\mathcal{K}$ is a k-arc and P is a point with $\mathcal{K} \cup \{P\}$ a $(k + 1)$-arc, we say that P is an *extending point* of $\mathcal{K}$. An arc is said to be *complete* if it possesses no extending points. A line ℓ is said to be a *tangent* (resp. *secant*) of an arc $\mathcal{K}$ if ℓ meets exactly one point (resp. two points) of $\mathcal{K}$. In a plane of even (resp. odd) order an arc can have size at most $n + 2$ (resp. $n + 1$) in which case it is an *hyperoval* (resp. *oval*). For background on arcs in projective planes see [3] or [5]. Given a k-arc $\mathcal{K}$ in π we define a parameter δ associated with $\mathcal{K}$ as follows: $k + \delta = n + 2$. We consider the following two conditions in π:

Condition A: Every arc of size $k \geq \frac{n+4}{2}$ is contained in a unique complete arc.

Condition B: If $\mathcal{K}$ is a complete k-arc, then no point of π lies on as many as $\delta + 2$ tangents of $\mathcal{K}$.

For $k \leq \frac{n}{2} + 3$ then Condition B is met trivially, so the condition is one on complete arcs of reasonable size. It is well known that the classical planes $PG(2, q)$ where q is even

*The author acknowledges support from the N.S.E.R.C. of Canada

meet Condition A ([7],[3]). Not all finite planes of even order satisfy Condition A; we mention a class of counterexamples due to Menichetti [4].

Our first result is that in π conditions A and B are equivalent. We call a plane meeting conditions A or B a *AB-plane* In [5], Segre proved the following famous result:

Theorem 1. *In $PG(2,q)$, q even, an arc of size $k > q - \sqrt{q} + 1$ is contained in an unique hyperoval.*

In his proof, Segre used the very deep Hasse-Weil theorem. No elementary proof of Theorem 1 is known. Twenty years later, Thas [8] provided a proof of the following by employing elementary methods of algebraic and projective geometry.

Theorem 2. *In $PG(2,q)$, q even, an arc of size $k > q - \sqrt{q + \frac{1}{4}} + \frac{3}{2}$ is contained in an unique hyperoval.*

Based on this result Thas was able to give embedding results matching the bound of Segre in most cases. Here, we improve the bound in Theorem 2 in classical planes, and match the bound in the broader context of a general AB-plane. Our methods are elementary, combinatorial and self contained.

It is a long standing conjecture that complete q-arcs exist in all non-classical planes [2]. The weaker conjecture, that the bound in Theorem 1 holds only in classical planes, has also stood for twenty years [6]. Hence, it is of interest to investigate the existence of AB-planes that are non-classical. This question seems difficult and may be beyond present day techniques.

2 AB-Planes

Theorem 3. *In a finite projective plane, conditions A and B are equivalent.*

Proof. Let π be a projective plane of order n contain the complete k-arc $\mathcal{K}$. Suppose π is a AB-plane. Observe that each point of $\mathcal{K}$ lies on $k-1$ secants and hence on exactly δ tangents. Suppose by way of contradiction that $P \notin \mathcal{K}$ lies on at least $\delta+2$ tangents. Then P is on say $\alpha \leq \frac{k-(\delta+2)}{2}$ secants. Form two new arcs $\mathcal{K}'$ and $\mathcal{K}''$ in the following manner. On each secant on P pick a point of $\mathcal{K}$, say $P_1, P_2, \ldots P_\alpha$. Let $\mathcal{K}' = \mathcal{K} - \{P_1, P_2, \ldots P_\alpha\}$ and $\mathcal{K}'' = \mathcal{K}' \cup \{P\}$. Then both $\mathcal{K}$ and $\mathcal{K}''$ contain $\mathcal{K}'$, and

$$|\mathcal{K}'| = k - \alpha \geq k - \frac{k-(\delta+2)}{2} = \frac{k+\delta+2}{2} = \frac{n+4}{2} \tag{1}$$

Since π is a AB-plane, the unique complete arc containing $\mathcal{K}'$ must contain both $\mathcal{K}$ and $\mathcal{K}''$. We conclude $\{P\} \cup \mathcal{K}$ is an arc. This is a contradiction.

For the converse we assume π is not a AB-plane. So we have two complete arcs $\mathcal{K}_1 \neq \mathcal{K}_2$ of size k_1 and k_2 respectively with $|\mathcal{K}_1 \cap \mathcal{K}_2| \geq \frac{n}{2} + 2$. Let δ_2 be defined by $k_2 + \delta_2 = n + 2$ and choose $P \in \mathcal{K}_1 - \mathcal{K}_2$. Then P is on at least $\frac{n}{2} + 2 \geq \delta_2 + 2$ tangents of $\mathcal{K}_2$ and is not an extending point of $\mathcal{K}_2$. $\square$

3 Arc Extending

Lemma 1. *Let $\mathcal{K}$ be a k-arc in a plane π of order n where $k > n - \sqrt{n + \frac{1}{4}} + \frac{3}{2}$. Then on any tangent to $\mathcal{K}$ there is a point P which lies on at least $\delta + 1$ tangents. Furthermore, if n is even then P lies on at least $\delta + 2$ tangents.*

Proof. Let ℓ be a tangent of $\mathcal{K}$ at say $Q \in \mathcal{K}$. There are exactly $\binom{k-1}{2}$ secants intersecting ℓ in points other than Q. As such there exists a point of ℓ off of $\mathcal{K}$ for which the maximum number of secants through that point is:

$$\binom{k-1}{2}\frac{1}{n} = \frac{(k-1)(k-2)}{2n} \tag{2}$$

We then have the number of tangents through some point of ℓ off of $\mathcal{K}$ is at least:

$$k - 2\frac{(k-1)(k-2)}{2n} = (n - \delta + 2) - \frac{(n-\delta+1)(n-\delta)}{n} = \delta + 1 - \frac{\delta(\delta - 1)}{n} \tag{3}$$

Thus, if $n > \delta(\delta - 1)$ there is a point P of ℓ off of $\mathcal{K}$ which lies on at least $\delta + 1$ tangents. A simple calculation with the restriction that δ is non-negative then gives $n > \delta(\delta - 1)$ iff $\delta < \sqrt{n + \frac{1}{4}} + \frac{1}{2}$. For the second part of the proof we observe that if n is even then $k + \delta + 1 = n + 3$ is odd. So exactly one of k and $\delta + 1$ is even. If k is even then P must be on an even number of tangents forcing P to be on at least $\delta + 2$ tangents. If k is odd then P must be on an odd number of tangents forcing P to be on at least $\delta + 2$ tangents. $\square$

Theorem 4. *In an AB-plane of even order n a k-arc with $k > n - \sqrt{n + \frac{1}{4}} + \frac{3}{2}$ is contained in a unique hyperoval.*

Proof. This follows immediately from Theorem 3 and Lemma 1. $\square$

Lemma 2. *Let $\mathcal{K}$ be an arc of size $k > n - \sqrt{n + \frac{9}{4}} + \frac{3}{2}$ in a projective plane of order $n = 2^t > 2$. Then on any tangent to $\mathcal{K}$ there is a point P which lies on at least $\delta + 2$ tangents.*

Proof. Assume by way of contradiction that no point lies on as many as $\delta + 2$ tangents. As in the proof of Lemma 1 we get

$$n \leq \delta(\delta - 1) \tag{4}$$

Since $n = 2^t > 2$, equality in (4) is not possible. Since both n and $\delta(\delta - 1)$ are even, we get

$$n \leq \delta(\delta - 1) - 2$$

Simple calculations then give

$$n \leq \delta(\delta - 1) - 2 \Rightarrow k \leq n - \sqrt{n + \frac{9}{4}} + \frac{3}{2}$$

$\square$

Theorem 5. *In a AB-plane of order $n = 2^t$ a k-arc with $k > n - \sqrt{n + \frac{9}{4}} + \frac{3}{2}$ is contained in a hyperoval. If $t > 1$ then the hyperoval is uniquely determined.*

Proof. In the case $n = 2$ simple counting gives the result. The case $t > 1$ follows immediately from Theorem 3 and Lemma 2. □

Corollary 1. *In $PG(2, q)$, q even a k-arc with $k > q - \sqrt{q + \frac{9}{4}} + \frac{3}{2}$ is contained in an unique hyperoval.*

Remark 1. *With reference to Theorem 1 we note the following inequality:*

$$\left(q - \sqrt{q + \frac{9}{4}} + \frac{3}{2}\right) - (q - \sqrt{q} + 1) = \frac{1}{2} - \frac{\frac{9}{4}}{\sqrt{q} + \sqrt{q + \frac{9}{4}}} < \frac{1}{2} \tag{5}$$

As in [8] we are able to improve our bounds in certain cases. For the sake of completeness we effectively reproduce the proofs in [8] for the following Lemmas.

Lemma 3. *If π is an AB-plane of even order n where n is a square, then every arc of size $k > n - \sqrt{n} + 1$ is contained in a unique hyperoval.*

Proof. If n is a square, we have

$$k > n - \sqrt{n} + 1 \Rightarrow k > n - \sqrt{n} + \frac{3}{2} \Rightarrow k > n - \sqrt{n + \frac{9}{4}} + \frac{3}{2}.$$

□

Corollary 2. *In $PG(2, q)$, $q = 2^{2t}$ every arc of size $k > q - \sqrt{q} + 1$ is contained in a unique hyperoval.*

Lemma 4. *In a projective AB-plane π of even order n, every arc of odd size k satisfying $k > n - \sqrt{n} + 1$ is contained in a unique hyperoval.*

Proof. Suppose that $\mathcal{K}$ is a complete arc of odd size $k < n + 2$. By Theorems 1 and 3 no point of $\pi - \mathcal{K}$ is on as many as $\delta + 1$ tangents. Since k is odd, every point of $\pi - \mathcal{K}$ must meet an odd number of tangent points. For each point $P_i \notin \mathcal{K}$, $1 \le i \le n^2 + n + 1 - k$, denote by t_i the number of tangents on P_i. $\mathcal{K}$ has $k\delta$ tangents, each of which contains n points not from $\mathcal{K}$. So we have $\Sigma_i t_i = nk\delta$. By counting ordered triples (l, l', P) where l and l' are tangents through $P \notin \mathcal{K}$ we obtain $\Sigma_i t_i(t_i - 1) = k\delta(k\delta - \delta)$. Since t_i is odd and less than $\delta + 1$ we have $\Sigma_i(t_i - 1)(t_i - \delta) \le 0$ whence $\Sigma_i t_i(t_i - 1) - \delta\Sigma_i t_i + \delta\Sigma_i 1 \le 0$. This gives $k\delta(k\delta - \delta) - \delta^2 kn + \delta(n^2 + n + 1 - k) \le 0$. Substituting $\delta = n + 2 - k$ and simplifying gives $(k - 1)(k - 2 - n)(k^2 - 2k - 2kn + 1 + n^2 + n) \le 0$ which in turn gives $(k - 1)(n + 2 - k)(k - (n + \sqrt{n} + 1))(k - (n - \sqrt{n} + 1)) \ge 0$. We conclude that $k \le n - \sqrt{n} + 1$. □

Corollary 3. *In $PG(2, q)$, $q = 2^t$, every arc of odd size k satisfying $k > q - \sqrt{q} + 1$ is contained in a unique hyperoval.*

References

[1] T. L. Alderson, On MDS Codes and Bruen-Silverman Codes, Ph.D. Dissertation, University of Western Ontario, January, 2002.

[2] Barlotti, A. *Un'osservazione intorno ad un teorema di B. Segre sui q-archi.* (Italian) Matematiche (Catania) 21 1966 23–29.

[3] J.W.P. Hirschfeld, Projective geometries over finite fields, Claredon Press, Oxford,1979.

[4] G. Menichetti, q-Archi completi nei paini di Hall di ordine $q = 2^k$. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **8**, 56, 1974, no. 4, 518–525.

[5] B. Segre, Introduction to Galois Geometries (ed.J.W.P Hirschfeld), *Mem. Accad. Naz. Lincei* **8**, 1967, 133-263.

[6] T. Szönyi, Complete arcs in non-Desarguesian planes, *Ars Combin.*, 25, 1988, C, 169–178.

[7] T. Szönyi, k-Sets in PG(2,q) Having Large Sets of Internal Nuclei, Combinatorics '88, volume 2,Mediterranean Press, Rende, 1991, 449-458.

[8] J.A. Thas, Elementary Proofs of Two Fundamental Theorems of B. Segre without Using the Hasse-Weil Theorem, *J. Comb. Theory Ser. A*, **34**, 1983, 381–384.