Rank–three matroids are Rayleigh

David G. Wagner*

Department of Combinatorics and Optimization
University of Waterloo, Waterloo, Ontario, Canada
dgwagner@math.uwaterloo.ca

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Abstract

A Rayleigh matroid is one which satisfies a set of inequalities analogous to the Rayleigh monotonicity property of linear resistive electrical networks. We show that every matroid of rank three satisfies these inequalities.

1 Introduction.

For the basic concepts of matroid theory we refer the reader to Oxley's book [5].

A linear resistive electrical network can be represented as a graph $G = (V, E)$ together with a set of positive real numbers $\mathbf{y} = \{y_e : \ e \in E\}$ that specify the conductances of the corresponding elements. In 1847 Kirchhoff [3] determined the effective conductance of the network measured between vertices $a, b \in V$ as a rational function $\mathcal{Y}_{ab}(G; \mathbf{y})$ of the conductances $\mathbf{y}$. This formula can be generalized directly to any matroid.

For electrical networks the following property is physically intuitive: if $y_c > 0$ for all $c \in E$ then for any $e \in E$,

$$\frac{\partial}{\partial y_e} \mathcal{Y}_{ab}(G; \mathbf{y}) \geq 0.$$

That is, by increasing the conductance of the element e we cannot decrease the effective conductance of the network as a whole. This is known as the *Rayleigh monotonicity* property.

Informally, a matroid has the Rayleigh property if it satisfies inequalities analogous to the Rayleigh monotonicity property of linear resistive electrical networks. While there are non–Rayleigh matroids of rank four or more, we show here that every matroid of rank (at most) three is Rayleigh, answering a question left open by Choe and Wagner [1].

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Let $\mathcal{M}$ be a matroid with ground–set E, and fix indeterminates $\mathbf{y} := \{y_e : \ e \in E\}$ indexed by E. For a basis B of $\mathcal{M}$ let $\mathbf{y}^B := \prod_{e \in B} y_e$, and let $M(\mathbf{y}) := \sum_{B \in \mathcal{M}} \mathbf{y}^B$ with the sum over all bases of $\mathcal{M}$. (As usual, an empty sum has value 0 and an empty product has value 1.) Since $M(\mathbf{y})$ is insensitive to the presence of loops we generally consider only loopless matroids, and regard $\mathcal{M}$ as its set of bases.

For disjoint subsets I, J of E, let $\mathcal{M}_I^J$ denote the minor of $\mathcal{M}$ obtained by contracting I and deleting J. We use the nonstandard convention that if I is dependent then $\mathcal{M}_I^J$ is empty, so that in general

$$\mathcal{M}_I^J := \{B \smallsetminus I : \ B \in \mathcal{M} \text{ and } I \subseteq B \subseteq E(\mathcal{M}) \smallsetminus J\}.$$

The matroid $\mathcal{M}$ is a *Rayleigh* matroid provided that whenever $y_c > 0$ for all $c \in E$, then for every pair of distinct $e, f \in E$,

$$\Delta M\{e, f\}(\mathbf{y}) := M_e^f(\mathbf{y}) M_f^e(\mathbf{y}) - M_{ef}(\mathbf{y}) M^{ef}(\mathbf{y}) \geq 0.$$

See Section 3 of Choe and Wagner [1] for more detailed motivation of this definition. Rayleigh matroids are "balanced" in the sense of Feder and Mihail [2], and for binary matroids these conditions are equivalent. For example, every sixth–root of unity matroid – in particular every regular matroid – is Rayleigh (Proposition 5.1 and Corollary 4.9 of [1]). Since graphic matroids are regular this generalizes the physical assertion that linear resistive electrical networks satisfy Rayleigh monotonicity. One of the main questions left open in [1] is whether or not every matroid of rank three is Rayleigh. Here we show that this is indeed the case.

Theorem 1.1 *Every matroid of rank three is Rayleigh.*

In contrast to this theorem there are several matroids of rank four that are known not to be Rayleigh, among them the matroids $\mathcal{S}_8$ and $\mathcal{J}'$ discussed in [1].

As a concrete but fairly representative consequence of Theorem 1.1, let E be a finite non–collinear set of points in a projective plane, and let $\mathcal{M}$ be the set of unordered non–collinear triples of points in E. Assign a positive real number y_c to each $c \in E$, and consider the probability space $\Omega(\mathcal{M}, \mathbf{y})$ which assigns to each $B \in \mathcal{M}$ the probability $\mathbf{y}^B / M(\mathbf{y})$. Since $\mathcal{M}$ is a rank–three matroid it is Rayleigh, by Theorem 1.1. A short calculation shows that for distinct $e, f \in E$:

$$\frac{M_{ef}(\mathbf{y})}{M_e(\mathbf{y})} \leq \frac{M_f(\mathbf{y})}{M(\mathbf{y})}.$$

That is, in $\Omega(\mathcal{M}, \mathbf{y})$ the probability that a random basis $B \in \mathcal{M}$ contains f, given that it contains e, is at most the probability that a random basis contains f. In short, the events $e \in B$ and $f \in B$ are *negatively correlated* for any distinct $e, f \in E$. This probabilistic point of view is carried further by Feder and Mihail [2] and Lyons [4].

Several conversations and correspondences with Jim Geelen, Sandra Kingan, and Bruce Reznick helped to clarify my thoughts on this problem, for which I thank them sincerely.

2 Preliminaries.

To simplify notation, when calculating with Rayleigh matroids we will henceforth usually omit reference to the variables $\mathbf{y}$ – writing M_I^J instead of $M_I^J(\mathbf{y})$ *et cetera* – unless a particular substitution of variables requires emphasis. We will also write "$\mathbf{y} > \mathbf{0}$" as shorthand for "$y_c > 0$ for all $c \in E$".

We require the following facts from [1].

Proposition 2.1 (Section 3 of [1]) *The class of Rayleigh matroids is closed by taking duals and minors.*

Sketch of Proof. For the matroid $\mathcal{M}^*$ dual to $\mathcal{M}$ and for $e, f \in E(\mathcal{M}^*)$,

$$\Delta M^*\{e,f\}(\mathbf{y}) = \mathbf{y}^{2E} \Delta M\{e,f\}(\mathbf{1}/\mathbf{y})$$

in which $\mathbf{1}/\mathbf{y} := \{1/y_c : \ c \in E\}$. From this it follows that $\mathcal{M}^*$ is Rayleigh if $\mathcal{M}$ is.

For distinct $e, f, g \in E(\mathcal{M})$,

$$\Delta M^g\{e,f\} = \lim_{y_g \to 0} \Delta M\{e,f\}$$

and

$$\Delta M_g\{e,f\} = \lim_{y_g \to \infty} \frac{1}{y_g^2} \Delta M\{e,f\}.$$

From this it follows that if $\mathcal{M}$ is Rayleigh then the deletion $\mathcal{M}^g$ and the contraction $\mathcal{M}_g$ are also Rayleigh. The case of a general minor follows by iteration of these two cases. $\square$

(The class of Rayleigh matroids is also closed by 2-sums, but we will not use this fact.)

For polynomials $A(\mathbf{y})$ and $B(\mathbf{y})$ in $\mathbb{R}[y_c : \ c \in E]$, we write $A(\mathbf{y}) \gg B(\mathbf{y})$ to mean that every coefficient of $A(\mathbf{y}) - B(\mathbf{y})$ is nonnegative. Certainly, if $A(\mathbf{y}) \gg 0$ then $A(\mathbf{y}) \geq 0$ for all $\mathbf{y} > \mathbf{0}$, but not conversely. Making the substitution $y_c = x_c^2$ for each $c \in E$, we have $A(\mathbf{y}) \geq 0$ for all $\mathbf{y} > \mathbf{0}$ if and only if $A(\mathbf{x}^2) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^E$; such a form $A(\mathbf{x}^2)$ is said to be *positive semidefinite*. Artin's solution to Hilbert's 17th problem asserts that every positive semidefinite form can be written as a positive sum of squares of rational functions, but the proof is nonconstructive. Reznick [6] gives an excellent survey of Hilbert's 17th problem. To prove Theorem 1.1 we will write $\Delta M\{e,f\}(\mathbf{y})$ as a positive sum of monomials and squares of polynomials in $\mathbf{y}$.

Regarding the Rayleigh property, one may restrict attention to the class of simple matroids (although it is not always useful to do so) for the following reason. We may assume that $\mathcal{M}$ is loopless, as remarked above. If $a, a_1, \ldots, a_k$ are parallel elements in $\mathcal{M}$, then let $\mathcal{N}$ be obtained from $\mathcal{M}$ by deleting $a_1, \ldots, a_k$. Letting $w_c := y_c$ if $c \in E(\mathcal{N}) \smallsetminus \{a\}$ and $w_a := y_a + y_{a_1} + \cdots + y_{a_k}$, one sees that $M(\mathbf{y}) = N(\mathbf{w})$. A little calculation shows that $\mathcal{M}$ is Rayleigh if and only if $\mathcal{N}$ is Rayleigh. Repeating this reduction as required, we find a simple matroid $\mathcal{L}$ and a substitution of variables $\mathbf{z} = \mathbf{z}(\mathbf{y})$ such that $M(\mathbf{y}) = L(\mathbf{z})$, and such that $\mathcal{M}$ is Rayleigh if and only if $\mathcal{L}$ is Rayleigh.

It is very easy to see that matroids of rank one or two are Rayleigh.

Proposition 2.2 *If $\mathcal{M}$ has rank at most two then $\Delta M\{e,f\} \gg 0$ for all distinct $e,f \in E(\mathcal{M})$. Consequently, $\mathcal{M}$ is Rayleigh.*

Proof. By the above remarks, we may assume that $\mathcal{M}$ is simple. Let the ground–set of $\mathcal{M}$ be $E = \{1, 2, \ldots, m\}$.

If $\mathcal{M}$ has rank one then $M(\mathbf{y}) = y_1 + y_2 + \cdots + y_m$, so $M_{ef} = 0$ for all distinct $e, f \in E$, and hence $\Delta M\{e,f\} = M_e^f M_f^e = 1 \gg 0$.

If $\mathcal{M}$ has rank two then $M(\mathbf{y}) = \sum_{1\leq i<j\leq m} y_i y_j$ is the second elementary symmetric function of $\mathbf{y}$. By symmetry we only need to show that $\Delta M\{1,2\} \gg 0$. Since $M_1^2 = M_2^1 = y_3 + y_4 + \cdots + y_m$ and $M_{12} = 1$ and

$$M^{12} = \sum_{3\leq i<j\leq m} y_i y_j,$$

it follows that

$$\Delta M\{1,2\} = \sum_{3\leq i\leq j\leq m} y_i y_i,$$

proving that $\Delta M\{1,2\} \gg 0$. $\square$

The case of rank–three matroids is much more interesting – the polynomial $\Delta M\{e,f\}$ can have terms with negative coefficients, as happens already for the graphic matroid $\mathcal{K}$ of the complete graph K_4 on four vertices. With the ground–set of $\mathcal{K}$ labelled as in Figure 3(IV), we have

$$\Delta K\{1,2\} = (y_3 y_4 - y_5 y_6)^2.$$

As will be seen in Table 3, however, in some sense this is the worst that can happen in rank three.

3 A reduction lemma for any rank.

For distinct elements $e, f, g \in E(\mathcal{M})$, a short calculation shows that

$$\Delta M\{e,f\} = y_g^2 \Delta M_g\{e,f\} + y_g \Theta M\{e,f|g\} + \Delta M^g\{e,f\}$$

in which

$$\begin{aligned} \Delta M_g\{e,f\} &= M_{eg}^f M_{fg}^e - M_{efg} M_g^{ef}, \\ \Delta M^g\{e,f\} &= M_e^{fg} M_f^{eg} - M_{ef}^g M^{efg}, \end{aligned}$$

and the *central term for $\{e,f\}$ and g in* $\mathcal{M}$ is defined by

$$\Theta M\{e,f|g\} := M_e^{fg} M_{fg}^e + M_f^{eg} M_{eg}^f - M_g^{ef} M_{ef}^g - M_{efg} M^{efg}.$$

For a subset S of $E(\mathcal{M})$, we use $\overline{S}$ to denote the closure of S in $\mathcal{M}$.

Lemma 3.1 *Let $\mathcal{M}$ be a matroid, and let $e, f, g \in E(\mathcal{M})$ be distinct elements. If $\{e, f, g\}$ is dependent in $\mathcal{M}$ then $\Theta M\{e, f|g\} \gg 0$.*

Proof. To prove this we exhibit an injective function

$$\left(\mathcal{M}_g^{ef} \times \mathcal{M}_{ef}^{g}\right) \cup \left(\mathcal{M}_{efg} \times \mathcal{M}^{efg}\right) \longrightarrow \left(\mathcal{M}_e^{fg} \times \mathcal{M}_{fg}^{e}\right) \cup \left(\mathcal{M}_f^{eg} \times \mathcal{M}_{eg}^{f}\right)$$

such that if $(B_1, B_2) \mapsto (A_1, A_2)$ then $\mathbf{y}^{A_1}\mathbf{y}^{A_2} = \mathbf{y}^{B_1}\mathbf{y}^{B_2}$.

Since $\{e, f, g\}$ is dependent it follows that $\mathcal{M}_{efg} = \varnothing$, so let $B_1 \in \mathcal{M}_g^{ef}$ and $B_2 \in \mathcal{M}_{ef}^{g}$. Let $L := \overline{B_1 \smallsetminus \{g\}}$. We claim that either $e \notin L$ or $f \notin L$. To see this, suppose not – then $g \in \overline{\{e, f\}} \subseteq L$, which contradicts the fact that B_1 is a basis. If $e \notin L$ then let $A_1 := B_1 \cup \{e\} \smallsetminus \{g\}$ and $A_2 := B_2 \cup \{g\} \smallsetminus \{e\}$. If $e \in L$ then $f \notin L$, so let $A_1 := B_1 \cup \{f\} \smallsetminus \{g\}$ and $A_2 := B_2 \cup \{g\} \smallsetminus \{f\}$. It is easy to see that in either case both A_1 and A_2 are bases of $\mathcal{M}$.

Notice that for (A_1, A_2) in the image of this function, $A_1 \in \mathcal{M}_e^{fg} \cup \mathcal{M}_f^{eg}$ and this union is disjoint. If $\mathcal{A}_1 \in \mathcal{M}_e^{fg}$ then let $B_1' := A_1 \cup \{g\} \smallsetminus \{e\}$ and $B_2' := A_2 \cup \{e\} \smallsetminus \{g\}$, while if $\mathcal{A}_1 \in \mathcal{M}_f^{eg}$ then let $B_1' := A_1 \cup \{g\} \smallsetminus \{f\}$ and $B_2' := A_2 \cup \{f\} \smallsetminus \{g\}$. In either case we have $(B_1', B_2') = (B_1, B_2)$ showing that the function $(B_1, B_2) \mapsto (A_1, A_2)$ is injective.

This construction provides the desired weight–preserving injection. □

Lemma 3.1 has the following consequence which might be helpful in the investigation of Rayleigh matroids of rank four or more.

Proposition 3.2 *Let $\mathcal{M}$ be a minor–minimal non–Rayleigh matroid, and let $e, f \in E(\mathcal{M})$ and $\mathbf{y} > \mathbf{0}$ be such that $\Delta M\{e, f\} < 0$. Then $\{e, f\}$ is closed in $\mathcal{M}$.*

Proof. If $g \in E(\mathcal{M}) \smallsetminus \{e, f\}$ is such that $\{e, f, g\}$ is dependent, then $\Theta M\{e, f|g\} \gg 0$ by Lemma 3.1. From this it follows that if $\mathbf{y} > \mathbf{0}$ then

$$\Delta M\{e, f\} = y_g^2 \Delta M_g\{e, f\} + y_g \Theta M\{e, f|g\} + \Delta M^g\{e, f\} \geq 0,$$

since every proper minor of $\mathcal{M}$ is Rayleigh. As this contradicts the hypothesis we conclude that $\{e, f\}$ is closed in $\mathcal{M}$. □

The following consequence of Lemma 3.1 is relevant to the present purpose.

Lemma 3.3 *Let $\mathcal{M}$ be a matroid of rank three, and let $e, f \in E(\mathcal{M})$. If $g \in E(\mathcal{M}) \smallsetminus \{e, f\}$ is such that $\{e, f, g\}$ is dependent in $\mathcal{M}$ then $\Delta M\{e, f\}(\mathbf{y}) \gg \Delta M^g\{e, f\}(\mathbf{y})$.*

Proof. Since

$$\Delta M\{e, f\} - \Delta M^g\{e, f\} = y_g^2 \Delta M_g\{e, f\} + y_g \Theta M\{e, f|g\},$$

the inequality follows directly from Proposition 2.2 and Lemma 3.1. □

Figure 1: The four–element rank–three simple matroids.

4 Matroids of rank three.

The proof of Theorem 1.1 is completed by means of the following lower bound for $\Delta M\{e,f\}(\mathbf{y})$, which was found mainly by trial and error.

For $a \in E(\mathcal{M}) \smallsetminus \{e,f\}$ let $L(a,e) := \overline{\{a,e\}} \smallsetminus \{a,e\}$, let $L(a,f) := \overline{\{a,f\}} \smallsetminus \{a,f\}$, and let $U(a) := E(\mathcal{M}) \smallsetminus (\overline{\{a,e\}} \cup \overline{\{a,f\}} \cup \overline{\{e,f\}})$. Define the linear polynomials $B(a) := \sum_{b \in U(a)} y_b$, $C(a) := \sum_{c \in L(a,e)} y_c$, and $D(a) := \sum_{d \in L(a,f)} y_d$, and the quartic polynomials

$$T(\mathcal{M}; e, f, a; \mathbf{y}) := (y_a B(a) - C(a) D(a))^2$$

for each $a \in E(\mathcal{M}) \smallsetminus \{e,f\}$. Finally, define

$$P(\mathcal{M}; e, f; \mathbf{y}) := \frac{1}{4} \sum_{a \in E(\mathcal{M}) \smallsetminus \overline{\{e,f\}}} T(\mathcal{M}; e, f, a; \mathbf{y}).$$

Proposition 4.1 *Let $\mathcal{M}$ be a simple matroid of rank three, and let $e, f \in E(\mathcal{M})$ be distinct. With the notation above,*

$$\Delta M\{e,f\}(\mathbf{y}) \gg P(\mathcal{M}; e, f; \mathbf{y}).$$

Proof. By repeated application of Lemma 3.3, if necessary, we may assume that $\{e,f\}$ is closed in $\mathcal{M}$, so we reduce to this case.

Both $\Delta := \Delta M\{e,f\}(\mathbf{y})$ and $P := P(\mathcal{M}; e, f; \mathbf{y})$ are homogeneous of degree four in the indeterminates $\{y_j : \ j \in E(\mathcal{M}) \smallsetminus \{e,f\}\}$, and the only monomials that occur with nonzero coefficient in either of these polynomials have shape $y_g^2 y_h^2$, $y_g^2 y_h y_i$, or $y_g y_h y_i y_j$ in which the subscripts are pairwise distinct. The coefficient of such a monomial in Δ depends only on the isomorphism type of the restriction $\mathcal{M}|\{e,f,g,h\}$, $\mathcal{M}|\{e,f,g,h,i\}$, or $\mathcal{M}|\{e,f,g,h,i,j\}$, the positions of e and f in this restriction, and, in the second case, the position of g relative to e and f in this restriction. (The coefficient of such a monomial in P can depend on more information, as we shall see.) Since $\{e,f\}$ is closed in $\mathcal{M}$, $\{e,f\}$ is also closed in any such restriction $\mathcal{N}$. The proposition is now proved by an exhaustive case analysis of these configurations in $\mathcal{M}$.

Figure 1 and Table 1 summarize the case analysis for monomials of shape $y_g^2 y_h^2$, Figure 2 and Table 2 summarize the case analysis for monomials of shape $y_g^2 y_h y_i$, and Figure 3

$\mathcal{N}\{e,f\}$	Δ	P	notes
I$\{1,2\}$	$0-0=0$	0	
II$\{1,2\}$	$1-0=1$	$1/2, 3/4, 1$	A.

Table 1: Monomials of shape $y_g^2 y_h^2$.

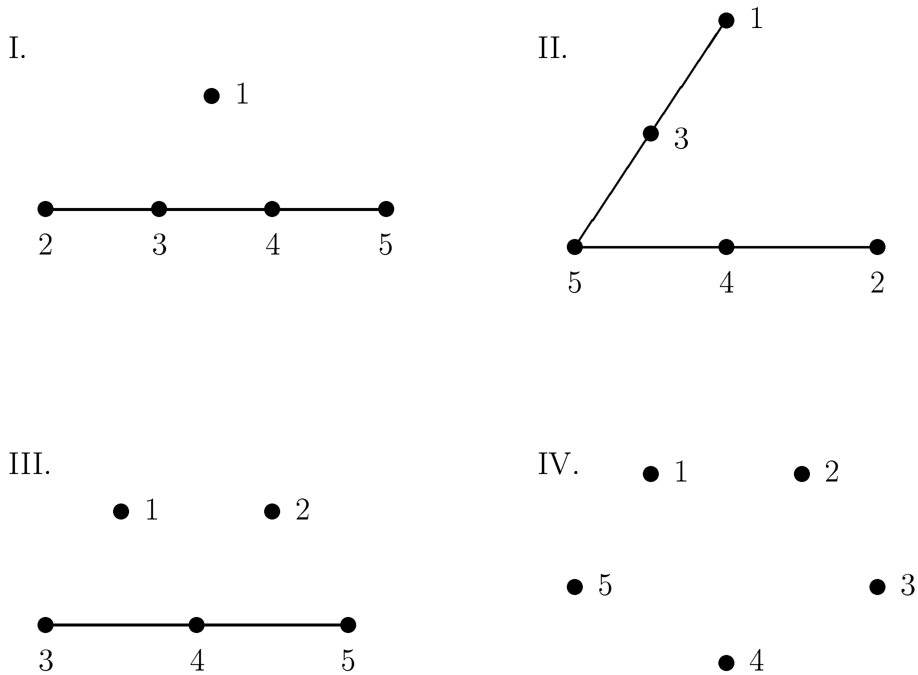

Figure 2: The five–element rank–three simple matroids.

$\mathcal{N}\{e,f\},g$	Δ	P	notes
I$\{1,2\},3$	$0-0=0$	0	
II$\{1,2\},3$	$1-1=0$	0	
II$\{1,2\},5$	$1-1=0$	0	
III$\{1,2\},3$	$2-0=2$	$1/2$	B.
III$\{1,3\},2$	$2-1=1$	$1/2, 1$	C.
III$\{1,3\},4$	$1-1=0$	0	
IV$\{1,2\},3$	$2-1=1$	$1/2$	B.

Table 2: Monomials of shape $y_g^2 y_h y_i$.

$\mathcal{N}\{e,f\}$	Δ	P	notes
$\mathrm{I}\{1,2\}$	$0-0=0$	0	
$\mathrm{II}\{1,2\}$	$3-3=0$	0	
$\mathrm{III}\{1,2\}$	$6-0=6$	0	
$\mathrm{III}\{1,3\}$	$3-3=0$	0	
$\mathrm{IV}\{1,2\}$	$2-4=-2$	-2	D.
$\mathrm{V}\{1,4\}$	$3-4=-1$	-1	E.
$\mathrm{V}\{4,5\}$	$4-3=1$	$-1/2$	F.
$\mathrm{VI}\{1,2\}$	$4-4=0$	$-1/2$	G.
$\mathrm{VI}\{1,3\}$	$5-3=2$	0	
$\mathrm{VI}\{3,6\}$	$4-4=0$	0	
$\mathrm{VII}\{1,2\}$	$5-4=1$	$0,1$	H.
$\mathrm{VIII}\{1,2\}$	$6-3=3$	0	
$\mathrm{VIII}\{1,4\}$	$5-4=1$	0	
$\mathrm{IX}\{1,2\}$	$6-4=2$	0	

Table 3: Monomials of shape $y_g y_h y_i y_j$.

and Table 3 summarize the case analysis for monomials of shape $y_g y_h y_i y_j$. In each table the first column indicates the isomorphism class (from the corresponding figure) of the restriction $\mathcal{N}$ of $\mathcal{M}$, the choice of $\{e,f\}$ in that restriction, and, in Table 2, the choice of g in $\mathcal{N}$. The second column in each table indicates the coefficient of the relevant monomial in each term of

$$M_e^f M_f^e - M_{ef} M^{ef} = \Delta M\{e,f\},$$

respectively. As remarked above these coefficients depend only on $\mathcal{N}$, $\{e,f\}$, and g and are computed from the definition by elementary counting. The third column in each table indicates the coefficient of the relevant monomial in P. Notes in the fourth column of each table refer to the following list of additional remarks regarding the coefficients of the monomials in P and (sometimes) in Δ. As a guide to the reasoning involved, we explain the cases A, C, D, and H in greater detail. It might help to note that

$$T(\mathcal{M}; e, f, a) = y_a^2 B(a)^2 - 2 y_a B(a) C(a) D(a) + C(a)^2 D(a)^2,$$

and that monomials with coefficients 1, 2 or 4 occur within the terms $y_a^2 B(a)^2$ and $C(a)^2 D(a)^2$.

• In general, when the coefficient in the third column is zero there is no possible location for an element $a \in E(\mathcal{M})$ such that the monomial occurs in $T(\mathcal{M}; e, f, a)$.

A. With $\mathcal{N}$ isomorphic to II in Figure 1 we take $e = 1$ and $f = 2$, and consider the coefficient of the monomial $y_3^2 y_4^2$ in Δ and in P. This monomial occurs with coefficient 1 in $M_e^f M_f^e$, and with coefficient 0 in $M_{ef} M^{ef}$.

The monomial occurs in $T(\mathcal{M}; 1, 2, a)$ in the term $y_a^2 B(a)^2$ when $a = 3$ or $a = 4$, and in the term $C(a)^2 D(a)^2$ when $\{a\}$ is one of $\overline{\{1,3\}} \cap \overline{\{2,4\}}$ or $\overline{\{1,4\}} \cap \overline{\{2,3\}}$. (Either of these

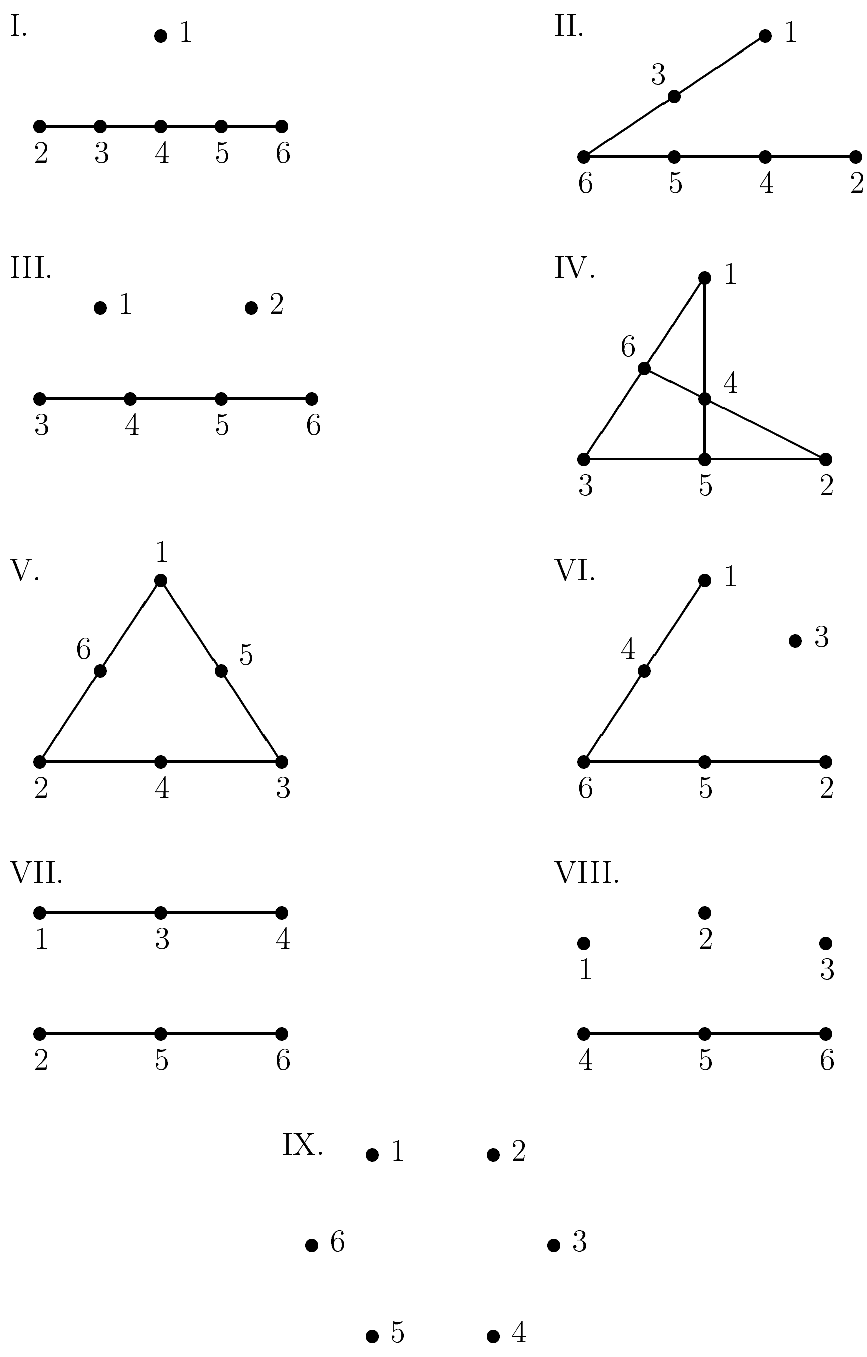

Figure 3: The six–element rank–three simple matroids.

last two sets might be empty instead, however. Since $\mathcal{M}$ is simple, these intersections have at most one element.)

B. The monomial occurs with coefficient 2 in the term $y_3^2 B(3)^2$ of $T(\mathcal{M};1,2,3)$.

C. With $\mathcal{N}$ isomorphic to III in Figure 2 we take $e=1$ and $f=3$ and $g=2$, and consider the coefficient of the monomial $y_2^2 y_4 y_5$ in Δ and in P. Writing ijk for the triple $\{i,j,k\}$, the pairs contributing to the coefficient of this monomial in Δ are $(124,235)$ and $(125,235)$ from $\mathcal{M}_e^f \times M_f^e$, and $(123,245)$ from $\mathcal{M}_{ef} \times M^{ef}$.

The monomial occurs with coefficient 2 in the term $y_2^2 B(2)^2$ of $T(\mathcal{M};1,3,2)$. If $\overline{\{1,2\}} \cap \overline{\{3,4\}} = \{a\}$ then the monomial also occurs with coefficient 2 in the term $C(a)^2 D(a)^2$ of $T(\mathcal{M};1,3,a)$. (The set $\overline{\{1,2\}} \cap \overline{\{3,4\}}$ might be empty instead, however. Since $\mathcal{M}$ is simple, this intersection has at most one element.)

D. With $\mathcal{N}$ isomorphic to IV in Figure 3 we take $e=1$ and $f=2$, and consider the coefficient of the monomial $y_3y_4y_5y_6$ in Δ and in P. The pairs contributing to the coefficient of this monomial in Δ are $(134,256)$ and $(156,234)$ from $\mathcal{M}_e^f \times M_f^e$, and $(123,456)$, $(124,356)$, $(125,346)$, and $(126,345)$ from $\mathcal{M}_{ef} \times M^{ef}$.

This monomial occurs in the term $-2y_aB(a)C(a)D(a)$ of $T(\mathcal{M};1,2,a)$ for each $a \in \{3,4,5,6\}$, for a total contribution of $-2y_3y_4y_5y_6$ to P. (That it occurs nowhere else in P can be verified by considering where in $\mathcal{M}$ the element a must be so that $y_3y_4y_4y_6$ occurs with nonzero coefficient in $T(\mathcal{M};1,2,a)$.)

E. This occurs in the term $-2y_aB(a)C(a)D(a)$ of $T(\mathcal{M};1,4,a)$ for $a=2$ and $a=3$.

F. This occurs in the term $-2y_3B(3)C(3)D(3)$ of $T(\mathcal{M};4,5,3)$.

G. This occurs in the term $-2y_6B(6)C(6)D(6)$ of $T(\mathcal{M};1,2,6)$.

H. With $\mathcal{N}$ isomorphic to VII in Figure 3 we take $e=1$ and $f=2$, and consider the coefficient of the monomial $y_3y_4y_5y_6$ in Δ and in P. The pairs contributing to the coefficient of this monomial in Δ are $(135,246)$, $(136,245)$, $(145,236)$, $(146,235)$, and $(156,234)$ from $\mathcal{M}_e^f \times M_f^e$, and $(123,456)$, $(124,356)$, $(125,346)$, and $(126,345)$ from $\mathcal{M}_{ef} \times M^{ef}$.

If $\overline{\{1,3\}} \cap \overline{\{2,5\}} = \{a\}$ then the monomial occurs with coefficient 4 in the term $C(a)^2D(a)^2$ of $T(\mathcal{M};1,2,a)$. If the above intersection is empty then the monomial does not occur in P. (This claim can be verified by considering where in $\mathcal{M}$ the element a must be so that $y_3y_4y_4y_6$ occurs with nonzero coefficient in $T(\mathcal{M};1,2,a)$.)

These remarks conclude the explanation of the various coefficients of $\Delta M\{e,f\}$ and $P(\mathcal{M};e,f)$, completing the proof that

$$\Delta M\{e,f\} \gg P(\mathcal{M};e,f). \qquad \square$$

Proof of Theorem 1.1. As seen in Section 2, we may assume that $\mathcal{M}$ is simple. Since $P(\mathcal{M};e,f;\mathbf{y})$ is a positive sum of squares it follows that $P(\mathcal{M};e,f;\mathbf{y}) \geq 0$ for all $\mathbf{y} \in \mathbb{R}^{E(\mathcal{M})}$. Since $\Delta M\{e,f\}(\mathbf{y}) \gg P(\mathcal{M};e,f;\mathbf{y})$ by Proposition 4.1 it follows that

$$\Delta M\{e,f\}(\mathbf{y}) \geq P(\mathcal{M};e,f;\mathbf{y}) \geq 0$$

for all $\mathbf{y} > \mathbf{0}$. Therefore, $\mathcal{M}$ is Rayleigh. $\square$

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