Sequentially perfect and uniform one-factorizations of the complete graph

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Abstract

In this paper, we consider a weakening of the definitions of uniform and perfect one-factorizations of the complete graph. Basically, we want to order the $2n-1$ one-factors of a one-factorization of the complete graph K_{2n} in such a way that the union of any two (cyclically) consecutive one-factors is always isomorphic to the same two-regular graph. This property is termed *sequentially uniform*; if this two-regular graph is a Hamiltonian cycle, then the property is termed *sequentially perfect*. We will discuss several methods for constructing sequentially uniform and sequentially perfect one-factorizations. In particular, we prove for any integer $n \geq 1$ that there is a sequentially perfect one-factorization of K_{2n}. As well, for any odd integer $m \geq 1$, we prove that there is a sequentially uniform one-factorization of $K_{2^t m}$ of type $(4, 4, \ldots, 4)$ for all integers $t \geq 2 + \lceil \log_2 m \rceil$ (where type $(4, 4, \ldots, 4)$ denotes a two-regular graph consisting of disjoint cycles of length four).

1 Introduction

A *one-factor* of a graph G is a subset of its edges which partitions the vertex set. A *one-factorization* of a graph G is a partition of its edges into one-factors. Any one-factorization of the complete graph K_{2n} has $2n-1$ one-factors, each of which has n edges. For a survey of one-factorizations of the complete graph, the reader is referred to [10], [14] or [15].

A one-factorization $\{F_0, \ldots, F_{2n-2}\}$ of K_{2n} is *sequentially uniform* if the one-factors can be ordered $(F_0, \ldots, F_{2n-2})$ so that the graphs with edge sets $F_i \cup F_{i+1}$ (subscripts taken modulo $2n-1$) are isomorphic for all $0 \leq i \leq 2n-2$. Since the union of two one-factors is a 2-regular graph which is 2-edge-colorable, it is isomorphic to a disjoint union of even cycles. We say the multiset $T = (k_1, \ldots, k_r)$ is the *type* of a sequentially uniform one-factorization if $F_i \cup F_{i+1}$ is isomorphic to the disjoint union of cycles of lengths $k_1, \ldots, k_r$, where $k_1 + \cdots + k_r = 2n$. When the union of every two consecutive one-factors is a Hamiltonian cycle, the one-factorization is said to be *sequentially perfect*.

The idea to consider orderings of the one-factors in a one-factorization of K_{2n} is not entirely academic. In fact, an ordered one-factorization of K_{2n} is a schedule of play for a round-robin tournament (played in $2n-1$ rounds). Round-robin tournaments possessing certain desired properties have been studied (see [15, Chapter 5], or [7]); however, to our knowledge, round robin tournaments with this "uniform" property have not been considered previously.

The definition above is a relaxation of the definition of uniform (perfect) one-factorization of K_{2n}, which requires that the union of *any* two one-factors be isomorphic (Hamiltonian, respectively). Much work has been done on perfect one-factorizations of K_{2n}; for a survey, see Seah [13]. Perfect one-factorizations of K_{2n} are known to exist whenever n or $2n-1$ is prime, and when $2n = 16$, 28, 36, 40, 50, 126, 170, 244, 344, 730, 1332, 1370, 1850, 2198, 3126, 6860, 12168, 16808, and 29792 (see [1]). Recently a few new perfect one-factorizations have been found, the smallest of which is in K_{530} (see [4, 9, 16]); however before this, no new perfect one-factorization of K_{2n} had been found since 1992 ([17]). The smallest value of $2n$ for which the existence of a perfect one-factorization of K_{2n} is unknown is $2n = 52$. We will show that sequentially perfect one-factorizations are much easier to produce and indeed we will produce a sequentially perfect one-factorization of K_{2n} for all $n \geq 1$.

Various uniform one-factorizations have been constructed from *Steiner triple systems*, [10]. For instance, when $n = 2^m$ for some positive m, the so-called binary projective Steiner triple systems provide a construction of uniform one-factorizations of K_{2n} of type $(4, 4, \ldots, 4)$. There are also sporadic examples of *perfect* Steiner triple systems, [8], which give rise to uniform one-factorizations of type $(2n-4, 4)$. When $v = 3^m$, uniform one-factorizations $(4, 6, \ldots, 6)$ exist (these are Steiner one-factorizations from Hall triple systems) and when p is an odd prime there is a uniform one-factorization of K_{p^s+1} of type $(p+1, 2p, \ldots, 2p)$ which arises from the elementary abelian p-group (see [10]).

The remainder of this paper is organized as follows. In Section 2, we review the classical "starter" construction for one-factorizations and we show that sequentially perfect one-factorizations of K_{2n} exist for all n. In Section 3, we summarize existence results obtained by computer for small orders. In Section 4, we investigate the construction of sequentially uniform one-factorizations from so-called quotient starters in noncyclic abelian groups. Here we obtain interesting number-theoretic conditions that determine if the resulting one-factorizations can be ordered so that they are sequentially uniform. In Section 5, we present a recursive product construction which yields infinite classes of sequentially uniform one-factorizations of $K_{2^t m}$ of type $(4, 4, \ldots, 4)$, for any odd integer m.

2 Starters

We describe our main tool for finding sequentially uniform one-factorizations. Let Γ be an abelian group of order $2n-1$, written additively. A *starter* in Γ is a set of $n-1$ pairs $S = \{\{x_1, y_1\}, \ldots, \{x_{n-1}, y_{n-1}\}\}$ such that every nonzero element of Γ appears as some x_i or y_i, and also as some difference $x_j - y_j$ or $y_j - x_j$. Let $S^* = S \cup \{\{0, \infty\}\}$ and define $x + \infty = \infty + x = \infty$ for all $x \in \Gamma$. Then $\{S^* + x : x \in \Gamma\}$ forms a one-factorization of K_{2n} (with vertex set $\Gamma \cup \{\infty\}$).

Many of the known constructions for (uniform and perfect) one-factorizations use starters in this way. In our first lemma we note the connection between starter-induced one-factorizations and sequentially uniform one-factorizations. Clearly, the order in which the 1-factors are listed is essential to the type of a sequentially uniform one-factorization. Thus we will sometimes refer to *ordered one-factorizations* in this context. Whenever we discuss sequentially uniform one-factorizations, we will always give the 1-factor ordering.

Lemma 2.1. *Let S be a starter in $\mathbb{Z}_{2n-1}$ with $n \geq 1$. Then the ordered one-factorization of K_{2n} generated by S, namely $(S^*, S^*+1, S^*+2, \ldots, S^*+(2n-2))$ is sequentially uniform.*

Proof: For any $x \in \mathbb{Z}_{2n-1}$, we have $(S^* + x) \cup (S^* + (x + 1)) = x + (S^* \cup (S^* + 1))$, so all unions of two consecutive one-factors in the given order are isomorphic. □

Remark: When $\gcd(k, 2n-1) = 1$, the ordering $(S^*, S^* + k, S^* + 2k, \ldots, S^* + (2n-2)k)$ of the same one-factorization is also sequentially uniform. Note, however, that it is not necessarily of the same type as the ordered one-factorization $(S^*, S^* + 1, S^* + 2, \ldots, S^* + (2n-2))$.

The most well-known one-factorization of K_{2n} (called $GK(2n)$) is generated from the *patterned* starter $P = \{\{x, -x\} : x \in \mathbb{Z}_{2n-1}\}$ in the cyclic group $\mathbb{Z}_{2n-1}$. It is known when $2n-1$ is prime that $GK(2n)$ is a perfect one-factorization and, in general, $GK(2n)$ is a uniform one-factorization for all $n \geq 1$. The cycle lengths in $P^* \cup (P^* + k)$ for $k \in \mathbb{Z}_{2n-1} \setminus \{0\}$ are now given.

Lemma 2.2. *Let P be the patterned starter in $\mathbb{Z}_{2n-1}$ with $n \geq 1$. Let $k \in \mathbb{Z}_{2n-1} \setminus \{0\}$ with $\gcd(2n-1, k) = d$. Then $P^* \cup (P^* + k)$ consists of a cycle of length $1 + (2n-1)/d$ and $(d-1)/2$ cycles of length $2(2n-1)/d$.*

Proof: The cycle through infinity is $(\infty, 0, 2k, -2k, 4k, -4k, \ldots, -k, k)$, which has length $1 + (2n-1)/d$. All other cycles (if any) are of the form

$$(i, -i, 2k + i, -2k - i, 4k + i, -4k - i, \ldots, -2k + i, 2k - i),$$

for $1 \leq i < d$. □

Combining Lemmas 2.1 and 2.2 (with $d = 1$) we have the following result.

Theorem 2.3. *For every $n \geq 1$ there exists a sequentially perfect one-factorization of K_{2n}.*

Contrast this with the known results for perfect one-factorizations: the sporadic small values mentioned in the Introduction, and only two infinite classes (each of density zero).

3 Small orders

The one-factorizations of K_4 and K_6 are unique and in each case they are perfect. Hence both are sequentially perfect (the only possible type in these small cases). The one-factorization of K_8 obtained from the unique Steiner triple system of order 7 has type $(4,4)$ while $GK(8)$ is a perfect one-factorization. Hence there exist sequentially uniform one-factorizations of K_8 of all possible types.

We have checked all starters in $\mathbb{Z}_9$ by computer and report that no ordering of the translates of any of these starters yields a sequentially uniform one-factorization of K_{10} of type $(4,6)$. However, there does exist a uniform one-factorization of type $(4,6)$ (it is one-factorization #1 in the list of all 396 non-isomorphic one-factorizations of K_{10} given in [1, p. 655]). Clearly this is also sequentially uniform of type $(4,6)$ under any ordering of the one-factors. From Theorem 2.3 there exists a sequentially perfect one-factorization of K_{10}. Thus sequentially uniform one-factorizations of K_{10} exist for both possible types.

Obviously, the ordering of the one-factors can affect the type of the 2-factors formed from consecutive 1-factors in an ordered one-factorization. Given a starter S in $\mathbb{Z}_{2n-1}$, let $F_S(k)$ denote the ordered one-factorization $(S^*, S^*+k, S^*+2k, \dots, S^*+(2n-2)k)$ of K_{2n}. In the following examples we discuss sequentially uniform one-factorizations in K_{12} and K_{14}. In $\mathbb{Z}_{13}$ we will give one starter which induces all possible types of ordered one-factorizations when different orderings are imposed on translates of that starter.

Example 3.1. *Given the following starter in $\mathbb{Z}_{11}$,*

$$S = \{\{1,2\},\{3,8\},\{4,6\},\{5,9\},\{7,10\}\},$$

$F_S(1)$ is sequentially uniform of type $(6,6)$, $F_S(2)$ is sequentially uniform of type $(4,8)$ and $F_S(3)$ is sequentially uniform of type (12).

By checking all starters in $\mathbb{Z}_{11}$, we found that no ordering of any of the one-factorizations formed by these starters gave a sequentially uniform one-factorization of type $(4,4,4)$. However, Figure 1 provides a non-starter-induced ordered one-factorization which is sequentially uniform of this type.

In [12] it is found that there exist exactly five nonisomorphic perfect one-factorizations of K_{12} and in [2] a uniform one-factorization of type $(6,6)$ is given. From the enumeration in [5], it is known that there exist no other uniform one-factorizations of K_{12}. Hence it is noteworthy that $F_S(2)$ (defined in Example 3.1) gives a sequentially uniform one-factorization of K_{12} of type $(8,4)$ and Figure 1 gives a sequentially uniform one-factorization of type $(4,4,4)$.

Figure 1: A sequentially uniform one-factorization of K_{12} with type $(4,4,4)$

$$
\begin{array}{l}
F_0 : \{\{0,1\},\{2,6\},\{3,4\},\{7,9\},\{8,10\},\{5,11\}\} \\
F_1 : \{\{0,2\},\{1,6\},\{3,9\},\{4,7\},\{5,10\},\{8,11\}\} \\
F_2 : \{\{0,3\},\{1,4\},\{5,8\},\{6,7\},\{2,9\},\{10,11\}\} \\
F_3 : \{\{0,4\},\{1,3\},\{2,8\},\{7,10\},\{6,11\},\{5,9\}\} \\
F_4 : \{\{0,5\},\{1,2\},\{3,8\},\{4,9\},\{6,10\},\{7,11\}\} \\
F_5 : \{\{0,8\},\{1,7\},\{2,11\},\{3,5\},\{4,6\},\{9,10\}\} \\
F_6 : \{\{0,6\},\{1,5\},\{2,10\},\{4,8\},\{3,7\},\{9,11\}\} \\
F_7 : \{\{0,7\},\{1,10\},\{2,5\},\{3,6\},\{4,11\},\{8,9\}\} \\
F_8 : \{\{0,9\},\{1,11\},\{2,3\},\{4,10\},\{5,6\},\{7,8\}\} \\
F_9 : \{\{0,11\},\{1,9\},\{2,4\},\{3,10\},\{6,8\},\{5,7\}\} \\
F_{10} : \{\{0,10\},\{1,8\},\{2,7\},\{3,11\},\{4,5\},\{6,9\}\}
\end{array}
$$

Example 3.2. *The following starter in* $\mathbb{Z}_{13}$*,*

$$S = \{\{1,10\},\{2,3\},\{4,9\},\{5,7\},\{6,12\},\{8,11\}\},$$

yields sequentially uniform one-factorizations of K_{14} *of all possible types: namely* (14)*,* $(10,4)$*,* $(8,6)$*, and* $(6,4,4)$*. Specifically,* $F_S(3)$ *is sequentially uniform of type* $(6,4,4)$*,* $F_S(1)$ *is sequentially uniform of type* $(8,6)$*,* $F_S(2)$ *is sequentially uniform of type* $(10,4)$ *and* $F_S(5)$ *is sequentially uniform of type* (14)*.*

For large n, there are many more possible types than there are translates, so the starter in Example 3.2 is of particular interest. In the Appendix we give examples of sequentially uniform one-factorizations of K_{2n} of all possible types, for $14 \leq 2n \leq 24$.

4 Starters in non-cyclic groups

Many uniform and perfect one-factorizations are known to be starter-induced over a non-cyclic group; for example, see [6]. So it is natural to also expect sequentially uniform one-factorizations where the ordering is not cyclic. In this section we give a numerical condition that determines when certain starter-induced one-factorizations over non-cyclic groups are sequentially uniform.

Let q be an odd prime-power (not a prime) and write $q = 2rt + 1$, where t is odd. In order to eliminate trivial cases, we will assume that $t > 1$. Suppose ω is a generator of the multiplicative group of $\mathbb{F}_q$ and let Q be the subgroup (of order t) generated by ω^{2r}. Suppose the cosets of Q are $C_i = \omega^i Q$, $i = 0, \ldots, 2r-1$. A starter S in $\mathbb{F}_q$ is said to be an r*-quotient starter* if, whenever $\{x,y\},\{x',y'\} \in S$ with $x, x' \in C_i$, it holds that $y/x = y'/x'$. An r-quotient starter S can be completely described by a list of *quotients* $(a_0, \ldots, a_{r-1})$, such that

$$S = \{\{x, a_i x\} : (a_i - 1)x \in C_i, i = 0, \ldots, r-1\}.$$

It is not hard to see that $S^* \cup (S^* + x)$ is isomorphic to $S^* \cup (S^* + y)$ whenever $x/y \in C_0 \cup C_r$. It follows that every 1-quotient starter yields a uniform one-factorization. We now show that, although r-quotient starters might not generate uniform one-factorizations when $r > 1$, [6], the resulting one-factorizations usually can be ordered in such a way that they are sequentially uniform.

Theorem 4.1. *Suppose $q = p^d$ is an odd prime-power (with p prime and $d > 1$) such that $q = 2rt + 1$ and $t > 1$ is odd. Let S be any r-quotient starter in $\mathbb{F}_q$. Then the one-factorization generated by S can be ordered to be sequentially uniform if and only if the multiplicative order of p modulo t is equal to d.*

Proof: Let $q = p^d = 2rt + 1$ with t odd. C_0 is the multiplicative subgroup of $\mathbb{F}_q^*$ generated by a primitive tth root of 1 in $\mathbb{F}_q$, say α. The splitting field of $x^t - 1$ over $\mathbb{F}_p$ is $\mathbb{F}_{p^e}$, where e is the smallest positive integer such that $p^e \equiv 1 \pmod{t}$. Hence the extension field $\mathbb{F}_p(\alpha) = \mathbb{F}_q$ if and only if the multiplicative order of p modulo t, which we denote by $\mathsf{ord}_t(p)$, is equal to d.

Suppose that $\mathsf{ord}_t(p) = d$. Then $\mathbb{F}_p(\alpha) = \mathbb{F}_q$ and $1, \alpha, \ldots, \alpha^{d-1}$ is a basis of $\mathbb{F}_q$ over $\mathbb{F}_p$. Therefore, every element $x \in \mathbb{F}_q$ can be expressed uniquely as a d-tuple $(x_1, \ldots, x_d) \in (\mathbb{Z}_p)^d$, where

$$x = \sum_{i=1}^{n} x_i \alpha^{i-1}.$$

Now, consider the graph on vertex set $(\mathbb{Z}_p)^d$ in which two vertices are adjacent if and only if they agree in $d-1$ coordinates and their values in the remaining coordinate differ by 1 modulo p (this is a *Cayley graph* of the elementary abelian group of order p^d). It is not hard to check that this graph has a hamiltonian cycle, say $C = (y_1, y_2, \ldots, y_{p^d}, y_1)$. The cycle C provides the desired ordering of $\mathbb{F}_q$ because the difference between any two consecutive elements y_i and y_{i+1} is in $C_0 \cup C_r$ (note that one of $y_i - y_{i+1}$ and $y_{i+1} - y_i$ is a power of α and hence in C_0, while the other is in C_r).

Conversely, suppose that $\mathsf{ord}_t(p) = e < d$. Then $\mathbb{F}_p(\alpha) = \mathbb{F}_{p^e}$ which is a strict subfield of $\mathbb{F}_q$. Clearly $C_0 \cup C_r \subseteq \mathbb{F}_{p^e}$. Suppose that $y_1, y_2, \ldots$ is an ordering of the elements of $\mathbb{F}_q$ such that adjacent elements always have a difference that is an element of $C_0 \cup C_r$. Without loss of generality we can take $y_1 = 0$. But then every element y_i is in the subfield $\mathbb{F}_{p^e}$, which is a contradiction. Hence, the desired ordering cannot exist. $\square$

It is interesting to note that the proof above does not depend on the structure of the starter S. Either all r-quotient starters in $\mathbb{F}_q$ yield sequentially uniform one-factorizations or they all do not do so.

Example 4.2. *Let $q = 25$ so that $t = 3$ and $r = 4$. We have $\mathsf{ord}_t(p) = 2 = d$, so Theorem 4.1 asserts that any 4-quotient starter will yield a sequentially uniform one-factorization. In particular, if we take $\mathbb{F}_{25} = \mathbb{Z}_5[x]/(x^2 + x + 2)$ then C_0 contains a basis $\{1, \alpha\}$ for the field, where $\alpha = x^8 = 3x + 1$. The field elements can be cyclically ordered*

$$0, 1, 2, 3, 4, 3x, \ldots, 3x + 4, x, \ldots, x + 4, 4x, \ldots, 4x + 4, 2x, \ldots, 2x + 4, 0$$

so that the difference of consecutive elements is either 1 or α.

Most applications of r-quotient starters use values of r that are powers of two (see, for example, [6]). It is interesting to determine the conditions under which the hypotheses of Theorem 4.1 are satisfied in this case. This is done in Lemma 4.3.

Lemma 4.3. *Suppose $q = p^d$ is an odd prime-power (with p prime and $d > 1$) such that $q = 2^k t + 1$ and $t > 1$ is odd. Then one of the two following conditions hold:*

1. $\mathsf{ord}_t(p) = d$, *or*
2. $p = 2^j - 1$ *for some integer j (i.e., p is a Mersenne prime) and $d = 2$. (In this case, $\mathsf{ord}_t(p) = 1$ is less than d.)*

Proof: Suppose that $p^e \equiv 1 \pmod t$ for some positive integer $e < d$ (note that $e|d$). Let $p^e = bt + 1$ where b is a positive integer. Then

$$2^k t + 1 = q = (p^e)^{d/e} = (bt + 1)^{d/e} = cbt + 1$$

for some integer c. Hence, $b|2^k$, and therefore $b = 2^\ell$ for some positive integer $\ell \le k$. So we have that $p^e = 2^\ell t + 1$.

Let $\rho = p^e$ and $f = d/e$. Then we have that

$$t = \frac{\rho^f - 1}{2^k} = \frac{\rho - 1}{2^\ell}.$$

Removing common factors, we obtain

$$\rho^{f-1} + \rho^{f-2} + \cdots + \rho + 1 = 2^{k-\ell}. \tag{1}$$

Suppose that $k = \ell$. Then the right side of (1) is equal to 1, so $f = 1$ and $d = e$. This contradicts the assumption that $d > e$. Therefore $k > \ell$ and the right side of (1) is even.

Now, suppose that f is odd. Then the left side of (1) is odd, and we have a contradiction. Therefore f is even, and $\rho + 1$ is a factor of the left side of (1). This implies that $2^{k-\ell} \equiv 0 \pmod{\rho + 1}$, and hence $\rho = 2^j - 1$ for some integer $j \ge 2$. Then, after dividing (1) by the factor $\rho + 1$, we obtain the following equation:

$$\rho^{f-2} + \rho^{f-4} + \cdots + \rho^2 + 1 = 2^{k-\ell-j}. \tag{2}$$

Suppose that $j < k - \ell$. Then the right side of (2) is even and $\rho^2 + 1$ is a factor of the left side of (2), so $\rho^2 = 2^i - 1$ for some integer $i \ge 2$. But $\rho = 2^j - 1$ where $j \ge 2$, so $\rho \equiv 3 \pmod 4$. Then $\rho^2 \equiv 1 \pmod 4$, which contradicts the fact that $\rho^2 = 2^i - 1$ where $i \ge 2$. Therefore we have that $j = k - \ell$. This implies that $f = 2$ and so $d = 2e$. So $\rho = 2^j - 1$ for some integer j and $q = \rho^2$.

However, it is easy to prove that the Diophantine equation $2^u - y^v = 1$ has no solution in positive integers with $u, v > 1$[†]. See, for example, Cassels [3, Corollary 2]. Therefore,

[†]This result is a special case of *Catalan's Conjecture*, which states that the Diophantine equation $x^u - y^v = 1$ has no solution in positive integers with $u, v > 1$ except for $3^2 - 2^3 = 1$. Catalan's Conjecture was proven correct in 2002 by Mihăilescu (see Metsänkylä [11] for a recent exposition of the proof).

we can conclude that ρ is prime. Hence, $p = 2^j - 1$ is a Mersenne prime, $e = 1$ and $d = 2$. In this case, we have $q = p^2$. Then we have that

$$q - 1 = (p-1)(p+1) = (p-1)2^j \equiv 0 \pmod{t}.$$

But t is odd, so $p \equiv 1 \pmod{t}$. Therefore $\mathsf{ord}_t(p) = 1$. $\square$

Example 4.4. *Let $q = 961 = 31^2$ so $p = 31$ and $d = 2$. Here $p = 31 = 2^5 - 1$ is a Mersenne prime. We can write $q = 2^5 15 + 1$, so $t = 15$. We see that $\mathsf{ord}_t(p) = 1 < 2$, as asserted by Lemma 4.3.*

The following corollary is an immediate consequence of Theorem 4.1 and Lemma 4.3.

Corollary 4.5. *Suppose $q = p^d$ is an odd prime-power (with p prime and $d > 1$) such that $q = 2^k t + 1$ and $t > 1$ is odd. Let S be any 2^{k-1}-quotient starter in $\mathbb{F}_q$. Then the one-factorization generated by S can be ordered to be sequentially uniform if and only if it is not the case that p is a Mersenne prime and $d = 2$.*

5 Product construction

We now recall the usual product construction for one-factorizations, and apply it to determine another infinite class of sequentially uniform one-factorizations.

Suppose that F is a one-factor on X and G is a one-factor on Y, where $|X| = 2n$ and $|Y| = 2m$. Define various one-factors of $X \times Y$ by

$$\begin{aligned} F^* &= \big\{\{(x_i, y), (x_i', y)\} : \{x_i, x_i'\} \in F, y \in Y\big\}, \\ G^* &= \big\{\{(x, y_j), (x, y_j')\} : x \in X, \{y_j, y_j'\} \in G\big\}, \\ FG &= \big\{\{(x_i, y_j), (x_i', y_j')\} : \{x_i, x_i'\} \in F, \{y_j, y_j'\} \in G\big\}. \end{aligned}$$

Given one-factorizations $\mathcal{F} = \{F_0, \ldots, F_{2n-2}\}$ and $\mathcal{G} = \{G_0, \ldots, G_{2m-2}\}$ of K_{2n} and K_{2m} on the points X and Y, respectively, it is easy to see that

$$\begin{aligned} \mathcal{F}\mathcal{G} = {} & \big\{F_i G_j : i = 0, \ldots, 2n-2 \text{ and } j = 0, \ldots, 2m-2\big\} \\ & \bigcup \big\{F_i^* : i = 0, \ldots, 2n-2\big\} \bigcup \big\{G_j^* : j = 0, \ldots, 2m-2\big\} \end{aligned}$$

is a one-factorization of $X \times Y$.

The following are easy lemmas about the cycle types of pairs of one-factors in $\mathcal{F}\mathcal{G}$.

Lemma 5.1. *For any $i \in \{0, \ldots, 2n-2\}$ and $j \in \{0, \ldots, 2m-2\}$, the following all have cycle type $(4, 4, \ldots, 4)$:*

(i) $F_i^* \cup G_j^*$,

(ii) $F_i G_j \cup F_i^*$, *and*

(iii) $F_i G_j \cup G_j^*$.

Lemma 5.2. *If* $(F_0, F_1, \ldots, F_{2n-2})$ *is sequentially uniform of type* $(4, 4, \ldots, 4)$*, then the following all have cycle type* $(4, 4, \ldots, 4)$*:*

(i) $F_i^* \cup F_{i+1}^*$*, and*

(ii) $F_i G_j \cup F_{i+1} G_j$*,*

for any i, j*, where the subscripts* $i + 1$ *are reduced modulo* $2n - 1$*.*

We can use the above results to give a product construction for sequentially uniform one-factorizations of type $(4, 4, \ldots, 4)$.

Theorem 5.3. *Suppose there exists a sequentially uniform one-factorization of* K_{2n} *of type* $(4, 4, \ldots, 4)$*. Let* $m \leq n$*. Then there is a sequentially uniform one-factorization of* K_{4mn} *of type* $(4, 4, \ldots, 4)$*.*

Proof: We use all the notation above, with $(F_0, F_1, \ldots, F_{2n-2})$ sequentially uniform of type $(4, 4, \ldots, 4)$ and $\mathcal{G}$ any one-factorization of K_{2m}. The ordered one-factorization

$$\begin{gathered}
\big(G_0^*, F_0G_0, F_1G_0, \ldots, F_{2n-2}G_0, F_{2n-2}^*, \\
G_1^*, F_1G_1, F_2G_1, \ldots, F_0G_1, F_0^*, \\
G_2^*, F_2G_2, F_3G_2, \ldots, F_1G_2, F_1^*, \\
\vdots \\
G_{2m-2}^*, F_{2m-2}G_{2m-2}, \ldots, F_{2m-3}G_{2m-2}, F_{2m-3}^*, \\
F_{2m-2}^*, F_{2m-1}^*, \ldots, F_{2n-3}^*\big)
\end{gathered}$$

of K_{4mn} is sequentially uniform of type $(4, 4, \ldots, 4)$ by Lemmas 5.1 and 5.2. □

By applying the above product construction with $2n$ a power of 2 — for which the existence of *uniform* one-factorizations of type $(4, 4, \ldots, 4)$ are known — one immediately has the following corollary.

Corollary 5.4. *For any odd integer* $m \geq 1$*, there is a sequentially uniform one-factorization of* $K_{2^t m}$ *of type* $(4, 4, \ldots, 4)$ *for all integers* $t \geq 2 + \lceil \log_2 m \rceil$*.*

Let $t_0 = t_0(m)$ denote the smallest integer such that there is a sequentially uniform one-factorization of $K_{2^t m}$ of type $(4, 4, \ldots, 4)$ for all integers $t \geq t_0$. Corollary 5.4 provides an explicit upper bound on $t_0(m)$; however, for a particular value of m, we might be able to give a better bound on $t_0(m)$. For example, the sequentially perfect one-factorization of K_4 shows that $t_0(1) = 2$, the sequentially uniform one-factorization of K_{12} of type $(4, 4, 4)$ given in Figure 1 yields $t_0(3) = 2$, and the sequentially uniform one-factorization of K_{20} of type $(4, 4, 4, 4, 4)$ exhibited in the Appendix gives $t_0(5) = 2$. In fact, we conjecture that $t_0(m) = 2$ for all odd integers $m \geq 1$.

As a final note, we observe that the existence results for sequentially uniform one-factorizations of $K_{2^t m}$ of type $(4, 4, \ldots, 4)$ provide an interesting contrast to those for uniform one-factorizations of $K_{2^t m}$ of type $(4, 4, \ldots, 4)$, which exist only when $m = 1$ (see Cameron [2, Proposition 4.3]).

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Appendix

Below is a table giving all possible types for sequentially uniform one-factorizations of K_{2n}, $14 \le 2n \le 24$. Each type is realized by the ordered 1-factorization $F_S(1)$ corresponding to the starter $S = \{\{x_i, x_i + i\} : i = 1, \dots, n-1\}$ in $\mathbb{Z}_{2n-1}$.

type	$(x_1, \dots, x_{n-1})$
(14)	$(1,5,8,6,12,3)$
$(10,4)$	$(1,9,4,6,3,12)$
$(8,6)$	$(1,5,9,6,3,11)$
$(6,4,4)$	$(7,2,9,1,6,10)$
(16)	$(1,4,8,9,7,14,3)$
$(12,4)$	$(1,4,8,9,5,12,7)$
$(10,6)$	$(1,7,11,4,5,12,6)$
$(8,8)$	$(1,3,7,9,6,8,12)$
$(8,4,4)$	$(2,11,9,4,5,1,14)$
$(6,6,4)$	$(3,11,2,6,7,8,9)$
$(4,4,4,4)$	$(3,6,11,12,5,7,2)$
(18)	$(1,3,7,12,8,9,4,6)$
$(14,4)$	$(1,3,11,8,4,10,6,7)$
$(12,6)$	$(1,5,8,12,9,4,13,15)$
$(10,8)$	$(1,3,6,11,8,10,7,4)$
$(10,4,4)$	$(1,13,5,7,9,4,16,12)$
$(8,6,4)$	$(1,4,12,5,8,10,7,3)$
$(6,6,6)$	$(1,6,10,5,11,15,7,12)$
$(6,4,4,4)$	$(3,7,12,2,8,10,11,14)$
(20)	$(1,3,6,11,13,8,10,4,7)$
$(16,4)$	$(1,3,9,13,10,8,4,18,16)$
$(14,6)$	$(1,3,9,7,13,8,10,15,16)$
$(12,8)$	$(1,3,9,13,6,10,8,18,14)$
$(12,4,4)$	$(1,5,12,6,9,17,11,8,13)$
$(10,10)$	$(1,3,12,9,11,4,7,17,18)$
$(10,6,4)$	$(1,4,10,11,7,18,9,14,8)$
$(8,8,4)$	$(1,3,12,10,6,7,16,9,18)$
$(8,6,6)$	$(1,4,5,13,9,10,11,7,3)$
$(8,4,4,4)$	$(2,5,11,4,10,12,13,9,16)$
$(6,6,4,4)$	$(2,14,8,13,7,4,18,1,15)$
$(4,4,4,4,4)$	$(6,16,17,11,4,8,3,5,12)$

type	$(x_1, \dots, x_{n-1})$
(22)	$(1,3,4,10,11,13,8,12,9,17)$
$(18,4)$	$(1,3,4,13,11,12,8,6,10,20)$
$(16,6)$	$(1,3,4,10,11,12,13,9,6,19)$
$(14,8)$	$(1,3,4,12,9,11,13,10,6,19)$
$(14,4,4)$	$(1,4,7,11,12,14,19,8,9,3)$
$(12,10)$	$(1,3,6,11,13,10,12,20,8,4)$
$(12,6,4)$	$(1,3,7,15,11,8,13,4,9,17)$
$(10,8,4)$	$(1,3,7,11,14,6,13,9,16,8)$
$(10,6,6)$	$(1,3,7,16,12,8,6,11,9,15)$
$(10,4,4,4)$	$(1,5,11,13,19,6,8,10,16,20)$
$(8,8,6)$	$(1,3,12,6,13,14,4,9,7,19)$
$(8,6,4,4)$	$(1,3,10,16,12,9,4,6,19,8)$
$(6,6,6,4)$	$(1,8,6,11,12,19,13,18,7,14)$
$(6,4,4,4,4)$	$(1,11,5,16,9,6,17,10,19,15)$
(24)	$(1,3,4,9,11,15,12,14,8,10,18)$
$(20,4)$	$(1,3,4,13,10,16,12,6,11,8,21)$
$(18,6)$	$(1,3,4,10,16,13,8,9,11,12,18)$
$(16,8)$	$(1,3,4,10,12,15,11,8,13,19,9)$
$(16,4,4)$	$(1,3,6,10,16,11,15,12,4,8,19)$
$(14,10)$	$(1,3,4,13,16,8,11,12,6,9,22)$
$(14,6,4)$	$(1,3,6,10,12,16,13,11,21,8,4)$
$(12,12)$	$(1,3,4,10,12,16,13,11,6,8,21)$
$(12,8,4)$	$(1,3,4,13,10,12,9,14,20,11,8)$
$(12,6,6)$	$(1,3,4,15,9,10,11,12,13,21,6)$
$(12,4,4,4)$	$(1,3,13,15,4,6,14,10,8,20,11)$
$(10,10,4)$	$(1,3,6,15,16,8,11,12,4,7,22)$
$(10,8,6)$	$(1,3,4,11,9,16,12,13,8,10,18)$
$(10,6,4,4)$	$(1,3,4,12,15,13,10,6,9,21,11)$
$(8,8,8)$	$(1,3,4,11,16,8,10,12,13,9,18)$
$(8,8,4,4)$	$(1,3,19,14,10,7,4,12,8,6,21)$
$(8,6,6,4)$	$(1,3,9,13,22,15,7,10,11,6,8)$
$(8,4,4,4,4)$	$(1,4,16,10,13,9,5,22,17,11,20)$
$(6,6,6,6)$	$(1,5,6,12,15,21,10,11,13,8,3)$
$(6,6,4,4,4)$	$(1,6,14,7,13,15,3,12,19,22,16)$
$(4,4,4,4,4,4)$	$(5,17,9,11,13,21,1,22,16,10,3)$