A Simple Proof of the Aztec Diamond Theorem

Sen-Peng Eu*

Department of Applied Mathematics
National University of Kaohsiung
Kaohsiung 811, Taiwan, ROC
speu@nuk.edu.tw

Tung-Shan Fu†

Mathematics Faculty
National Pingtung Institute of Commerce
Pingtung 900, Taiwan, ROC
tsfu@npic.edu.tw

Submitted: Apr 5, 2004; Accepted: Apr 8, 2005; Published: Apr 20, 2005
Mathematics Subject Classifications: 05A15; 05B45; 05C50; 05C20

Abstract

Based on a bijection between domino tilings of an Aztec diamond and non-intersecting lattice paths, a simple proof of the Aztec diamond theorem is given by means of Hankel determinants of the large and small Schröder numbers.

Keywords: Aztec diamond, domino tilings, Hankel matrices, Schröder numbers, lattice paths

1 Introduction

The *Aztec diamond* of order n, denoted by AD_n, is defined as the union of all the unit squares with integral corners (x, y) satisfying $|x| + |y| \leq n + 1$. A *domino* is simply a 1-by-2 or 2-by-1 rectangle with integral corners. A *domino tiling* of a region R is a set of non-overlapping dominoes the union of which is R. Figure 1 shows the Aztec diamond of order 3 and a domino tiling. The Aztec diamond theorem, first proved by Elkies *et al.* in [4], states that the number a_n of domino tilings of the Aztec diamond of order n is $2^{n(n+1)/2}$. They give four proofs, relating the tilings in turn to alternating sign matrices,

*Partially supported by National Science Council, Taiwan, ROC (NSC 93-2115-M-390-005).
†Partially supported by National Science Council, Taiwan, ROC (NSC 93-2115-M-251-001).

monotone triangles, representations of general linear groups, and domino shuffling. Other approaches to this theorem appear in [2, 3, 6]. Ciucu [3] derives the recurrence relation $a_n = 2^n a_{n-1}$ by means of perfect matchings of cellular graphs. Kuo [6] develops a method, called graphical condensation, to derive the recurrence relation $a_n a_{n-2} = 2a_{n-1}^2$, for $n \geq 3$. Recently, Brualdi and Kirkland [2] give a proof by considering a matrix of order $n(n+1)$ the determinant of which gives a_n. Their proof is reduced to the computation of the determinant of a Hankel matrix of order n that involves large Schröder numbers. In this note we give a proof by means of Hankel determinants of the large and small Schröder numbers based on a bijection between the domino tilings of an Aztec diamond and non-intersecting lattice paths.

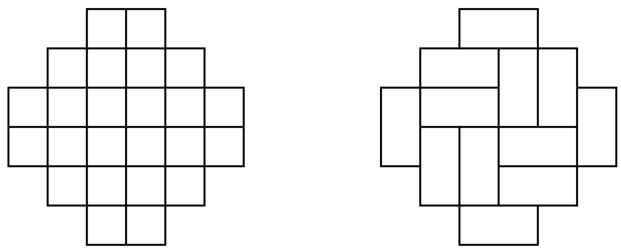

Figure 1: The AD_3 and a domino tiling

The *large Schröder numbers* $\{r_n\}_{n\geq 0} := \{1, 2, 6, 22, 90, 394, 1806, \ldots\}$ and the *small Schröder numbers* $\{s_n\}_{n\geq 0} := \{1, 1, 3, 11, 45, 197, 903, \ldots\}$ are registered in Sloane's On-Line Encyclopedia of Integer Sequences [7], namely A006318 and A001003, respectively. Among many other combinatorial structures, the nth large Schröder number r_n counts the number of lattice paths in the plane $\mathbb{Z} \times \mathbb{Z}$ from $(0, 0)$ to $(2n, 0)$ using *up* steps $(1, 1)$, *down* steps $(1, -1)$, and *level* steps $(2, 0)$ that never pass below the x-axis. Such a path is called a *large Schröder path* of length n (or a *large n-Schröder path* for short). Let U, D, and L denote an up, down, and level step, respectively. Note that the terms of $\{r_n\}_{n\geq 1}$ are twice of those in $\{s_n\}_{n\geq 1}$. It turns out that the nth small Schröder number s_n counts the number of large n-Schröder paths without level steps on the x-axis, for $n \geq 1$. Such a path is called a *small n-Schröder path*. Refer to [8, Exercise 6.39] for more information.

Our proof relies on the determinants of the following *Hankel matrices* of the large and small Schröder numbers

$$H_n^{(1)} := \begin{bmatrix} r_1 & r_2 & \cdots & r_n \\ r_2 & r_3 & \cdots & r_{n+1} \\ \vdots & \vdots & & \vdots \\ r_n & r_{n+1} & \cdots & r_{2n-1} \end{bmatrix}, \quad G_n^{(1)} := \begin{bmatrix} s_1 & s_2 & \cdots & s_n \\ s_2 & s_3 & \cdots & s_{n+1} \\ \vdots & \vdots & & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n-1} \end{bmatrix}.$$

Making use of a method of Gessel and Viennot [5], we associate the determinants of $H_n^{(1)}$ and $G_n^{(1)}$ with the numbers of n-tuples of non-intersecting large and small Schröder paths, respectively. Note that $H_n^{(1)} = 2G_n^{(1)}$. This relation bridges the recurrence relation (2) that leads to the result $\det(H_n^{(1)}) = 2^{n(n+1)/2}$ as well as the number of the required n-tuples of non-intersecting large Schröder paths (see Proposition 2.1). Our proof of the Aztec diamond theorem is completed by a bijection between domino tilings of an Aztec diamond and non-intersecting large Schröder paths (see Proposition 2.2).

We remark that Brualdi and Kirkland [2] use an algebraic method, relying on a J-fraction expansion of generating functions, to evaluate the determinant of a Hankel matrix of large Schröder numbers. Here we use a combinatorial approach that simplifies the evaluation of the Hankel determinants of large and small Schröder numbers significantly.

2 A proof of the Aztec diamond theorem

Let Π_n (resp. Ω_n) denote the set of n-tuples $(\pi_1, \ldots, \pi_n)$ of large Schröder paths (resp. small Schröder paths) satisfying the following two conditions.

(A1) Each path π_i goes from $(-2i+1, 0)$ to $(2i-1, 0)$, for $1 \leq i \leq n$.

(A2) Any two paths π_i and π_j do not intersect.

There is an immediate bijection ϕ between Π_{n-1} and Ω_n, for $n \geq 2$, which carries $(\pi_1, \ldots, \pi_{n-1}) \in \Pi_{n-1}$ into $\phi((\pi_1, \ldots, \pi_{n-1})) = (\omega_1, \ldots, \omega_n) \in \Omega_n$, where $\omega_1 = \mathsf{UD}$ and $\omega_i = \mathsf{UU}\pi_{i-1}\mathsf{DD}$ (i.e., ω_i is obtained from π_{i-1} with 2 up steps attached in the beginning and 2 down steps attached in the end, and then rises above the x-axis), for $2 \leq i \leq n$. For example, on the left of Figure 2 is a triple $(\pi_1, \pi_2, \pi_3) \in \Pi_3$. The corresponding quadruple $(\omega_1, \omega_2, \omega_3, \omega_4) \in \Omega_4$ is shown on the right. Hence, for $n \geq 2$, we have

$$|\Pi_{n-1}| = |\Omega_n|. \tag{1}$$

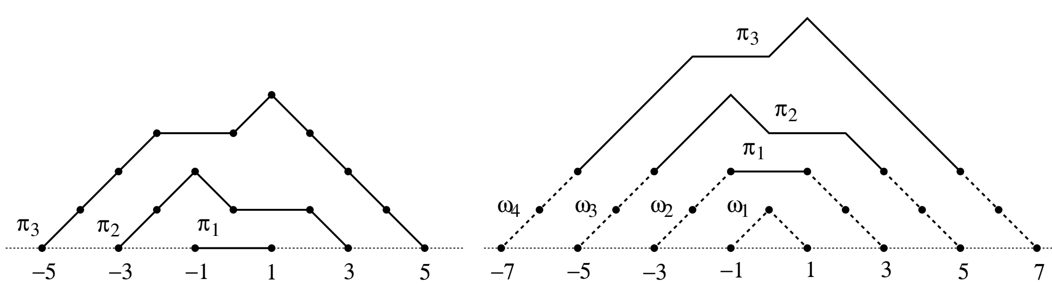

Figure 2: A triple $(\pi_1, \pi_2, \pi_3) \in \Pi_3$ and the corresponding quadruple $(\omega_1, \omega_2, \omega_3, \omega_4) \in \Omega_4$

For a permutation $\sigma = z_1 z_2 \cdots z_n$ of $\{1, \ldots, n\}$, the *sign* of σ, denoted by $\mathrm{sgn}(\sigma)$, is defined by $\mathrm{sgn}(\sigma) := (-1)^{\mathrm{inv}(\sigma)}$, where $\mathrm{inv}(\sigma) := \mathrm{Card}\{(z_i, z_j) \mid i < j \text{ and } z_i > z_j\}$ is the number of *inversions* of σ.

Using the technique of a sign-reversing involution over a signed set, we prove that the cardinalities of Π_n and Ω_n coincide with the determinants of $H_n^{(1)}$ and $G_n^{(1)}$, respectively. Following the same steps as [9, Theorem 5.1], a proof is given here for completeness.

Proposition 2.1 *For $n \geq 1$, we have*

(i) $|\Pi_n| = \det(H_n^{(1)}) = 2^{n(n+1)/2}$, *and*

(ii) $|\Omega_n| = \det(G_n^{(1)}) = 2^{n(n-1)/2}$.

Proof: For $1 \leq i \leq n$, let A_i denote the point $(-2i+1, 0)$ and let B_i denote the point $(2i-1, 0)$. Let h_{ij} denote the (i,j)-entry of $H_n^{(1)}$. Note that $h_{ij} = r_{i+j-1}$ is equal to the number of large Schröder paths from A_i to B_j. Let P be the set of ordered pairs $(\sigma, (\tau_1, \ldots, \tau_n))$, where σ is a permutation of $\{1, \ldots, n\}$, and $(\tau_1, \ldots, \tau_n)$ is an n-tuple of large Schröder paths such that τ_i goes from A_i to $B_{\sigma(i)}$. According to the sign of σ, the ordered pairs in P are partitioned into P^+ and P^-. Then

$$\det(H_n^{(1)}) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} h_{i,\sigma(i)} = |P^+| - |P^-|.$$

We show that there exists a sign-reversing involution φ on P, in which case $\det(H_n^{(1)})$ is equal to the number of fixed points of φ. Let $(\sigma, (\tau_1, \ldots, \tau_n)) \in P$ be such a pair that at least two paths of $(\tau_1, \ldots, \tau_n)$ intersect. Choose the first pair $i < j$ in lexicographical order such that τ_i intersects τ_j. Construct new paths τ_i' and τ_j' by switching the tails after the last point of intersection of τ_i and τ_j. Now τ_i' goes from A_i to $B_{\sigma(j)}$ and τ_j' goes from A_j to $B_{\sigma(i)}$. Since $\sigma \circ (ij)$ carries i into $\sigma(j)$, j into $\sigma(i)$, and k into $\sigma(k)$, for $k \neq i, j$, we define

$$\varphi((\sigma, (\tau_1, \ldots, \tau_n))) = (\sigma \circ (ij), (\tau_1, \ldots, \tau_i', \ldots, \tau_j', \ldots, \tau_n)).$$

Clearly, φ is sign-reversing. Since this first intersecting pair $i < j$ of paths is not affected by φ, φ is an involution. The fixed points of φ are the pairs $(\sigma, (\tau_1, \ldots, \tau_n)) \in P$, where $\tau_1, \ldots, \tau_n$ do not intersect. It follows that τ_i goes from A_i to B_i, for $1 \leq i \leq n$ (i.e., σ is the identity) and $(\tau_1, \ldots, \tau_n) \in \Pi_n$. Hence $\det(H_n^{(1)}) = |\Pi_n|$. By the same argument, we have $\det(G_n^{(1)}) = |\Omega_n|$. It follows from (1) and the relation $H_n^{(1)} = 2G_n^{(1)}$ that

$$|\Pi_n| = \det(H_n^{(1)}) = 2^n \cdot \det(G_n^{(1)}) = 2^n |\Omega_n| = 2^n |\Pi_{n-1}|. \tag{2}$$

Note that $|\Pi_1| = 2$, and hence, by induction, assertions (i) and (ii) follow. □

To prove the Aztec diamond theorem, we shall establish a bijection between Π_n and the set of domino tilings of AD_n based on an idea, due to D. Randall, mentioned in [8, Solution of Exercise 6.49].

Proposition 2.2 *There is a bijection between the set of domino tilings of the Aztec diamond of order n and the set of n-tuples* $(\pi_1, \ldots, \pi_n)$ *of large Schröder paths satisfying conditions (A1) and (A2).*

Proof: Given a tiling T of AD_n, we associate T with an n-tuple $(\tau_1, \ldots, \tau_n)$ of non-intersecting paths as follows. Let the rows of AD_n be indexed by $1, 2, \ldots, 2n$ from bottom to top. For each i, $(1 \le i \le n)$ we define a path τ_i from the center of the left-hand edge of the ith row to the center of the right-hand edge of the ith row. Namely, each step of the path is from the center of a domino edge (where a domino is regarded as having six edges of unit length) to the center of another edge of the some domino D, such that the step is symmetric with respect to the center of D. One can check that for each tiling there is a unique such an n-tuple $(\tau_1, \ldots, \tau_n)$ of paths, moreover, any two paths τ_i, τ_j of which do not intersect. Conversely, any such n-tuple of paths corresponds to a unique domino tiling of AD_n.

Let Λ_n denote the set of such n-tuples $(\tau_1, \ldots, \tau_n)$ of non-intersecting paths associated with domino tilings of AD_n. We shall establish a bijection ψ between the set of domino tilings of AD_n to Π_n with Λ_n as the intermediate stage. Given a tiling T of AD_n, let $(\tau_1, \ldots, \tau_n) \in \Lambda_n$ be the n-tuple of paths associated with T. The mapping ψ is defined by carrying T into $\psi(T) = (\pi_1, \ldots, \pi_n)$, where $\pi_i = \mathsf{U}_1 \cdots \mathsf{U}_{i-1}\tau_i\mathsf{D}_{i-1} \cdots \mathsf{D}_1$ (i.e., the large Schröder path π_i is obtained from τ_i with $i-1$ up steps attached in the beginning of τ_i and with $i-1$ down steps attached in the end, and then rises above the x-axis), for $1 \le i \le n$. One can verify that $\pi_1, \ldots, \pi_n$ satisfy conditions (A1) and (A2), and hence $\psi(T) \in \Pi_n$.

To find ψ^{-1}, given $(\pi_1, \ldots, \pi_n) \in \Pi_n$, we can recover an n-tuple $(\tau_1, \ldots, \tau_n) \in \Lambda_n$ of non-intersecting paths from $(\pi_1, \ldots, \pi_n)$ by a reverse procedure. Then we retrieve the required domino tiling $\psi^{-1}((\pi_1, \ldots, \pi_n))$ of AD_n from $(\tau_1, \ldots, \tau_n)$. $\square$

For example, on the left of Figure 3 is a tiling T of AD_3 and the associated triple (τ_1, τ_2, τ_3) of non-intersecting paths. On the right of Figure 3 is the corresponding triple $\psi(T) = (\pi_1, \pi_2, \pi_3) \in \Pi_3$ of large Schröder paths.

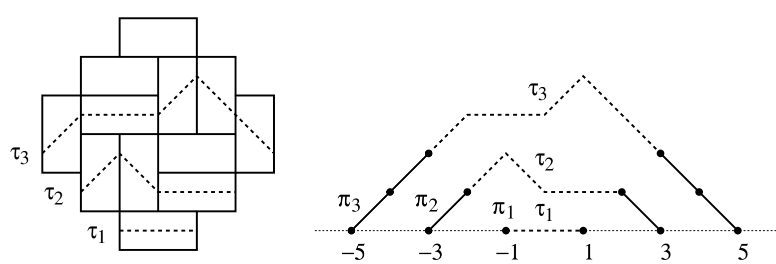

Figure 3: A tiling of AD_3 and the corresponding triple of non-intersecting Schröder paths

By Propositions 2.1 and 2.2, we deduce the Aztec diamond theorem anew.

Theorem 2.3 (Aztec diamond theorem) *The number of domino tilings of the Aztec diamond of order n is $2^{n(n+1)/2}$.*

Remark: The proof of Proposition 2.1 relies on the recurrence relation $\Pi_n = 2^n \Pi_{n-1}$ essentially, which is derived by means of the determinants of the Hankel matrices $H_n^{(1)}$ and $G_n^{(1)}$. We are interested to hear a purely combinatorial proof of this recurrence relation.

In a similar manner we derive the determinants of the Hankel matrices of large and small Schröder paths of the forms

$$H_n^{(0)} := \begin{bmatrix} r_0 & r_1 & \cdots & r_{n-1} \\ r_1 & r_2 & \cdots & r_n \\ \vdots & \vdots & & \vdots \\ r_{n-1} & r_n & \cdots & r_{2n-2} \end{bmatrix}, \quad G_n^{(0)} := \begin{bmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & s_2 & \cdots & s_n \\ \vdots & \vdots & & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-2} \end{bmatrix}.$$

Let Π_n^* (resp. Ω_n^*) be the set of n-tuples $(\mu_0, \mu_1, \ldots, \mu_{n-1})$ of large Schröder paths (resp. small Schröder paths) satisfying the following two conditions.

(B1) Each path μ_i goes from $(-2i, 0)$ to $(2i, 0)$, for $0 \le i \le n-1$.

(B2) Any two paths μ_i and μ_j do not intersect.

Note that μ_0 degenerates into a single point and that Π_n^* and Ω_n^* are identical since for any $(\mu_0, \mu_1, \ldots, \mu_{n-1}) \in \Pi_n^*$ all of the paths μ_i have no level steps on the x-axis. Moreover, for $n \ge 2$, there is a bijection ρ between Π_{n-1} and Π_n^* that carries $(\pi_1, \ldots, \pi_{n-1}) \in \Pi_{n-1}$ into $\rho((\pi_1, \ldots, \pi_{n-1})) = (\mu_0, \mu_1, \ldots, \mu_{n-1}) \in \Pi_n^*$, where μ_0 is the origin and $\mu_i = \mathsf{U}\pi_i\mathsf{D}$, for $1 \le i \le n-1$. Hence, for $n \ge 2$, we have

$$|\Pi_n^*| = |\Pi_{n-1}|. \tag{3}$$

For example, on the left of Figure 4 is a triple $(\pi_1, \pi_2, \pi_3) \in \Pi_3$ of non-intersecting large Schröder paths. The corresponding quadruple $(\mu_0, \mu_1, \mu_2, \mu_3) \in \Pi_4^*$ is shown on the right.

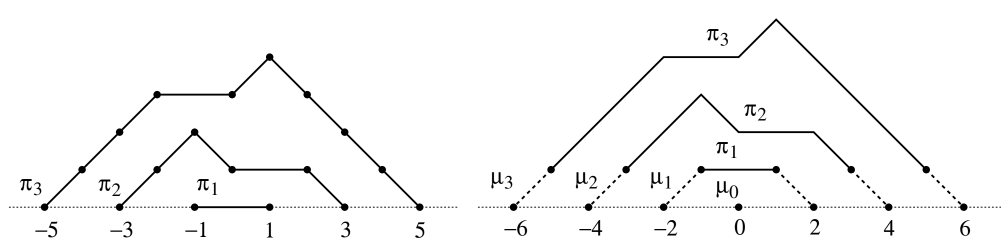

Figure 4: A triple $(\pi_1, \pi_2, \pi_3) \in \Pi_3$ and the corresponding quadruple $(\mu_0, \mu_1, \mu_2, \mu_3) \in \Pi_4^*$

By a similar argument to that of Proposition 2.1, we have $\det(H_n^{(0)}) = |\Pi_n^*| = |\Omega_n^*| = \det(G_n^{(0)})$. Hence, by (3) and Proposition 2.1(i), we have the following result.

Proposition 2.4 *For $n \geq 1$, $\det(H_n^{(0)}) = \det(G_n^{(0)}) = 2^{n(n-1)/2}$.*

Hankel matrices $H_n^{(0)}$ and $H_n^{(1)}$ may be associated with any given sequence of real numbers. As noted by Aigner in [1, Section 1(D)] that the sequence of determinants

$$\det(H_1^{(0)}), \det(H_1^{(1)}), \det(H_2^{(0)}), \det(H_2^{(1)}), \ldots$$

uniquely determines the original number sequence provided that $\det(H_n^{(0)}) \neq 0$ and $\det(H_n^{(1)}) \neq 0$, for all $n \geq 1$, we have a characterization of large and small Schröder numbers.

Corollary 2.5 *The following results hold.*

(i) *The large Schröder numbers $\{r_n\}_{n\geq 0}$ are the unique sequence with the Hankel determinants $\det(H_n^{(0)}) = 2^{n(n-1)/2}$ and $\det(H_n^{(1)}) = 2^{n(n+1)/2}$, for all $n \geq 1$.*

(ii) *The small Schröder numbers $\{s_n\}_{n\geq 0}$ are the unique sequence with the Hankel determinants $\det(G_n^{(0)}) = \det(G_n^{(1)}) = 2^{n(n-1)/2}$, for all $n \geq 1$.*

Acknowledgements

The authors would like to thank an anonymous referee for many helpful suggestions that improve the presentation of this article.

References

[1] M. Aigner, Catalan and other numbers: a recurrent theme, pp. 347-390 in: *Algebraic Combinatorics and Computer Science, a tribute to Gian-Carlo Rota* (H. Crapo and D. Senato, eds.), Springer, Milano, 2001.

[2] R. A. Brualdi and S. Kirkland, Aztec diamonds and diagraphs, and Hankel determinants of Schröder numbers, *J. Combin. Theory Ser. B*, to appear.

[3] M. Ciucu, Perfect matchings of cellular graphs, *J. Algebraic Combinatorics* **5** (1996), 87–103.

[4] N. Elkies, G. Kuperberg, M. Larsen, and J. Propp, Alternating-sign matrices and domino tilings (Part I), *J. Algebraic Combinatorics* **1** (1992), 111–132.

[5] I. Gessel and G. Viennot, Binomial determinants, paths, and hook length formulae, *Advances in Math.* **58** (1985), 300–321.

[6] E. H. Kuo, Applications of graphical condensation for enumerating matchings and tilings, *Theoret. Comput. Sci.* **319** (2004), 29–57.

[7] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at http://www.research.att.com/∼njas/sequences/.

[8] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge University Press, Cambridge, 1999.

[9] D. Stanton and D. White, *Constructive Combinatorics*, Springer-Verlag, Berlin/Heidelberg/New York, 1986.