Lower Bound for the Size of Maximal Nontraceable Graphs *

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Abstract

Let $g(n)$ denote the minimum number of edges of a maximal nontraceable graph of order n. Dudek, Katona and Wojda (2003) showed that $g(n) \geq \lceil \frac{3n-2}{2} \rceil - 2$ for $n \geq 20$ and $g(n) \leq \lceil \frac{3n-2}{2} \rceil$ for $n \geq 54$ as well as for $n \in I = \{22, 23, 30, 31, 38, 39, 40, 41, 42, 43, 46, 47, 48, 49, 50, 51\}$. We show that $g(n) = \lceil \frac{3n-2}{2} \rceil$ for $n \geq 54$ as well as for $n \in I \cup \{12, 13\}$ and we determine $g(n)$ for $n \leq 9$.
Keywords: maximal nontraceable, hamiltonian path, traceable, nontraceable, nonhamiltonian

1 Introduction

We consider only simple, finite graphs G and denote the vertex set, the edge set, the order and the size of G by $V(G)$, $E(G)$, $v(G)$ and $e(G)$, respectively. The *open neighbourhood* of a vertex v in G is the set $N_G(v) = \{x \in V(G) : vx \in E(G)\}$. If U is a nonempty subset of $V(G)$ then $\langle U \rangle$ denotes the subgraph of G induced by U.

A graph G is *hamiltonian* if it has a *hamiltonian cycle* (a cycle containing all the vertices of G), and *traceable* if it has a *hamiltonian path* (a path containing all the vertices of G). A graph G is *maximal nonhamiltonian* (MNH) if G is not hamiltonian, but $G + e$ is hamiltonian for each $e \in E(\overline{G})$, where $\overline{G}$ denotes the complement of G. A graph G is *maximal nontraceable* (MNT) if G is not traceable, but $G + e$ is traceable for each $e \in E(\overline{G})$.

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In 1978 Bollobás [1] posed the problem of finding the least number of edges, $f(n)$, in a MNH graph of order n. Bondy [2] had already shown that a MNH graph with order $n \geq 7$ that contained m vertices of degree 2 had at least $(3n+m)/2$ edges, and hence $f(n) \geq \lceil 3n/2 \rceil$ for $n \geq 7$. Combined results of Clark, Entringer and Shapiro [3], [4] and Lin, Jiang, Zhang and Yang [7] show that $f(n) = \lceil 3n/2 \rceil$ for $n \geq 19$ and for $n = 6, 10, 11, 12, 13, 17$. The values of $f(n)$ for the remaining values of n are also given in [7].

Let $g(n)$ denote the minimum number of edges in a MNT graph of order n. Dudek, Katona and Wojda [5] proved that

$$g(n) \geq \lceil \tfrac{3n-2}{2} \rceil - 2 \text{ for } n \geq 20$$

and showed, by construction, that

$$g(n) \leq \lceil \tfrac{3n-2}{2} \rceil \text{ for } n \geq 54$$

$$\text{as well as for } n \in I = \{22, 23, 30, 31, 38, 39, 40, 41, 42, 43, 46, 47, 48, 49, 50, 51\}.$$

We prove, using a method different from that in [5], that

$$g(n) \geq \lceil \tfrac{3n-2}{2} \rceil \text{ for } n \geq 10.$$

We also construct graphs of order $n = 12, 13$ with $\lceil \frac{3n-2}{2} \rceil$ edges and thus show that

$$g(n) = \lceil \tfrac{3n-2}{2} \rceil \text{ for } n \geq 54 \text{ as well as for } n \in I \cup \{12, 13\}.$$

We also determine $g(n)$ for $n \leq 9$.

2 Auxiliary Results

In this section we present some results concerning MNT graphs, which we shall use, in the next section, to prove that a MNT graph of order $n \geq 10$ has at least $\frac{3n-2}{2}$ edges. The first one concerns the lower bound for the number of edges of MNH graphs. It is the combination of results proved in [2] and [7].

Theorem 1 *(Bondy and Lin, Jiang, Zhang and Yang) If G is a MNH graph of order n, then $e(G) \geq \frac{3n}{2}$ for $n \geq 6$.*

The following lemma, which we proved in [6], will be used frequently.

Lemma 2 *Let Q be a path in a MNT graph G. If $\langle V(Q) \rangle$ is not complete, then some internal vertex of Q has a neighbour in $G - V(Q)$.*

Proof. Let u and v be two nonadjacent vertices of Q. Then $G+uv$ has a hamiltonian path P. Let x and y be the two endvertices of Q and suppose no internal vertex of Q has a neighbour in $G-V(Q)$. Then P has a subpath R in $\langle V(Q)\rangle + uv$ and R has either one or both endvertices in $\{x,y\}$. If R has only one endvertex in $\{x,y\}$, then P has an endvertex in Q. In either case the path obtained from P by replacing R with Q is a hamiltonian path of G. ∎

The following lemma is easy to prove.

Lemma 3 *Suppose T is a cutset of a connected graph G and $A_1, ..., A_k$ are components of $G-T$.*

(a) If $k \geq |T|+2$, then G is nontraceable.

(b) If G is MNT then $k \leq |T|+2$.

(c) If G is MNT and $k = |T|+2$, then $\langle T \cup A_i\rangle$ is complete for $i = 1, 2, ..., k$.

Proof. (a) and (b) are obvious. If (c) is not true, then there is an i such that $\langle T \cup A_i\rangle$ has two nonadjacent vertices x and y. But then T is a cutset of the graph $G+xy$ and $(G+xy)-T$ has $|T|+2$ components and hence $G+xy$ is nontraceable, by (a). ∎

The proof of the following lemma is similar to the previous one.

Lemma 4 *Suppose B is a block of a connected graph G.*

(a) If B has more than two cut-vertices, then G is nontraceable.

(b) If G is MNT, then B has at most three cut-vertices.

(c) If G is MNT and B has exactly three cut-vertices, then G consists of exactly four blocks, each of which is complete.

In [6] we proved some results concerning the degrees of the neighbours of the vertices of degree 2 in a 2-connected MNT graph, which enabled us to show that the average degree of the vertices in a 2-connected MNT graph is at least 3. We now restate those results in a form that is applicable also to MNT graphs which are not 2-connected. (Note that in a 2-connected graph no two vertices of degree 2 are adjacent to one another.)

Lemma 5 *If G is a connected MNT graph and $v \in V(G)$ with $d(v) = 2$, then the neighbours of v are adjacent. Also, one of the neighbours has degree at least 4 and the other neighbour has degree 2 or at least 4.*

Proof. Let $N_G(v) = \{x_1, x_2\}$ and let Q be the path x_1vx_2. Since $N_G(v) \subseteq Q$, it follows from Lemma 2 that $\langle V(Q)\rangle$ is a complete graph; hence x_1 and x_2 are adjacent.

Since G is connected and nontraceable, at least one of x_1 and x_2 has degree bigger that 2. Suppose $d(x_1) > 2$ and let $z \in N(x_1) - \{v, x_2\}$. If Q is the path zx_1vx_2 then, since $d(v) = 2$, the graph $\langle V(Q)\rangle$ is not complete and hence it follows from Lemma 2 that $d(x_1) \geq 4$. Similarily if $d(x_2) > 2$, then $d(x_2) \geq 4$. ∎

Lemma 6 *Suppose G is a connected MNT graph with distinct nonadjacent vertices v_1 and v_2 such that $d(v_1) = d(v_2) = 2$.*

(a) If v_1 and v_2 have exactly one common neighbour x, then $d(x) \geq 5$.

(b) If v_1 and v_2 have the same two neighbours x_1 and x_2, then $N_G(x_1) - \{x_2\} = N_G(x_2) - \{x_1\}$ and $d(x_1) = d(x_2) \geq 5$.

Proof. (a) Let $N(v_i) = \{x, y_i\}$; $i = 1, 2$. It follows from Lemma 5 that x is adjacent to y_i; $i = 1, 2$. Let Q be the path $y_1 v_1 x v_2 y_2$. Since $\langle V(Q) \rangle$ is not complete, it follows from Lemma 2 that x has a neighbour in $G - V(Q)$. Hence $d(x) \geq 5$.

(b) From Lemma 5 it follows that x_1 and x_2 are adjacent. Let Q be the path $x_2 v_1 x_1 v_2$. $\langle V(Q) \rangle$ is not complete since v_1 and v_2 are nonadjacent. Thus it follows from Lemma 2 that x_1 has a neighbour in $G - V(Q)$. Now suppose $p \in N_{G-V(Q)}(x_1)$ and $p \notin N_G(x_2)$. Then a hamiltonian path P in $G + px_2$ contains a subpath of either of the forms given in the first column of Table 1. Note that $i, j \in \{1, 2\}$; $i \neq j$ and that L represents a subpath of P in $G - \{x_1, x_2, v_1, v_2, p\}$. If each of the subpaths is replaced by the corresponding subpath in the second column of the table we obtain a hamiltonian path P' in G, which leads to a contradiction.

Subpath of P	Replace with
$v_i x_1 v_j x_2 p$	$v_i x_2 v_j x_1 p$
$v_i x_1 L p x_2 v_j$	$v_i x_2 v_j x_1 L p$

Table 1

Hence $p \in N_G(x_2)$. Thus $N_G(x_1) - \{x_2\} \subseteq N_G(x_2) - \{x_1\}$. Similarly $N_G(x_2) - \{x_1\} \subseteq N_G(x_1) - \{x_2\}$. Thus $N_G(x_1) - \{x_2\} = N_G(x_2) - \{x_1\}$ and hence $d(x_1) = d(x_2)$. Now let Q be the path $p x_1 v_1 x_2 v_2$. Since $\langle V(Q) \rangle$ is not complete, it follows from Lemma 2 that x_1 or x_2 has a neighbour in $G - V(Q)$. Hence $d(x_1) = d(x_2) \geq 5$. ■

Lemma 7 *Suppose G is a connected MNT graph of order $n \geq 6$ and that v_1, v_2 and v_3 are vertices of degree 2 in G having the same neighbours, x_1 and x_2. Then $G - \{v_1, v_2, v_3\}$ is complete and hence $e(G) = \frac{1}{2}(n^2 - 7n + 24)$.*

Proof. The set $\{x_1, x_2\}$ is a cutset of G. Thus according to Lemma 3 $G - \{v_1, v_2, v_3\} = K_{n-3}$. Hence $e(G) = \frac{1}{2}(n-3)(n-4) + 6$. ■

By combining the previous three results we obtain

Theorem 8 *Suppose G is a connected MNT graph without vertices of degree 1 or adjacent vertices of degree 2. If G has order $n \geq 7$ and m vertices of degree 2, then $e(G) \geq \frac{1}{2}(3n + m)$.*

Proof. If G has three vertices of degree 2 having the same two neighbours then, by Lemma 7, $m = 3$ and

$$e(G) = \frac{1}{2}(n^2 - 7n + 24) \geq \frac{1}{2}(3n + m) \text{ when } n \geq 7.$$

We now assume that G does not have three vertices of degree 2 that have the same two neighbours. Let $v_1, ..., v_m$ be the vertices of degree 2 in G and let $H = G - \{v_1, ..., v_m\}$. Then by Lemmas 5 and 6 the minimum degree, $\delta(H)$ of H is at least 3. Hence

$$e(G) = e(H) + 2m \geq \tfrac{3}{2}(n - m) + 2m = \tfrac{1}{2}(3n + m). \qquad \blacksquare$$

3 The minimum size of a MNT graph

Our aim is to determine the exact value of $g(n)$. By consulting the Atlas of Graphs [8], one can see, by inspection, that $g(2) = 0$, $g(3) = 1$, $g(4) = 2$, $g(5) = 4$, $g(6) = 6$ and $g(7) = 8$ (see Fig. 3).

We now give a lower bound for $g(n)$ for $n \geq 8$.

Theorem 9 *If G is a MNT graph of order n, then*

$$e(G) \;\; \geq \;\; \begin{cases} 10 & \text{if } n = 8 \\ 12 & \text{if } n = 9 \\ \frac{3n-2}{2} & \text{if } n \geq 10. \end{cases}$$

Proof. If G is not connected, then $G = K_k \cup K_{n-k}$, for some positive integer $k < n$ and then, clearly, $e(G) > \frac{3n-2}{2}$ for $n \geq 8$. Thus we assume that G is connected.

We need to prove that the sum of the degrees of the vertices of G is at least $3n - 2$. In view of Theorem 8, we let

$$M = \{v \in V(G) \mid d(v) = 2 \text{ and no neighbour of } v \text{ has degree } 2\}.$$

The remaining vertices of degree 2 can be dealt with simultaneously with the vertices of degree 1. We let

$$S = \{v \in V(G) - M \mid d(v) = 2 \text{ or } d(v) = 1\}.$$

If $S = \emptyset$, then it follows from Theorem 8 that $e(G) \geq \frac{1}{2}(3n + m)$. Thus we assume that $S \neq \emptyset$.

We observe that, if H is a component of the graph of $\langle S \rangle$, then either $H \cong K_1$ or $H \cong K_2$ and $N_G(H) - V(H)$ consists of a single vertex, which is a cut-vertex of G.

An example of such a graph G is depicted in the figure below.

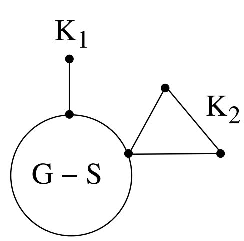

Fig. 1

Let $s = |S|$. By Lemma 4 the graph $\langle S \rangle$ has at most three components. We thus have three cases:

CASE 1. $\langle S \rangle$ has exactly three components, say H_1, H_2, H_3:

In this case the neighbourhoods of H_1, H_2, H_3 are pairwise disjoint; hence G has three cut-vertices. Hence it follows from Lemma 4 that $G - S$ is a complete graph of order at least 3. Futhermore, for every possible value of s, the number of edges in G incident with the vertices in S is $2s - 3$. Thus

$$e(G) = \binom{n-s}{2} + 2s - 3 \text{ for } s = 3, 4, 5 \text{ or } 6;\ \ s \leq n - 3.$$

An easy calculation shows that, for each possible value of s,

$$e(G) \ \geq \ \begin{cases} 10 & \text{if } n = 8 \\ 12 & \text{if } n = 9 \\ \frac{3n-2}{2} & \text{if } n \geq 10. \end{cases}$$

This case is a Zelinka Type II construction, cf. [9]. The graphs of smallest size of order 8 and 9 given by this construction are depicted in Fig. 3.

CASE 2. $\langle S \rangle$ has exactly two components, say H_1, H_2:

In this case the number of edges in G incident with the vertices in S is $2s - 2$.

Subcase 2.1. $N_G(H_1) = N_G(H_2)$:

Then it follows from Lemma 3 that $G - S$ is a complete graph. Hence

$$e(G) = \binom{n-s}{2} + 2s - 2 \text{ for } s = 2, 3 \text{ or } 4.$$

Thus

$$e(G) \ \geq \ \begin{cases} 12 & \text{if } n = 8 \\ 16 & \text{if } n = 9 \\ \frac{3n-2}{2} & \text{if } n \geq 10 \end{cases}$$

This case is a Zelinka Type I construction, cf. [9].

Subcase 2.2. $N_G(H_1) \neq N_G(H_2)$:

Let $N_G(H_i) = y_i$, $i = 1, 2$ and $y_1 \neq y_2$.

If $y_1y_2 \notin E(G)$ then $G + y_1y_2$ has a hamiltonian path P. But then P has one endvertex in H_1 and the other in H_2 and contains the edge y_1y_2; hence $V(G - S) = \{y_1, y_2\}$. But then G is disconnected. This contradiction shows that $y_1y_2 \in E(G)$.

Now $G - S$ is not complete, otherwise G would be traceable. Since $G + vw$, where v and w are nonadjacent vertices in $V(G - S)$, contains a hamiltonian path with one endvertex in H_1 and the other in H_2 and $y_1y_2 \in E(G)$, it follows that $(G - S) + vw$ has a hamiltonian cycle. Hence $G-S$ is either hamiltonian or MNH. We consider these two cases separately:

Subcase 2.2.1. $G-S$ is hamiltonian:
Then no hamiltonian cycle in $G-S$ contains y_1y_2, otherwise G would be traceable. Thus $d_{G-S}(y_i) \geq 3$ for $i=1,2$.

It also follows from Lemma 3 that no vertex $v \in M$ can be adjacent to both y_1 and y_2 since the graph $\langle V(H_i) \cup T \rangle$, where $T=\{y_1,y_2\}$ is not complete, for $i=1,2$. If $v \in M$ is adjacent to to one of the y_i's for $i=1,2$, say y_1, then, since the neighbours of v are adjacent, it follows that $d_{G-M-S}(y_1) \geq 3$.

It follows from our definition of M and S that $N_G(M) \cap S = \emptyset$. Since $G-M$ is not a complete graph, it follows from Lemma 7 that M does not have three vertices that have the same neighbourhood in G. Hence, by Lemmas 5 and 6, the minimum degree of the graph $G-M-S$ is at least 3.

Now, for $n \geq 8$

$$\begin{aligned} e(G) &= e(G-M-S)+2m+2s-2 \\ &\geq \frac{1}{2}\left(3\left(n-m-s\right)\right)+2m+2s-2 \\ &= \frac{1}{2}\left(3n+m+s-4\right) \\ &\geq \frac{3n-2}{2}, \text{ since } s \geq 2. \end{aligned}$$

Subcase 2.2.2. $G-S$ is nonhamiltonian:
Then $G-S$ is MNH (as shown above); hence it follows from Theorem 1, that $e(G-S) \geq \frac{3}{2}(n-s)$ for $n-s \geq 6$.

Thus, for $n-s \geq 6$ and $n \geq 8$

$$\begin{aligned} e(G) &= e(G-S)+2s-2 \\ &\geq \frac{1}{2}(3(n-s))+2s-2 \\ &= \frac{1}{2}(3n+s-4) \\ &\geq \frac{3n-2}{2}, \text{ since } s \geq 2. \end{aligned}$$

From [7] we have

$$e(G-S) \geq \begin{cases} 6 & \text{for } n-s=5 \\ 4 & \text{for } n-s=4. \end{cases}$$

Thus

$$e(G) \geq \begin{cases} 12 & \text{for } n=9 \text{ and } n-s=5 \\ 10 & \text{for } n=8 \text{ and } n-s=5 \text{ or } n-s=4. \end{cases}$$

The smallest MNH graphs F_4 and F_5 of order 4 and 5 respectively, are depicted in Fig. 2; cf. [7]. The graphs G_8 and G_9 (see Fig. 3) are obtained, respectively, by using F_4 with $s = 4$ or F_5 with $s = 3$, and F_5 with $s = 4$.

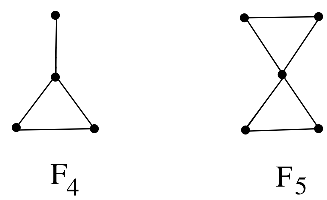

Fig. 2

CASE 3. $\langle S \rangle$ has exactly one component, say H:
Since

$$\sum_{v \in S} d_G(v) = 3s - 2, \text{ for } s = 1, 2$$

it follows that

$$\begin{aligned} e(G) &= e(G - M) + 2m \\ &= \frac{1}{2}\left(\sum_{v \in V(G-M)-S} d_{G-M}(v) + \sum_{v \in S} d_{G-M}(v) \right) + 2m \\ &\geq \frac{1}{2}\left(3\left(n - m - s\right) + 3s - 2\right) + 2m \\ &= \frac{1}{2}\left(3n + m - 2\right) \\ &\geq \frac{3n - 2}{2}. \end{aligned}$$

■

From the previous theorem we have $g(8) = 10$, $g(9) = 12$ and $g(n) \geq \lceil \frac{3n-2}{2} \rceil$ for $n \geq 10$. The MNT graphs G_n of order n with $g(n)$ edges, for $n \leq 9$ are given in Fig. 3.

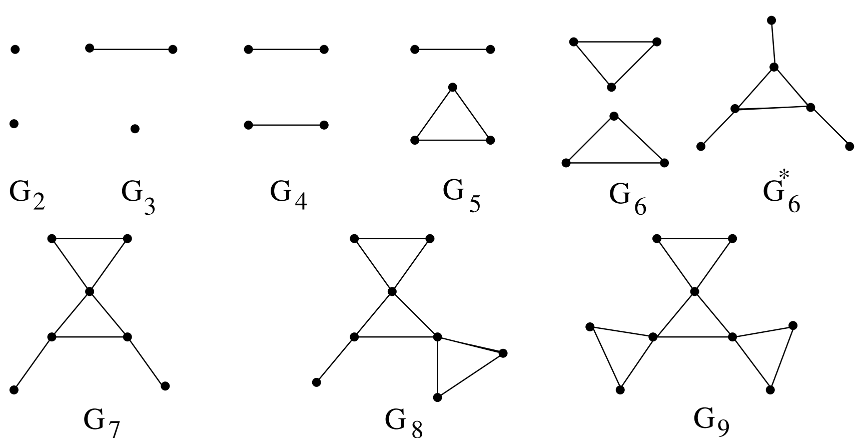

Fig. 3

In [5] Dudek, Katona and Wojda constructed, for every $n \geq 54$ as well as for every $n \in I = \{22, 23, 30, 31, 38, 39, 40, 41, 42, 43, 46, 47, 48, 49, 50, 51\}$, a MNT graph of size $\lceil \frac{3n-2}{2} \rceil$ in the following way: Consider a cubic MNH graph G with the property that
(1) there is an edge y_1y_2 of G, such that $N(y_1) \cap N(y_2) = \emptyset$, and
(2) $G + e$ has a hamiltonian cycle containing y_1y_2 for every $e \in E(\overline{G})$.

Now take two graphs H_1 and H_2, with $H_1 \cong K_1$ and $H_2 \cong K_1$ or $H_2 \cong K_2$ and join each vertex of H_i to y_i; $i = 1, 2$. The new graph is a MNT graph of order $v(G) + 2$ and size $e(G) + 2$ or of order $v(G) + 3$ and size $e(G) + 4$.

It follows from results in [3] and [4] that for every even $n \geq 52$ as well as for $n \in \{20, 28, 36, 38, 40, 44, 46, 48\}$ there exists a cubic MNH graph of order n that satisfies (1) and (2). Thus this construction provides MNT graphs of order n and size $\lceil \frac{3n-2}{2} \rceil$ for every $n \geq 54$ as well as for every $n \in I$.

We determined, by using the Graph Manipulation Package developed by Siqinfu and Sheng Bau*, that the Petersen graph also satisfies the above property. Hence, according to the above construction, there are also MNT graphs of order n and size $\lceil \frac{3n-2}{2} \rceil$ for $n = 12, 13$.

Thus $g(n) = \lceil \frac{3n-2}{2} \rceil$ for $n \geq 54$ as well as for every $n \in I \cup \{12, 13\}$.

It remains an open problem to find $g(n)$ for $n = 10, 11$ and those values of n between 13 and 54 which are not in I.

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