Indecomposable tilings of the integers with exponentially long periods

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Abstract

Let A be a finite multiset of integers. A second multiset of integers T is said to be an *A-tiling of level d* if every integer can be expressed in exactly d ways as the sum of an element of A and of an element of T. The set T is *indecomposable* if it cannot be written as the disjoint union of two proper subsets that are also A-tilings. In this paper we show how to construct indecomposable tilings that have exponentially long periods. More precisely, we give a sequence of multisets $(A_k)_{k=1}^{\infty}$ such that each A_k admits an indecomposable tiling T_k of period greater than $e^{c\sqrt[3]{n_k \log(n_k)}}$ where $n_k = \mathrm{diam}(A_k) = \max\{j \in A_k\} - \min\{j \in A_k\}$ tends to infinity and where $c > 0$ is some constant independent of k.

Introduction

Let A be a finite multiset of integers (which we shall call a *tile*) and let d be a nonnegative integer. Another multiset T of integers is said to be an *A-tiling of level d* if every integer can be written in exactly d ways as the sum of an element of T and an element of A.

For example if $A = \{0, 2\}$ then $T = \{4k, 4k+1 : k \in \mathbb{Z}\}$ is an A-tiling of level $d = 1$. One can understand the set T as specifying a set of positions for translates of the set A such that each integer is included exactly d times in the union of all the translates. We illustrate this for the above example in Fig. 1, where we shade the original copy of $A = \{0, 2\}$ and show which points belong to the same translate of $\{0, 2\}$ by connecting

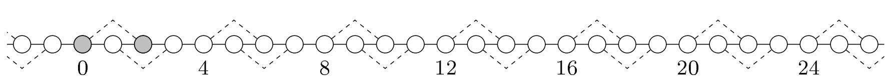

Figure 1: A $\{0, 2\}$-tiling of level 1 ($T = \{4k, 4k+1 : k \in \mathbb{Z}\}$).

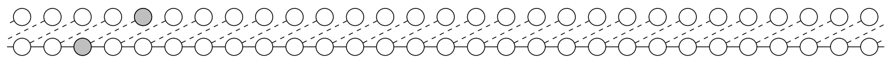

Figure 2: A $\{0,2\}$-tiling of level 2 ($T = \mathbb{Z}$).

them with a dashed line. Tilings of level greater than 1 can be similarly illustrated by using more than one row of dots, as in Fig. 2. In such a figure the vertical dimension serves only to alleviate clutter, and the reader should not be fooled into thinking there is any formal "under-over" relationship between elements of different translates that occupy the same position. Tilings of level greater than 1 are traditionally called "multiple tilings", but we shall not emphasize this distinction here.

A simple pigeonhole argument (see e.g. [7]) shows that all A-tilings of level d are periodic with period less than $(d+1)^{\mathrm{diam}(A)}$ where $\mathrm{diam}(A) = \max\{j \in A\} - \min\{j \in A\}$. In fact, Ruzsa [6] and Kolountzakis [3] have shown there is an upper bound on the longest period of an A-tiling that is independent of the level d. Ruzsa gives the explicit upper bound

$$\mathcal{H}(A) < e^{c_R\sqrt{\mathrm{diam}(A)\ln(\mathrm{diam}(A))}}$$

where $\mathcal{H}(A)$ stands for the longest minimal period of an A-tiling, $c_R > 0$ is a constant and where $\mathrm{diam}(A)$ is sufficiently large. If we define a function

$$D(n) = \max\{\mathcal{H}(A) : \mathrm{diam}(A) \leq n\}$$

then Ruzsa's upper bound can be more succinctly restated as saying that

$$D(n) \; < \; e^{c_R\sqrt{n\ln(n)}} \tag{1}$$

for all n sufficiently large. Ruzsa's upper bound is tight in the sense that the exists some constant $c_S > 0$ such that

$$D(n) \; > \; e^{c_S\sqrt{n\ln(n)}} \tag{2}$$

for all n sufficiently large. The lower bound (2) is derived in a previous paper of ours [9].

The tilings which are used in [9] to derive the lower bound (2) have the major aesthetical drawback of being so-called "decomposable tilings". An A-tiling is "decomposable" if it can be written as the disjoint union of two A-tilings of lower level (thus the $\{0,2\}$-tiling of Fig. 2 is decomposable, unlike the $\{0,2\}$-tiling of Fig. 1 which is *de facto* indecomposable because it has level 1). The construction in [9] essentially functions by finding tiles A that admit many different tilings of small period length and then taking the disjoint union of these tilings to form a large decomposable A-tiling whose period is the lcm of all the smaller periods. The purpose of this paper is to show that indecomposable tilings can also have long periods. More precisely, if we let $\mathcal{H}'(A)$ stand for the longest minimal period of an indecomposable A-tiling and if we let $D'(n) = \max\{\mathcal{H}'(A) : \mathrm{diam}(A) \leq n\}$, then we show that

$$D'(n) \; > \; e^{c_T\sqrt[3]{n\ln(n)}} \tag{3}$$

for all n sufficiently large, and where $c_T > 0$ is another constant independent of n. This is the paper's main result.

We shall arrive at the lower bound (3) by a constructive approach, i.e. by exhibiting specific tilings with long periods. The tiles which we use for this construction are closely related to those used in [9] to establish the lower bound (2). In particular, these tiles have the property of admitting many different tilings of small period such that the lcm of the different periods is exponentially large compared to the diameter of the tile. The principal difference between the approach of this paper and the approach in [9] is that, rather than superimposing all the tilings with small period lengths such as to obtain a tiling with long period (which does not yield an indecomposable tiling), we shall instead take a *linear combination* of the tilings with small period in such a way as to scramble the periods while ending up with an indecomposable tiling.

Naturally, any level 1 tiling is indecomposable so any lower bound on the longest period of a level 1 tiling is automatically a lower bound for $D'(n)$. Kolountzakis [3] and Biró [1] hold respectively the best lower and upper bounds on the periods of level 1 tilings. Letting $D_1(n)$ be the analog of the function $D(n)$ for level 1 tilings (i.e. $D_1(n) = \max\{\mathcal{H}_1(A) : \operatorname{diam}(A) \leq n\}$ where $\mathcal{H}_1(A)$ is the longest minimal period of a level 1 A-tiling), then Kolountzakis shows there is some constant $c_K > 0$ such that

$$D_1(n) > c_K n^2$$

for all n sufficiently large, whereas Biró shows that

$$D_1(n) < e^{n^{\frac{1}{3}+\epsilon}}$$

for all $\epsilon > 0$ and all n sufficiently large. In particular the reader will notice that the current lower and upper bounds for $D_1(n)$ suffer from a huge gap. It seems that most researchers suspect there exists a polynomial upper bound for $D_1(n)$. Our contribution in this paper is to show that indecomposability is not the key factor which prevents tilings from having long periods.

Background and Ideas

It will be convenient to encode multisets of integers as power series. Let $A[i]$ denote the multiplicity of integer i in the multiset of integers A. We define

$$A(x) = \sum_{k=-\infty}^{\infty} A[k]x^k.$$

It is easy to verify that if A is a finite multiset then another multiset T is an A-tiling if and only if

$$T(x)A(x) = d \sum_{t=-\infty}^{\infty} x^t. \tag{4}$$

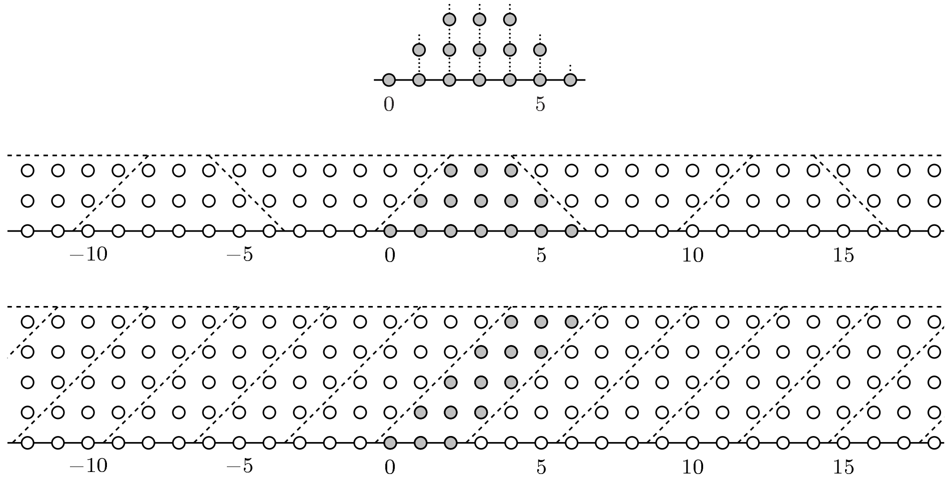

Figure 3: The tile $P_{3,5}$ (at top, where the number of times an integer appears in the tile is equal to the number of dots in the column above the integer) shown with a $P_{3,5}$-tiling of level 3 (middle) and a $P_{3,5}$-tiling of level 5 (bottom). The level 3 tiling corresponds to taking $T = 5\mathbb{Z}$ whereas the level 5 tiling corresponds to taking $T = 3\mathbb{Z}$.

For example, Fig. 1 simply bears testimony to the fact that

$$(\cdots + x^{-4} + x^{-3} + 1 + x + x^4 + x^5 + \cdots)(1 + x^2) = \sum_{t=-\infty}^{\infty} x^t.$$

We will start with the same class of tiles that are used in [9]. Let $P_{n_1,\ldots,n_k}$ be a tile parameterized by k natural numbers $n_1, \ldots, n_k$ and defined by

$$P_{n_1,\ldots,n_k}(x) \quad = \quad \prod_{j=1}^{k}(1 + x + \cdots + x^{n_j-1}).$$

Fig. 3 shows for example the tile $P_{3,5}$, together with a $P_{3,5}$-tiling of level 3 and a $P_{3,5}$-tiling of level 5. In general, the set $T_i = n_i\mathbb{Z}$ is a $P_{n_1,\ldots,n_k}$-tiling of level N/n_i where $N = n_1 n_2 \cdots n_k$ since

$$\begin{aligned} T_i(x)P_{n_1,\ldots,n_k}(x) &= \sum_{m=-\infty}^{\infty} x^{mn_i} \prod_{j=1}^{k}(1 + x + \cdots + x^{n_j-1}) \\ &= \prod_{\substack{j=1 \\ j\neq i}}^{k}(1 + x + \cdots + x^{n_j-1}) \sum_{t=-\infty}^{\infty} x^t \\ &= (N/n_i) \sum_{t=-\infty}^{\infty} x^t \end{aligned}$$

in accordance with (4).

If we take the disjoint unions of the $P_{n_1,\ldots,n_k}$-tilings $T_1,\ldots,T_k$ we obtain a $P_{n_1,\ldots,n_k}$-tiling of period $M = \mathrm{lcm}(n_1,\ldots,n_k)$. If the n_i's have few prime factors in common then M can become very large compared to $\mathrm{diam}(P_{n_1,\ldots,n_k})$. This simple observation leads to the lower bound (2) given in [9]. Taking disjoint unions, however, is a non-starter if we want indecomposable tilings. What we will do here instead is to construct a $P_{n_1,\ldots,n_k}$-tiling of minimal period M as a linear combination of (translates of) the k power series $T_1(x),\ldots,T_k(x)$. Finding such a linear combination may not be possible, as we will see, if the n_i's have too few prime factors in common, which accounts for the discrepancy between the lower bounds (2) and (3).

We first need to establish some general facts about $P_{n_1,\ldots,n_k}$-tilings. Let T be any $P_{n_1,\ldots,n_k}$-tiling. An elementary pigeonhole argument (cf. [7], for example) shows that all tilings in the sense discussed here are periodic, so that T must be periodic mod L for some $L > 0$. We can assume L is chosen large enough that $M = \mathrm{lcm}(n_1,\ldots,n_k)$ divides L (since we are not assuming that L is the minimal period of T, but simply that L is a period of T). Let T' be the restriction of T to the ground set $\{0,1,\ldots,L-1\}$ (i.e. $T'[i] = T[i]$ if $i \in \{0,\ldots,L-1\}$ and $T'[i] = 0$ otherwise). We then have

$$T'(x)P_{n_1,\ldots,n_k}(x) \equiv d(1+x+\cdots+x^{L-1}) \mod (1-x^L)$$

where d is the level of T, so

$$(1-x)T'(x)P_{n_1,\ldots,n_k}(x) \equiv 0 \mod (1-x^L). \tag{5}$$

It follows from (5) that every L-th root of unity except for '1' is either a root of $T'(x)$ or a root of $P_{n_1,\ldots,n_k}(x)$. But every root of $P_{n_1,\ldots,n_k}(x)$ is an M-th root of unity, so every L-th root of unity is a root of $T'(x)(1-x^M)$, i.e.

$$T'(x)(1-x^M) \equiv 0 \mod (1-x^L). \tag{6}$$

Since $M|L$ equation (6) states precisely that T' is periodic mod M. In other words, we have just proved that every $P_{n_1,\ldots,n_k}(x)$-tiling is periodic mod $M = \mathrm{lcm}(n_1,\ldots,n_k)$. The above argument is due to Kolountzakis [3]. A variant also appears in Ruzsa [6].

Knowing that $P_{n_1,\ldots,n_k}(x)$-tilings are periodic mod $M = \mathrm{lcm}(n_1,\ldots,n_k)$ allows us to study them in an essentially finite setting. Namely, $P_{n_1,\ldots,n_k}(x)$-tilings are in 1-to-1 correspondence with polynomials $T'(x) \in \mathbb{Z}[x]/(1-x^M)$ with nonnegative coefficients such that

$$T'(x)P_{n_1,\ldots,n_k}(x) \equiv d(1+x+\cdots+x^{M-1}) \mod (1-x^M)$$

or which is to say to such that

$$(1-x)T'(x)P_{n_1,\ldots,n_k}(x) \equiv 0 \mod (1-x^M).$$

We know in particular that every M-th root of unity except '1' which is not a root of $P_{n_1,\ldots,n_k}(x)$ must be a root of $T'(x)$. Let θ be a primitive M-th root of unity. Then θ^{M/n_i}, θ^{2M/n_i}, …, $\theta^{(n_i-1)M/n_i}$ are all the roots of $1+x+\cdots+x^{n_i-1}$, so all roots in the set

$$\mathcal{C} = \{\theta,\theta^2,\ldots,\theta^{M-1}\}\backslash\bigcup_{i=1}^{k}\{\theta^{M/n_i},\ldots,\theta^{(n_i-1)M/n_i}\}$$

must be roots of $T'(x)$. Conversely, if a polynomial $S(x) \in \mathbb{Z}[x]/(1-x^M)$ has all the roots $\mathcal{C}$ then $S(x)P_{n_1,\ldots,n_k}(x)$ has all M-th roots of unity except (maybe) for '1' so $S(x)P_{n_1,\ldots,n_k}(x) \equiv d(1+x+\cdots+x^{M-1}) \mod (1-x^M)$ for some d. We therefore have:

Proposition 1. *A polynomial $S(x) \in \mathbb{Z}[x]/(1-x^M)$ with nonnegative coefficients corresponds to a $P_{n_1,\ldots,n_k}$-tiling if and only if every element of $\mathcal{C}$ is a root of $S(x)$.*

Now consider the polynomials

$$T_i'(x) = 1 + x^{n_i} + x^{2n_i} + \cdots + x^{M-n_i}$$

defined for $1 \leq i \leq k$. Note the roots of $T_i'(x)$ are all M-th roots of unity *except* for those roots in the set $\{1, \theta^{M/n_i}, \ldots, \theta^{(n_i-1)M/n_i}\}$, so the roots of $R(x) = \gcd(T_1'(x), \ldots, T_k'(x))$ are precisely the roots in $\mathcal{C}$. By Proposition 1, therefore, $P_{n_1,\ldots,n_k}(x)$-tilings are in 1-to-1 correspondence with those polynomials in $\mathbb{Q}[x]/(1-x^M)$ with nonnegative integer coefficients that are in the ideal of $\mathbb{Q}[x]/(1-x^M)$ generated by $R(x)$, which is also equal to the ideal generated by the polynomials $T_1'(x), \ldots, T_k'(x)$.

The ideal generated by $T_1'(x), \ldots, T_k'(x)$ in $\mathbb{Q}[x]/(1-x^M)$ is therefore very much of interest to us. We now take a more geometric look at this ideal. For concreteness, suppose first that $n_1 = p_1, \ldots, n_k = p_k$ are distinct primes (note that in this case the ratio $M/\operatorname{diam}(P_{n_1,\ldots,n_k}) \approx n_1 \cdots n_k/(n_1 + \cdots + n_k)$ becomes quite large as $k \to \infty$). The numbers between 0 and $M-1$ are uniquely given by their value mod p_i for $1 \leq i \leq k$ by the Chinese Remainder Theorem so it makes sense to think of polynomials in $\mathbb{Q}[x]/(1-x^M)$ as arrays of size $p_1 \times \ldots \times p_k$ whereby the coefficient of x^n becomes the entry in the array with coordinate $(n \mod p_1, \ldots, n \mod p_k)$. Then the polynomial $T_i'(x)$ corresponds to the array whose $(j_1, \ldots, j_k)$-th entry is 1 if $j_i = 0$ and is 0 otherwise since the exponents with nonzero coefficients in $T_i'(x) = 1 + x^{p_i} + \ldots + x^{M-p_i}$ are precisely the numbers between 0 and $M-1$ equal to 0 mod p_i.

Say that a *slab* is an array whose entries are all 0 except for those entries with a given value of the i-th coordinate (for any i), which entries are set to 1. We have just remarked that the array corresponding to the polynomial $T_i'(x)$ is a slab. It is equally easy to see that the polynomials $x^j T_i'(x)$ for $1 \leq j < p_i$ also correspond to slabs—indeed these are just the $(p_i - 1)$ "translates" of the slab corresponding to $T_i'(x)$ along the i-th coordinate direction of the array. Thus a polynomial in the ideal generated by $T_1'(x), \ldots, T_k'(x)$ in $\mathbb{Q}[x]/(1-x^M)$ simply corresponds to an array of size $p_1 \times \ldots \times p_k$ that can be written as a linear combination of slabs. We shall call such an array a "C1 array" where "C1" stands for "codimension 1" (which somehow reflects our intuition that slabs are codimension 1 objects).

To reformulate the above observations, $P_{p_1,\ldots,p_k}$-tilings are in 1-to-1 correpondence with nonnegative, integer-valued C1 arrays of dimension $p_1 \times \ldots \times p_k$. We will say that a C1 array is *minimal* if it is nonzero, nonnegative and integer-valued and if it cannot be written as the sum of two other nonzero, nonnegative, integer-valued C1 arrays. It is clear from the relevant definitions that a nonempty $P_{p_1,\ldots,p_k}$-tiling is indecomposable if and only its associated C1 array is minimal. (Connoisseurs may also note that the set of minimal C1 arrays forms the so-called “Hilbert basis” of the cone obtained by intersecting the space of all C1 arrays with the nonnegative orthant in $\mathbb{R}^{p_1 \cdots p_k}$.)

A $P_{p_1,\ldots,p_k}$-tiling has a minimal period of M if and only if it is not periodic mod M/p_i for every $1 \le i \le k$. In terms of the associated array this means that for every $1 \le i \le k$ there are two coordinates $(j_1,\ldots,j_k)$ and $(h_1,\ldots,h_k)$ differing only in the i-th position such that the $(j_1,\ldots,j_k)$-th entry of the associated array is not equal to the $(h_1,\ldots,h_k)$-th entry. We will say for shortness that an array is “non-periodic” if it possesses this property. Thus a $P_{p_1,\ldots,p_k}$-tiling has a minimal period of M if and only if its associated array is non-periodic. Our quest for tilings with long periods therefore leads us to ask whether there exist minimal non-periodic C1 arrays. Unfortunately, the following theorem puts an end to such hopes:

Theorem 1. *The only minimal C1 arrays are slabs.*

Proof. Let C be a minimal C1 array of size $n_1 \times \ldots \times n_k$. We assume by contradiction that C is not equal to a slab. We write $C_{i_1,\ldots,i_k}$ for the $(i_1,\ldots,i_j)$-th entry of C where $i_j \in \mathbb{Z}_{n_j} = \{0,1,\ldots,n_j-1\}$ for $1 \le j \le k$ (note that we are indexing coordinates of the array starting from 0 instead of from 1).

Remark that if A is any slab of size $n_1 \times \ldots \times n_k$ then for any $(j_1,\ldots,j_k) \in \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_k}$ we have

$$A_{0,\ldots,0} - A_{j_1,0,\ldots,0} - A_{0,j_2,\ldots,j_k} + A_{j_1,\ldots,j_k} = 0. \tag{7}$$

Since C is a linear combination of slabs we then likewise have

$$C_{0,\ldots,0} - C_{j_1,0,\ldots,0} - C_{0,j_2,\ldots,j_k} + C_{j_1,\ldots,j_k} = 0 \tag{8}$$

for any $(j_1,\ldots,j_k) \in \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_k}$.

Since C does not dominate any slab C must have some zero entry. Because permuting the coordinates of an array maps slabs to slabs (and thus maps minimal C1 arrays to minimal C1 arrays) we can assume that $C_{0,\ldots,0} = 0$. Take $j_1 \in \mathbb{Z}_{n_1}$. Again because C does not dominate any slab, there must be $(j_2,\ldots,j_k) \in \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_k}$ such that $C_{j_1,\ldots,j_k} = 0$. Applying Eq. 8 we get that

$$-C_{j_1,0,\ldots,0} - C_{0,j_2,\ldots,j_k} = 0$$

but C is nonnegative, so we get (in particular) that $C_{j_1,0,\ldots,0} = 0$. Since j_1 was arbitrary, we thus have $C_{j,0,\ldots,0} = 0$ for all $j \in \mathbb{Z}_{n_1}$. Treating other indices symmetrically we get that all entries in any line containing a zero are zero, which implies that $C = 0$, a contradiction. □

In a sense, Theorem 1 reflects our intuition that slabs are too clumsy a set of generators to construct interesting arrays. An immediate corollary of Theorem 1 is that the only indecomposable $P_{p_1,\ldots,p_k}$-tilings are translates of $p_1\mathbb{Z},\ldots,p_k\mathbb{Z}$. One can apply the same argument to show that the only indecomposable $P_{n_1,\ldots,n_k}$-tilings are the translates

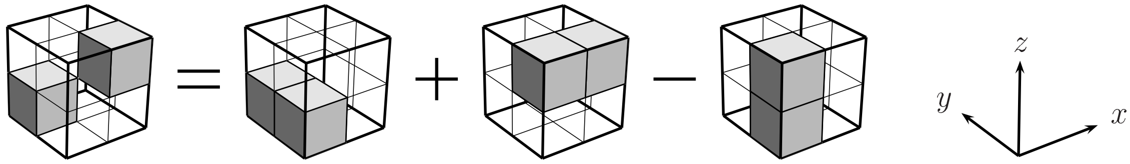

Figure 4: A $2 \times 2 \times 2$ cyclotomic array shown decomposed as a linear combination of lines of 1's; shaded cubes denote 1's, other entries are 0.

of $n_1\mathbb{Z}, \ldots, n_k\mathbb{Z}$ when the n_i's are pairwise coprime (this is a special case of one of the main results of [9]).

We now take a look at the structure of the ideal generated by $T_1'(X),\ldots, T_k'(x)$ when the n_i's have many prime factors in common. For concreteness, say that $n_1 = M/p_1, \ldots, n_k = M/p_k$ where $p_1 < \ldots < p_k$ are distinct primes and where $M = p_1 \cdots p_k$. In this case the ratio $M/\operatorname{diam}(P_{n_1,\ldots,n_k})$ is very poor but we consider these tiles nonetheless for the sake of example.

We again think of polynomials in $\mathbb{Q}[x]/(1-x^M)$ as arrays of size $p_1 \times \ldots \times p_k$ under the map given by the Chinese Remainder Theorem. This time we have

$$T_i'(x) = 1 + x^{M/p_i} + x^{2M/p_i} + \cdots + x^{(p_i-1)M/p_i}$$

so the exponents of $T_i'(x)$ with nonzero coefficients are precisely those numbers between 0 and $M-1$ which are 0 mod p_j for all $j \neq i$. This means that $T_i'(x)$ corresponds to the $p_1 \times \ldots \times p_k$ array whose $(l_1,\ldots,l_k)$-th entry is 1 if $l_j = 0$ for all $j \neq i$ and 0 otherwise. This type of array looks like a single line of 1's running parallel to the i-th coordinate axis, and running the full length of the array.

Call a *fiber* any array consisting of a single line of 1's running parallel to one of the coordinate axes and running the full length of the array. By the above remarks, a polynomial of the form $x^j T_i'(x) \mod 1-x^M$ maps to a fiber running parallel to the i-th coordinate axis. Thus elements of the ideal generated by $T_1'(x),\ldots,T_k'(x)$ in $\mathbb{Q}[x]/(1-x^M)$ map to arrays that are linear combinations of fibers and vice-versa. We might call such arrays "D1 arrays" by analogy with our previous terminology ("D1" for "dimension 1" as opposed to "codimension 1") but these kinds of arrays have already been tagged "cyclotomic" elsewhere in the literature ([8], [2]), and we will adhere to the latter terminology. As above, we say that a nonnegative, integer-valued cyclotomic array is "minimal" if it cannot be written as the sum of two other nonzero, nonnegative, integer-valued cyclotomic arrays. Note that an array may be minimal as a C1 array but not minimal as a cyclotomic array (it will be clear in each context which we mean). Indecomposable $P_{M/p_1,\ldots,M/p_k}$-tilings are thus in 1-to-1 correspondence with minimal cyclotomic arrays of dimension $p_1 \times \ldots \times p_k$.

As for $P_{p_1,\ldots,p_k}$-tilings a $P_{M/p_1,\ldots,M/p_k}$-tiling has minimal period M if and only its associated array is non-periodic. Since we are looking for tilings with long periods we are thus again led to ask whether there exist minimal non-periodic cyclotomic arrays. This time (and by opposition with C1 arrays) the answer is yes, provided the dimension k of

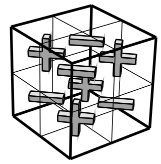

Figure 5: A $2 \times 2 \times 2$ array that is orthogonal to all fibers, and thus to all $2 \times 2 \times 2$ cyclotomic arrays. Each entry of the array is ± 1; a '+' sign denotes an entry of 1 and a '-' signs denotes an entry of -1.

the array is greater than or equal to 3 and also provided the sidelengths of the array are all greater than or equal to 2 (which latter condition is a trivial requirement for an array to be non-periodic). A construction for a minimal non-periodic $2 \times 2 \times 2$ cyclotomic array is shown in Fig. 4, where the array is shown on the left and its decomposition as a linear combination of fibers is shown on the right. The Fig. 4 cyclotomic array is obviously non-periodic since for each coordinate direction there is a line in the array containing different values. It is maybe not quite so obvious to see the same array is minimal. We give a formal proof that the Fig. 4 is minimal in the next proposition.

Proposition 2. *The Fig. 4 cyclotomic array is minimal.*

Proof. Let C denote the $2 \times 2 \times 2$ array of Fig. 4. We will write the coordinates of entries in C as binary strings of length 3 instead of as triplets (i, j, k), putting $C_{ijk} = C_{i,j,k}$. If the lower front corner of the Fig. 4 array has coordinate 000 and the axes are ordered as on the right of Fig. 4 we then have $C_{101} = C_{010} = 1$ and $C_{000} = C_{001} = C_{011} = C_{111} = C_{110} = C_{100} = 0$. Let A be an integer-valued $2 \times 2 \times 2$ cyclotomic array such that $0 \leq A \leq C$. We need to show that $A = 0$ or $A = C$. Consider the $2 \times 2 \times 2$ array of Fig. 5 with entries of ± 1. The Fig. 5 array is orthogonal in $\mathbb{R}^8$ to any $2 \times 2 \times 2$ fiber so it is also orthogonal to A, which is by assumption a linear combination of fibers. We therefore have

$$A_{000} - A_{001} - A_{010} - A_{100} + A_{011} + A_{101} + A_{110} - A_{111} = 0 \tag{9}$$

but $A_{000} = A_{001} = A_{011} = A_{111} = A_{110} = A_{100} = 0$ so we get $A_{010} = A_{101}$. Therefore A is a scalar multiple of C and thus, since A is integer-valued, $A = 0$ or $A = C$, as desired. $\square$

It might seem to the reader that constructing a $2 \times 2 \times 2$ non-periodic minimal cyclotomic array is a waste of breath when our bijection is only between indecomposable $P_{M/p_1,\ldots,M/p_k}$-tilings and minimal cyclotomic arrays of size $p_1 \times \ldots \times p_k$ for distinct primes $p_1, \ldots, p_k$. However, larger cyclotomic arrays of unequal sidelengths can easily be obtained from smaller cyclotomic arrays of same sidelength by using a process called *inflation*. We say that an array C' of size $n'_1 \times \ldots \times n'_k$ is an *inflate* of an array C of size $n_1 \times \ldots n_k$ if there exists surjections $\kappa_1 : \mathbb{Z}_{n'_1} \to \mathbb{Z}_{n_1}, \ldots, \kappa_k : \mathbb{Z}_{n'_k} \to \mathbb{Z}_{n_k}$ such that $C'_{i_1,\ldots,i_k} = C_{\kappa_1(i_1),\ldots,\kappa_k(i_k)}$ for all $(i_1, \ldots, i_k) \in \mathbb{Z}_{n'_1} \times \ldots \times \mathbb{Z}_{n'_k}$. The basic idea behind the process of inflation is shown in Fig. 6.

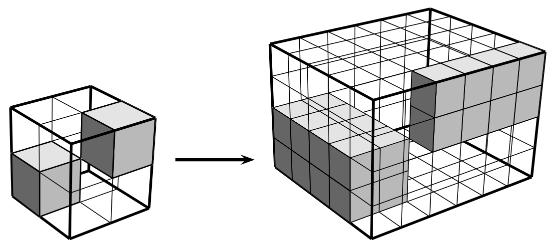

Figure 6: Inflating a $2 \times 2 \times 2$ array.

1	
1	

$\longrightarrow$

1	1	
1	1	

Figure 7: A non-minimal inflate of a minimal cyclotomic array.

It is quite easy to check that the inflate of a cyclotomic array is again a cyclotomic array (and likewise for C1 arrays) and that inflates of non-periodic arrays are non-periodic. On the other hand inflation does not always preserve minimality, as shown by Fig. 7. Say that an $n_1 \times \ldots \times n_k$ array C is "full-dimensional" if there does not exist $1 \leq j \leq k$ and $q \in \mathbb{Z}_{n_j}$ such that $C_{i_1,\ldots,i_k} \neq 0 \implies i_j = q$. We have the following proposition from [8] concerning the minimality of inflates of cyclotomic arrays:

Proposition 3. ([8] Cor. 1) *If C is a minimal cyclotomic array of size $n_1 \times \ldots \times n_k$ and $n'_1 \geq n_1, \ldots, n'_k \geq n_k$ then there exists an inflate C' of C of size $n'_1 \times \ldots \times n'_k$ that is also minimal. Moreover, if C is full-dimensional then any inflate of C is minimal.*

By Proposition 3, any inflate of the Fig. 4 array is minimal. Consider in particular the $2 \times 3 \times 5$ inflate shown in Fig. 8. If the lower corner closest to the viewer has coordinate $(0, 0, 0)$ then this array corresponds to the polynomial $x^5 + x^6 + x^{12} + x^{18} + x^{24} + x^{25}$ in $\mathbb{Q}[x]/(1 - x^{30})$ under the map given by the Chinese Remainder Theorem (whereby the coefficient of x^n becomes the value of the entry $(n \mod 2, n \mod 3, n \mod 5)$). Thus the

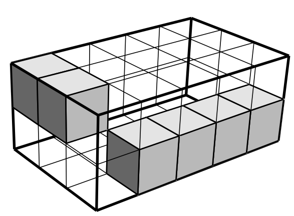

Figure 8: A $2 \times 3 \times 5$ inflate of the $2 \times 2 \times 2$ array of Fig. 4.

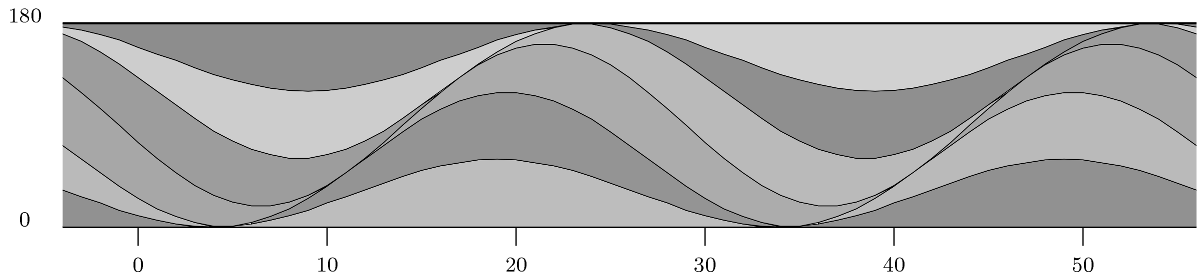

Figure 9: The $P_{15,10,6}$-tiling corresponding to the cyclotomic array of Fig. 8. Each shaded region corresponds to one translate of $P_{15,10,6}$.

subset T of $\mathbb{Z}$ with power series

$$T(x) = \sum_{k=-\infty}^{\infty} x^{30k}(x^5 + x^6 + x^{12} + x^{18} + x^{24} + x^{25})$$

is an indecomposable $P_{15,10,6}$-tiling of minimal period 30 (where $15 = M/p_1 = 30/2$, $10 = M/p_2 = 30/3$, etc). Notice the period of T is longer than the period of any of the "obvious" $P_{15,10,6}$-tilings $15\mathbb{Z}$, $10\mathbb{Z}$ and $6\mathbb{Z}$. Since there are 6 copies of $P_{15,10,6}$ per interval of length 30 the level of T is $6 \cdot (15 \cdot 10 \cdot 6)/30 = 180$. If we illustrate the $P_{15,10,6}$-tiling T in the style of Fig. 3 we get something like Fig. 9, where sets of points are approximated by shaded regions because of the large scale involved. (Recall that the layering of the tiles in such a figure is arbitrary.)

Thus far, our discussion has mainly served to illustrate the following points:

- If $n_1, \ldots, n_k$ are pairwise coprime, then there do not exist indecomposable $P_{n_1,\ldots,n_k}$-tilings of minimal period $M = \mathrm{lcm}(n_1, \ldots, n_k)$ (assuming that $M \neq n_i$ for some i, which means assuming that $k \geq 2$ and that $n_j > 1$ for all j)

- In certain cases when the n_i's have many factors in common, it is possible to construct indecomposable $P_{n_1,\ldots,n_k}$-tilings of minimal period $M = \mathrm{lcm}(n_1, \ldots, n_k)$

- The ratio $M/\operatorname{diam}(P_{n_1,\ldots,n_k})$ is best maximized when the n_i's have few prime factors in common

Given the above comments, the next most logical tile to look at is $P_{p_1p_2,p_2p_3,\ldots,\,p_{k-1}p_k,\,p_kp_1}$ where $p_1, \ldots, p_k$ are distinct primes. This is indeed what works, as $P_{p_1p_2,\ldots,p_kp_1}$ admits an indecomposable tiling of minimal period $M = \mathrm{lcm}(p_1p_2,\ p_2p_3,\ \ldots,\ p_kp_1) = p_1 \cdots p_k$. Choosing $p_1, \ldots, p_k$ to be the first k primes and letting k tend to infinity and examining the growth of M compared to the growth of $\operatorname{diam}(P_{p_1p_2,\ldots,p_kp_1}) \approx p_1p_2 + \ldots + p_kp_1$ yields the lower bound (3). We do this asymptotical computation at the end of the section.

As the asymptotical computation is easy our main job is really to explain how to construct the indecomposable $P_{p_1p_2,\ldots,p_kp_1}$-tiling of period M. We do this for $k = 4$ in this section and give a general construction and proof in the next section (note that

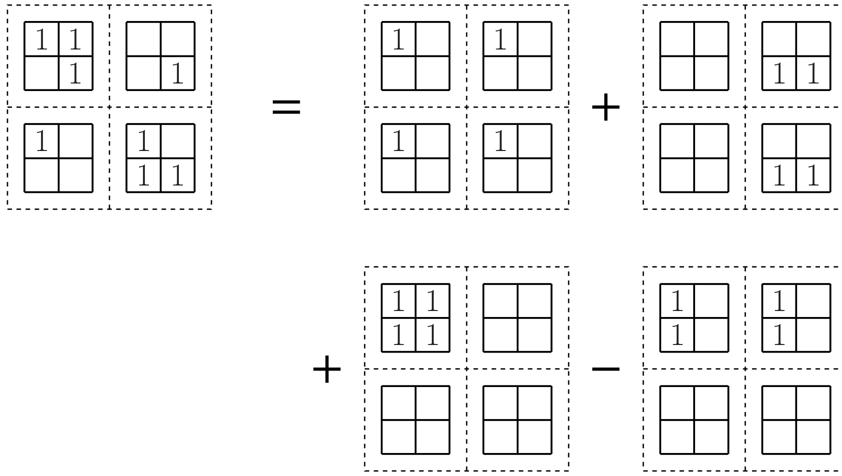

Figure 10: A minimal non-periodic CC2 array of size $2 \times 2 \times 2 \times 2$.

$P_{p_1p_2,\ldots,p_kp_1} = P_{M/p_1,\ldots,M/p_k}$ when $k = 3$ so we already know how to construct an indecomposable $P_{p_1p_2,\ldots,p_kp_1}$-tiling of minimal period M in this case).

Since all $P_{p_1p_2,\ldots,p_kp_1}$-tilings are periodic mod $M = p_1 \cdots p_k$ we can again understand $P_{p_1p_2,\ldots,p_kp_1}$-tilings as arrays of size $p_1 \times \ldots \times p_k$ (under the same bijective map as usual). Now that $n_1 = p_1p_2$, $n_2 = p_2p_3$, $\ldots$, $n_k = p_kp_1$, the exponents of $T_i'(x)$ with nonzero coefficients are the integers between 0 and $M-1$ that are divisible by p_ip_{i+1} (where we put $p_{k+1} = p_1$; indices referring to numbers in the set $\{1, 2, \ldots, k\}$ will be taken in 'wrap-around fashion' from now on). Thus a polynomial of the type $x^nT_i'(x)$ becomes an array whose $(j_1, \ldots, j_k)$-th entry is 1 if $j_i \equiv n \bmod p_i$, $j_{i+1} \equiv n \bmod p_{i+1}$ and is 0 otherwise. We will call an array of this type an *adjacent index co-slab*. Put otherwise, an adjacent index co-slab is a 0-1 array whose support is the set of all entries with given i-th and $(i+1)$-th coordinates. An array that can be written as a linear combination of adjacent index co-slabs will be called a 'CC2 array' (for "Cyclic Codimension 2").

$P_{p_1p_2,\ldots,p_kp_1}$-tilings are therefore in 1-to-1 correspondence with nonnegative, integer-valued CC2 arrays of size $p_1 \times \ldots \times p_k$. What we want is to be able to construct minimal non-periodic CC2 arrays for arbitrarily large k. For $k = 2$ an array with a single entry of 1 is an adjacent index co-slab, so gives us a minimal non-periodic CC2 array, whereas when $k = 3$ CC2 arrays are the same as cyclotomic arrays, which we have already discussed. The first case of interest to us is therefore $k = 4$. We can simplify our task by using the following analog of Theorem 3 for CC2 arrays:

Proposition 4. *If C is a minimal CC2 array of size $n_1 \times \ldots \times n_k$ and $n_1' \geq n_1$, $\ldots$, $n_k' \geq n_k$ then there exists an inflate C' of C of size $n_1' \times \ldots \times n_k'$ that is also minimal. Moreover, if C is full-dimensional then any inflate of C is minimal.*

We omit the proof of Proposition 4 since it is exactly the same as its counterpart for cyclotomic arrays, which is given in [8].

Proposition 4 implies it is sufficient to construct a 4-dimensional minimal non-periodic CC2 array of any size, say, $2 \times 2 \times 2 \times 2$, in order to establish the existence of a minimal

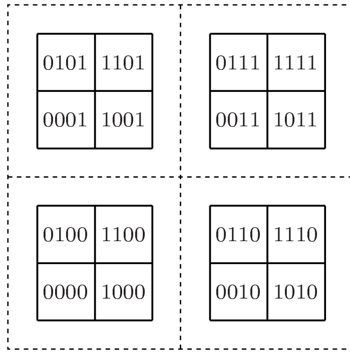

Figure 11: The coordinatization for the $2 \times 2 \times 2 \times 2$ array of Fig. 10 (and for subsequent $2 \times 2 \times 2 \times 2$ arrays).

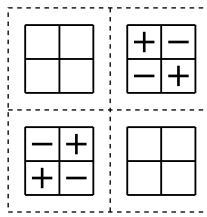

Figure 12: An array orthogonal to all $2 \times 2 \times 2 \times 2$ CC2 arrays. Pluses denote entries of 1, minuses entries of -1.

non-periodic CC2 array whose sides are distinct primes p_1, p_2, p_3, p_4. Fig. 10 shows a promising candidate for such a $2 \times 2 \times 2 \times 2$ CC2 array (the coordinatization of this array is shown in Fig. 11). While the array of Fig. 10 is obviously non-periodic, it is somewhat touchier to tell whether it is minimal. We devote a proposition to this. The proof we give is slightly more complicated than necessary as we are setting up a blueprint for the k-dimensional case.

Proposition 5. *The CC2 array of Fig. 10 is minimal.*

Proof. Let C denote the CC2 array of Fig. 10. As in the proof of Proposition 2 we write

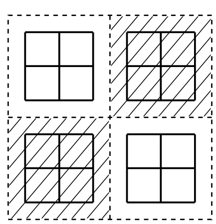

Figure 13: The ground set $Q = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \{0\} \times \{0\} \cup \mathbb{Z}_2 \times \mathbb{Z}_2 \times \{1\} \times \{1\} \subseteq (\mathbb{Z}_2)^4$.

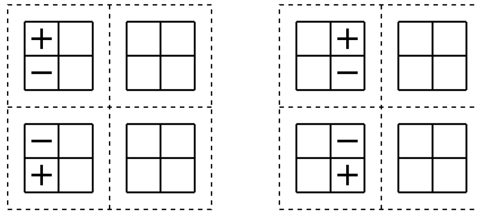

Figure 14: Two more arrays that are orthogonal to all $2 \times 2 \times 2 \times 2$ CC2 arrays.

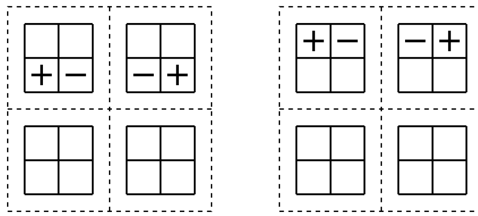

Figure 15: Two more.

the coordinates of entries in C as binary strings of length 4, putting C_{ijkh} for $C_{i,j,k,h}$. Let A be an integer-valued CC2 array such that $0 \leq A \leq C$. We need to show that $A = 0$ or $A = C$. Since A is integer-valued it is sufficient to show that A is a scalar multiple of C.

Consider the $2 \times 2 \times 2 \times 2$ array of Fig. 12 with entries of 0 and ± 1. It is easy to check this array is orthogonal to every adjacent index co-slab (the 4 basic types of which appear in Fig. 10), so must be orthogonal to A, which is a linear combination of such co-slabs. Let us restrict our attention for a moment to the portion of A with ground set $Q = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \{0\} \times \{0\} \cup \mathbb{Z}_2 \times \mathbb{Z}_2 \times \{1\} \times \{1\} \subseteq (\mathbb{Z}_2)^4$ (shown in Fig. 13). If we consider $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \{0\} \times \{0\}$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \{1\} \times \{1\}$ as the two 2×2 layers of a $2 \times 2 \times 2$ array then C restricted to Q looks exactly like the Fig. 4 array whereas the Fig. 12 array restricted to Q looks exactly like the Fig. 5 array. It follows from the same argument as in the proof of Proposition 2 that $A_{0100} = A_{1011}$. Thus there exists a $\lambda \in \mathbb{R}$ such that $A_{ijkh} = \lambda C_{ijkh}$ for all $(i, j, k, h) \in Q$. All that we have left to show is that $A_{ij01} = \lambda C_{ij01}$ and that $A_{ij10} = \lambda C_{ij10}$ for all $(i, j) \in \mathbb{Z}_2 \times \mathbb{Z}_2$.

Say for now that we know of some pair $(i_0, j_0) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $A_{i_0 j_0 01} = \lambda C_{i_0 j_0 01}$. Let $i'_0 = 1 - i_0$, $j'_0 = 1 - j_0$. Note the two $2 \times 2 \times 2 \times 2$ arrays of Fig. 14 are orthogonal (like the array of Fig. 12) to all adjacent co-slabs, so are orthogonal to A and C. It follows that

$$A_{i_0 j_0 00} - A_{i_0 j'_0 00} - A_{i_0 j_0 01} + A_{i_0 j'_0 01} = 0$$

and also

$$\lambda C_{i_0 j_0 00} - \lambda C_{i_0 j'_0 00} - \lambda C_{i_0 j_0 01} + \lambda C_{i_0 j'_0 01} = 0$$

but $A_{i_0 j_0 00} = \lambda C_{i_0 j_0 00}$, $A_{i_0 j'_0 00} = \lambda C_{i_0 j'_0 00}$ and $A_{i_0 j_0 01} = \lambda C_{i_0 j_0 01}$ so we get $A_{i_0 j'_0 01} = \lambda C_{i_0 j'_0 01}$. A similar argument which uses the fact that the arrays of Fig. 15 are orthogonal to all adjacent index co-slabs shows that $A_{i'_0 j_0 01} = \lambda C_{i'_0 j_0 01}$. We therefore have that both

$A_{i_0j_0'01} = \lambda C_{i_0j_0'01}$ and $A_{i_0'j_001} = \lambda C_{i_0'j_001}$ if $A_{i_0j_001} = \lambda C_{i_0j_001}$ from which it follows (by repeated applications of the same argument) that $A_{ij01} = \lambda C_{ij01}$ for all $(i,j) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ if $A_{i_0j_010} = \lambda C_{i_0j_010}$ for some $(i_0, j_0) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. But we know of such a pair (i_0, j_0), namely the pair $(0,0)$, since $A_{0001} = C_{0001} = 0$. Therefore $A_{ij01} = \lambda C_{ij01}$ for all $(i,j) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. A symmetric argument shows that $A_{ij10} = \lambda C_{ij10}$ for all $(i,j) \in \mathbb{Z}_2 \times \mathbb{Z}_2$, which completes the proof. □

It follows from Propositions 4 and 5 that there exist indecomposable $P_{p_1p_2,p_2p_3,p_3p_4,p_4p_1}$-tilings of minimal period $p_1p_2p_3p_4$ for any distinct primes p_1, p_2, p_3, p_4. In the next section we generalize the construction of Fig. 10 and the proof of Proposition 5 to show there exist non-periodic minimal k-dimensional CC2 arrays of size $2 \times \ldots \times 2$ for any $k \geq 2$, from which it likewise follows that there are indecomposable $P_{p_1p_2,\ldots,p_kp_1}$-tilings of minimal period $p_1 \cdots p_k$ for any distinct primes $p_1, \ldots, p_k$. Meanwhile we show that the lower bound (3) holds if we assume the existence of such tilings:

Proposition 6. *If there exist indecomposable $P_{p_1p_2,\ldots,p_kp_1}$-tilings of minimal period $p_1 \cdots p_k$ for any distinct primes $p_1, \ldots, p_k$ and any $k \geq 2$ then there is some constant $c > 0$ such that for all n sufficiently large there is a tile of diameter n or less admitting an indecomposable tiling of minimal period $e^{c\sqrt[3]{n\ln(n)}}$ or more.*

Proof. Recall that $D'(n)$ is defined as the longest minimal period of an indecomposable tiling of a tile of diameter n or less (so the proposition states that $D'(n) \geq e^{c\sqrt[3]{n\ln(n)}}$ for all n sufficiently large). Let $\pi(r)$ denote the number of primes at most r and let $p_1, \ldots, p_{\pi(r)}$ be the first $\pi(r)$ primes. Let $\sigma_r = (p_1 + \ldots + p_{\pi(r)})p_{\pi(r)}$. Note that $\sigma_r \geq \mathrm{diam}(P_{p_1p_2,p_2p_3,\ldots,p_{\pi(r)}p_1}) = p_1p_2 + \ldots + p_{\pi(r)}p_1 - \pi(r)$. By the prime number theorem $\pi(r) \approx \frac{r}{\ln(r)}$ so

$$\sigma_r \leq \frac{r^3}{\ln(r)}$$

for all r sufficiently large. If $r \leq \sqrt[3]{\sigma_r \ln(\sqrt[3]{\sigma(r)})}$ then we get

$$\sigma_r \leq \frac{r^3}{\ln(r)} \leq \frac{\sigma_r \ln(\sqrt[3]{\sigma(r)})}{\ln(\sqrt[3]{\sigma_r \ln(\sqrt[3]{\sigma(r)})})}$$

a contradiction, so

$$r \geq \sqrt[3]{\sigma_r \ln(\sqrt[3]{\sigma(r)})} = \frac{1}{\sqrt[3]{3}}\sqrt[3]{\sigma_r \ln(\sigma_r)}$$

for all r sufficiently large. Since $\sigma_r \geq \mathrm{diam}(P_{p_1p_2,\ldots,p_{\pi(r)}p_1})$ and $P_{p_1p_2,\ldots,p_{\pi(r)}p_1}$ admits (by assumption) an indecomposable tiling of period $p_1 \cdots p_{\pi(r)}$, we have

$$\begin{aligned} D'(\sigma_r) &\geq p_1 \cdots p_{\pi(r)} \\ &\geq \epsilon e^r \\ &\geq \epsilon e^{\frac{1}{\sqrt[3]{3}}\sqrt[3]{\sigma_r \ln(\sigma_r)}} \end{aligned}$$

for all $\epsilon < 1$, for all r sufficiently large. Thus, since for any $n \in \mathbb{N}$ there is some $r \in \mathbb{N}$ such that $n \leq \sigma_r \leq 4n$, there is a constant $c > 0$ such that

$$D'(n) \geq e^{c\sqrt[3]{n\ln(n)}}$$

for all n sufficiently large, as desired. $\square$

The main proof

We have so far reduced the proof of lower bound (3) to showing there exist minimal non-periodic k-dimensional CC2 arrays of size $2 \times \ldots \times 2$ for all k, say, greater than or equal to 3. We will index the entries of our $2 \times \ldots \times 2$ arrays as binary strings, like in the proofs of Proposition 2 and 5. We let

$$\delta_{i_1 i_2}^{j_1 j_2} = \delta_{i_1 j_1}\delta_{i_2 j_2}$$

where δ_{ij} is the Kronecker delta. We start with a simple lemma.

Lemma 1. *Let C be a $2 \times \ldots \times 2$ CC2 array of dimension $k \geq 3$ and let $\tau(C)$ be a $2 \times \ldots \times 2$ array of dimension $k-1$ given by*

$$(\tau(C))_{i_1 \ldots i_{k-1}} = C_{i_1 \ldots i_{k-1} i_{k-1}}$$

Then $\tau(C)$ is also a CC2 array.

Proof. It suffices to show that $\tau(C)$ is a CC2 array when C is an adjacent index co-slab. Say therefore that $C_{i_1 \ldots i_k} = \delta_{i_l i_{l+1}}^{z_1 z_2}$ where $z_1, z_2 \in \{0, 1\}$ and $1 \leq l \leq k$ (with $i_{k+1} = i_1$). If $1 \leq l \leq k-2$ then $(\tau(C))_{i_1 \ldots i_{k-1}}$ is also equal to $\delta_{i_l i_{l+1}}^{z_1 z_2}$, so $\tau(C)$ is an adjacent index co-slab and therefore a CC2 array. If $l = k-1$ then $(\tau(C))_{i_1 \ldots i_{k-1}} = \delta_{i_{k-1}}^{z_1} = \delta_{i_{k-2} i_{k-1}}^{0\, z_1} + \delta_{i_{k-2} i_{k-1}}^{1\, z_1}$ if $z_1 = z_2$ and is equal to 0 otherwise, so in either case $\tau(C)$ is a CC2 array. Lastly, if $l = k$ then $(\tau(C))_{i_1 \ldots i_{k-1}} = \delta_{i_{k-1} i_1}^{z_1 z_2}$ so $\tau(C)$ is again an adjacent index co-slab, completing the proof. $\square$

We now introduce our family $\{C^k : k \geq 3\}$ of minimal non-periodic CC2 arrays of size $2 \times \ldots \times 2$, where C^k has dimension k. The array C^k is defined by:

$$C_{i_1 \ldots i_k}^k = \left(\sum_{l=1}^{k-1} \delta_{i_l i_{l+1}}^{0\,1} \right) - \delta_{i_1 i_k}^{0\,1}. \tag{10}$$

Put more verbally, the value of $C_{i_1 \ldots i_k}^k$ is the number '01' substrings in the binary string $i_1 i_2 \ldots i_k$, minus one if the first and last characters of the string are respectively a 0 and a 1. The array C^3 is the one already shown in Fig. 4 (under the coordinatization of Proposition 2) whereas the array C^4 is the one shown in Fig. 10 and discussed in Proposition 5. It follows directly from (10) that C^k is a CC2 array and that C^k is integer-valued. We prove a few more easy things about the arrays C^k:

Proposition 7. *The array C^k is nonnegative for all $k \geq 3$.*

Proof. This simply follows from the fact that any binary string which starts with a '0' and ends with a '1' must contain a substring '01'. $\square$

Proposition 8. *The array C^k is non-periodic for all $k \geq 3$.*

Proof. Let $1 \leq l \leq k$. We need to find some $i_1, \ldots, i_{l-1}, i_{l+1}, \ldots, i_k \in \mathbb{Z}_2$ such that $C^k_{i_1 \ldots i_{l-1} 0 i_{l+1} \ldots i_k} \neq C^k_{i_1 \ldots i_{l-1} 1 i_{l+1} \ldots i_k}$. If $1 < l < k$ then we can take $i_1, \ldots, i_{l-1}, i_{l+1}, \ldots, i_k$ all equal to 0 since $0 = C^k_{0 \ldots 0} \neq C^k_{0 \ldots 010 \ldots 0} = 1$ if the '1' in the right-hand subscript appears neither as the first or last character. If $l = 1$ then we can take $i_2 = \ldots = i_{k-1} = 0$, $i_k = 1$ since $0 = C^k_{0 \ldots 01} \neq C^k_{10 \ldots 01} = 1$. Similarly if $l = k$ then we can take $i_1 = 1$, $i_2 = \ldots = i_{k-1} = 0$ since $0 = C^k_{10 \ldots 0} \neq C^k_{10 \ldots 01} = 1$. $\square$

Proposition 9. *Let τ be the projection from k-dimensional arrays of size $2 \times \ldots \times 2$ to $(k-1)$-dimensional arrays of size $2 \times \ldots \times 2$ defined in Lemma 1. Then $\tau(C^k) = C^{k-1}$ for all $k \geq 4$.*

Proof. This is simply because the number of '01' substrings in a binary string $i_1 \ldots i_{k-1}$ minus $\delta^{0\,1}_{i_1 i_{k-1}}$ is equal to the number of '01' substrings in the string $i_1 \ldots i_{k-1} i_{k-1}$ minus $\delta^{0\,1}_{i_1 i_{k-1}}$. $\square$

We have left to show that C^k is minimal for all $k \geq 3$. Note that C^k always contains some entries equal to 1, since for example $C^k_{010 \ldots 0} = 1$. It is therefore sufficient to show that any nonnegative CC2 array whose support is contained in the support of C^k is a scalar multiple of C^k. We will do this using an induction on k.

Theorem 2. *The array C^k is minimal for all $k \geq 3$.*

Proof. We prove by induction on k that any nonnegative CC2 array whose support is contained in the support of C^k is a scalar multiple of k. Our basis is the case $k = 3$, which was proved in Proposition 2 (the case $k = 4$ was proved in Proposition 5).

Therefore let $k \geq 4$ and let $A \geq 0$ be a k-dimensional $2 \times \ldots \times 2$ CC2 array whose support is contained in the support of C^k. Since by Lemma 1 $\tau(A)$ is a nonnegative $(k-1)$-dimensional CC2 array whose support is contained in the support of $\tau(C^k) = C^{k-1}$, it follows from the induction hypothesis that $\tau(A) = \lambda \tau(C^k)$ for some $\lambda \in \mathbb{R}$. Put another way, $A_{i_1 \ldots i_{k-2} 00} = \lambda C^k_{i_1 \ldots i_{k-2} 00}$ and $A_{i_1 \ldots i_{k-2} 11} = \lambda C^k_{i_1 \ldots i_{k-2} 11}$ for all $(i_1, \ldots, i_{k-2}) \in (\mathbb{Z}_2)^{k-2}$. We have left to show that $A_{i_1 \ldots i_{k-2} 01} = \lambda C^k_{i_1 \ldots i_{k-2} 01}$ and that $A_{i_1 \ldots i_{k-2} 10} = \lambda C^k_{i_1 \ldots i_{k-2} 10}$ for all $(i_1, \ldots, i_{k-2}) \in (\mathbb{Z}_2)^{k-2}$.

Assume that $A_{j_1 \ldots j_{k-2} 01} = \lambda C^k_{j_1 \ldots j_{k-2} 01}$ for some $(j_1, \ldots, j_{k-2}) \in (\mathbb{Z}_2)^{k-2}$. Let $j'_l = 1 - j_l$ for $1 \leq l \leq k-2$. We first wish to prove that $A_{j_1 \ldots j_{k-2} 01} = \lambda C^k_{j_1 \ldots j_{k-2} 01} \implies (A_{j_1 \ldots j_{l-1} j'_l j_{l+1} \ldots j_{k-2} 01} = \lambda C^k_{j_1 \ldots j_{l-1} j'_l j_{l+1} \ldots j_{k-2} 01}$ for all $1 \leq l \leq k-2)$. We distinguish between the cases $1 \leq l \leq k-3$ and $2 \leq l \leq k-2$ (the two cases overlap when $k \geq 5$, which does not bother us).

Assume first that $2 \leq l \leq k-2$. Let B be any k-dimensional adjacent index co-slab of size $2 \times \ldots \times 2$. It is easy to check that

$$B_{j_1 \ldots j_{k-2}00} - B_{j_1 \ldots j_{l-1}j'_l j_{l+1} \ldots j_{k-2}00} - B_{j_1 \ldots j_{k-2}01} + B_{j_1 \ldots j_{l-1}j'_l j_{l+1} \ldots j_{k-2}01} = 0$$

so it follows that

$$A_{j_1 \ldots j_{k-2}00} - A_{j_1 \ldots j_{l-1}j'_l j_{l+1} \ldots j_{k-2}00} - A_{j_1 \ldots j_{k-2}01} + A_{j_1 \ldots j_{l-1}j'_l j_{l+1} \ldots j_{k-2}01} = 0 \tag{11}$$

and that

$$\lambda C^k_{j_1 \ldots j_{k-2}00} - \lambda C^k_{j_1 \ldots j_{l-1}j'_l j_{l+1} \ldots j_{k-2}00} - \lambda C^k_{j_1 \ldots j_{k-2}01} + \lambda C^k_{j_1 \ldots j_{l-1}j'_l j_{l+1} \ldots j_{k-2}01} = 0. \tag{12}$$

But we know that $A_{j_1 \ldots j_{k-2}00} = \lambda C^k_{j_1 \ldots j_{k-2}00}$, $A_{j_1 \ldots j_{l-1}j'_l j_{l+1} \ldots j_{k-2}00} = \lambda C^k_{j_1 \ldots j_{l-1}j'_l j_{l+1} \ldots j_{k-2}00}$ and we are assuming $A_{j_1 \ldots j_{k-2}01} = \lambda C^k_{j_1 \ldots j_{k-2}01}$ so (11), (12) imply that $A_{j_1 \ldots j_{l-1}j'_l j_{l+1} \ldots j_{k-2}01} = \lambda C^k_{j_1 \ldots j_{l-1}j'_l j_{l+1} \ldots j_{k-2}01}$, as desired. The case $1 \leq l \leq k-3$ is treated similarly by observing that

$$B_{j_1 \ldots j_{k-2}11} - B_{j_1 \ldots j_{l-1}j'_l j_{l+1} \ldots j_{k-2}11} - B_{j_1 \ldots j_{k-2}01} + B_{j_1 \ldots j_{l-1}j'_l j_{l+1} \ldots j_{k-2}01} = 0$$

for all $2 \times \ldots \times 2$ k-dimensional adjacent index co-slabs B when $1 \leq l \leq k-3$.

We have thus shown that if $A_{j_1 \ldots j_{k-2}01} = \lambda C^k_{j_1 \ldots j_{k-2}01}$ for some $(j_1, \ldots, j_{k-2}) \in (\mathbb{Z}_2)^{k-2}$ then $A_{j_1 \ldots j_{l-1}j'_l j_{l+1} \ldots j_{k-2}01} = \lambda C^k_{j_1 \ldots j_{l-1}j'_l j_{l+1} \ldots j_{k-2}01}$ for all $1 \leq l \leq k-2$. To put things more graphically, say that a coordinate $(i_1, \ldots, i_{k-2}) \in (\mathbb{Z}_2)^{k-2}$ *has smallpox* if $A_{i_1 \ldots i_{k-2}01} = \lambda C^k_{i_1 \ldots i_{k-2}01}$. What we have just shown is that any coordinate in $(\mathbb{Z}_2)^{k-2}$ that differs in a single position from a coordinate with smallpox also has smallpox. Thus if a single coordinate in $(\mathbb{Z}_2)^{k-2}$ has smallpox, then all coordinates have smallpox. However we do know of one coordinate in $(\mathbb{Z}_2)^{k-2}$ with smallpox: namely the coordinate $(0, \ldots, 0)$, since $A_{0 \ldots 01} = \lambda C_{0 \ldots 01} = 0$. Thus $A_{i_1 \ldots i_{k-2}01} = \lambda C^k_{i_1 \ldots i_{k-2}01}$ for all $(i_1, \ldots, i_{k-2}) \in (\mathbb{Z}_2)^{k-2}$. A symmetric argument using the fact that $A_{1 \ldots 10} = \lambda C_{1 \ldots 10} = 0$ shows that $A_{i_1 \ldots i_{k-2}10} = \lambda C^k_{i_1 \ldots i_{k-2}10}$ for all $(i_1, \ldots, i_{k-2}) \in (\mathbb{Z}_2)^{k-2}$. Thus $A = \lambda C$ and we are done. $\square$

Further remarks

If k is even then $C^k_{0101 \ldots 01} = k/2 - 1$ and if k is odd $C^k_{0101 \ldots 11} = (k-1)/2 - 1$. Thus the maximum entry of C^k grows arbitrarily large with k, and it follows from our construction that there exist indecomposable tilings in which individual tile translates appear with arbitrarily high multiplicity (that is, for every $m \in \mathbb{N}$ there is some tile A and some indecomposable A-tiling T such that $T[i] \geq m$ for some $i \in \mathbb{Z}$). Even more, if we inflate C^k to an array of size $p_1 \times p_2 \times \cdots \times p_k$ such that $(p_1 - 1)(p_2 - 1) \cdots (p_k - 1)$ entries of the inflate are equal to the highest entry of C^k (which is obviously possible to do) then we obtain an indecomposable tiling which is "almost everywhere thick". We summarize this with a theorem:

Theorem 3. *For every $\epsilon > 0$ and every $m \in \mathbb{N}$ there some tile A and some indecomposable A-tiling T such that* $\lim_{c \to \infty} \frac{1}{c} |\{i : 1 \leq i \leq c, T[i] \geq m\}| > 1 - \epsilon$.

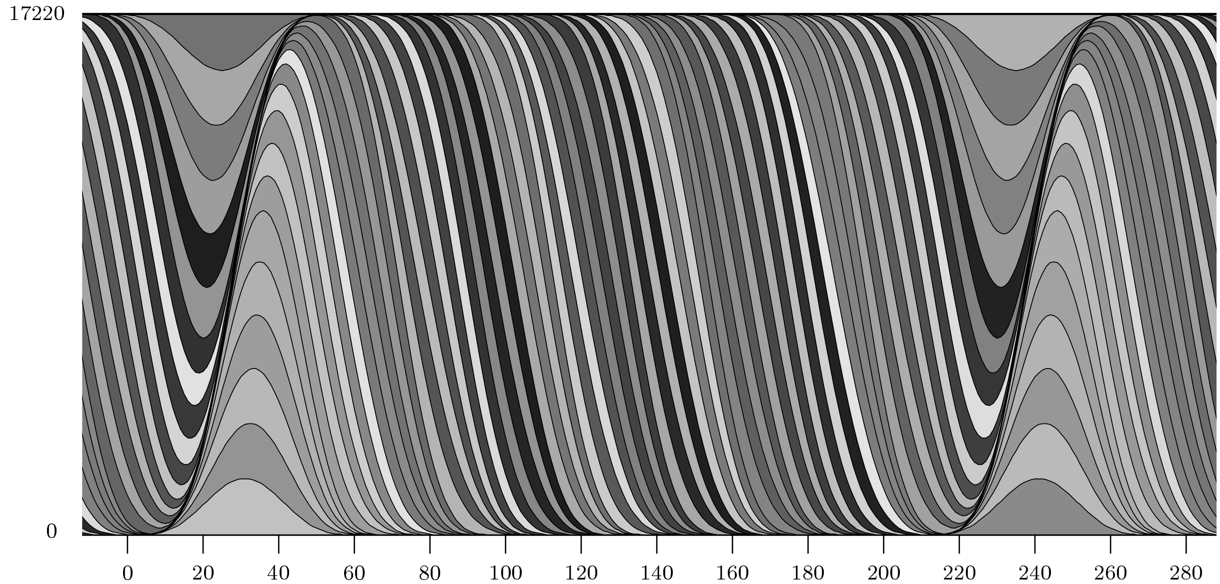

Figure 16: A $P_{2\cdot 7,7\cdot 3,3\cdot 5,5\cdot 2}$-tiling obtained from an inflate of C^4. The tile has diameter 56 and the tiling has period 210.

k	$\operatorname{diam}(P_{\dots})$	M
3	28	30
4	56	210
5	116	2310
10	1022	$> 10^9$
15	4202	$> 10^{17}$
20	11502	$> 10^{26}$
30	48142	$> 10^{46}$

Table 1

Some constructions of minimal cyclotomic arrays with arbitrarily large entries can be found in [8].

The ratio $M/\operatorname{diam}(P_{n_1,\dots,n_k})$ is slightly better maximized if we put $n_1 = p_1p_k, n_2 = p_kp_2, n_3 = p_2p_{k-1}, n_4 = p_{k-1}p_3, \dots$ (etc.) instead of $n_1 = p_1p_2, n_2 = p_2p_3, n_3 = p_3p_4, \dots$ where $p_1, \dots, p_k$ are the first k primes (this small change does not significantly affect the asymptotical computation). Fig. 16 shows, for the reader's amusement and for the satisfaction of our own curiosity, a $P_{2\cdot 7,7\cdot 3,3\cdot 5,5\cdot 2}$-tiling of period 210 obtained from an inflate of C^4. Table 1 also compares the period $M = p_1 \cdots p_k$ with the diameter of $P_{p_1p_k,p_kp_2,\dots,p_{\lceil (k+1)/2\rceil}p_1}$ for some larger values of k. Numerical experiments with $k \sim 5000$ suggest that the constant c of Proposition 6 may be taken greater than 1, but we cannot vouch for this value.

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