A Relationship Between the Major Index For Tableaux and the Charge Statistic For Permutations

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Submitted: Jul 13, 2005; Accepted: Aug 30, 2005; Published: Sep 5, 2005
Mathematics Subject Classifications: 05A15, 05E10

Abstract

The widely studied q-polynomial $f^\lambda(q)$, which specializes when $q = 1$ to f^λ, the number of standard Young tableaux of shape λ, has multiple combinatorial interpretations. It represents the dimension of the unipotent representation S_q^λ of the finite general linear group $GL_n(q)$, it occurs as a special case of the Kostka-Foulkes polynomials, and it gives the generating function for the *major index* statistic on standard Young tableaux. Similarly, the q-polynomial $g^\lambda(q)$ has combinatorial interpretations as the q-multinomial coefficient, as the dimension of the permutation representation M_q^λ of the general linear group $GL_n(q)$, and as the generating function for both the *inversion* statistic and the *charge* statistic on permutations in W_λ. It is a well known result that for λ a partition of n, $dim(M_q^\lambda) = \Sigma_\mu K_{\mu\lambda} dim(S_q^\mu)$, where the sum is over all partitions μ of n and where the Kostka number $K_{\mu\lambda}$ gives the number of semistandard Young tableaux of shape μ and content λ. Thus $g^\lambda(q) - f^\lambda(q)$ is a q-polynomial with nonnegative coefficients. This paper gives a combinatorial proof of this result by defining an injection f from the set of standard Young tableaux of shape λ, $SYT(\lambda)$, to W_λ such that $maj(T) = ch(f(T))$ for $T \in SYT(\lambda)$.

Key words: Young tableaux, permutation statistics, inversion statistic, charge statistic, Kostka polynomials.

1 Introduction

For λ any partition of n, f^λ gives the number of standard Young tableaux of shape λ. The q-version of f^λ is a polynomial that has many important combinatorial interpretations. In particular, $f^\lambda(q)$ is known to give the dimension of the unipotent representation S_q^λ

of the finite general linear group $GL_n(q)$. The polynomial $f^\lambda(q)$ can be computed as the generating function for the major index $maj(T)$ on the set of standard Young tableaux of shape λ, $SYT(\lambda)$.

$$f^\lambda(q) = \sum_{T \in SYT(\lambda)} q^{maj(T)}$$

In addition, the q-multinomial coefficient

$$g^\lambda(q) = \begin{bmatrix} & & n & & \\ \lambda_1, & \lambda_2, & \lambda_3, & \cdots & \lambda_k \end{bmatrix} = \frac{[n!]}{[\lambda_1!][\lambda_2!][\lambda_3!]\cdots[\lambda_k!]}$$

is known to give the dimension of the permutation representation M_q^λ of $GL_n(q)$. The polynomial $g^\lambda(q)$ also has a combinatorial interpretation as

$$g^\lambda(q) = \sum_{\pi \in W_\lambda} q^{inv(\pi)}$$

where W_λ is the subset of permutations in S_n of type λ and $inv(\pi)$ is the inversion statistic on π. The following is a well-known result on the representation of $GL_n(q)$:

Proposition 1. *For λ a partition of n,*

$$dim(M_q^\lambda) = \sum_{\mu \vdash n} K_{\mu\lambda} dim(S_q^\mu),$$

where $K_{\mu\lambda}$ is the Kostka number which counts the number of semi-standard tableaux of shape μ and content λ.

Thus we have

$$g^\lambda(q) = \sum_{\mu \vdash n} K_{\mu\lambda} f^\mu(q)$$

and in particular, since $K_{\lambda\lambda} = 1$ for all λ,

$$g^\lambda(q) = f^\lambda(q) + \sum_{\substack{\mu \vdash n \\ \mu \neq \lambda}} K_{\mu\lambda} f^\mu(q).$$

Thus

$$g^\lambda(q) - f^\lambda(q) = \sum_{\substack{\mu \vdash n \\ \mu \neq \lambda}} K_{\mu\lambda} f^\mu(q)$$

is a q-polynomial with non-negative coefficients. This implies that

$$g^\lambda(q) - f^\lambda(q) = \sum_{\pi \in W_\lambda} q^{inv(\pi)} - \sum_{T \in SYT(\lambda)} q^{maj(T)}$$

is a q-polynomial with non-negative coefficients. It is natural, then, to seek an injection from standard Young tableaux of shape λ to permutations in W_λ which takes the statistic

$maj(T)$ to the statistic $inv(\pi)$. Cho [2] has recently given such an injection for λ a two part partition, but the given injection does not hold for general λ and finding such an injection for all partitions λ is left as an open question. In Section 3 of this paper, we give explicit proofs for some known but not well documented results on the charge statistic, $ch(\pi)$, namely

$$\sum_{\pi\in W_\lambda} q^{inv(\pi)} = \sum_{\pi\in W_\lambda} q^{ch(\pi)}.$$

This implies that

$$g^\lambda(q) - f^\lambda(q) = \sum_{\pi\in W_\lambda} q^{ch(\pi)} - \sum_{T\in SYT(\lambda)} q^{maj(T)}.$$

The main result of this paper, in Section 4, is to answer Cho's open questions by giving a general injection h from $SYT(\lambda)$ to W_λ which takes $maj(T)$ to $ch(h(T))$. Section 2 of the paper contains necessary background and definitions.

2 Definitions and Background

We say $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a *partition of* n if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ and $\sum_{i=1}^k \lambda_i = n$. A partition is described pictorially by its *Ferrers diagram*, an array of n dots into k left-justified rows with row i containing λ_i dots for $1 \leq i \leq k$. For example, the Ferrers diagram for the partition $\lambda = (6, 5, 3, 3, 1)$ is:

$$T \;=\; \begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & & & \\ \bullet & \bullet & \bullet & & & \\ \bullet & & & & & \end{array}$$

A *standard Young tableau of shape* λ is a filling of the Ferrers diagram for λ with the numbers $1, 2, \ldots, n$ such that rows are strictly increasing from left to right and columns are strictly increasing from top to bottom. One example of a standard Young tableau for the partition $\lambda = 65331$ is shown below:

$$T \;=\; \begin{array}{cccccc} 1 & 2 & 6 & 7 & 9 & 14 \\ 3 & 5 & 8 & 15 & 17 & \\ 4 & 11 & 12 & & & \\ 10 & 16 & 18 & & & \\ 13 & & & & & \end{array}$$

Let f^λ denote the number of standard Young tableaux of shape λ.

For a standard Young tableau T, the major index of T is given by

$$maj(T) = \sum_{i\in D(T)} i$$

where $D(T) = \{\ i \mid i+1 \text{ is in a row strictly below that of } i \text{ in T}\}$. For the tableau T given in the previous example, $D(T) = \{2, 3, 7, 9, 12, 14, 15, 17\}$ and $maj(T) = 79$.

For a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$, define an *inversion* to be a pair (i, j) such that $i < j$ and $\pi_i > \pi_j$. Then the *inversion statistic*, $inv(\pi)$, is the total number of inversions in π.

For example, for $\pi = \ 3\ \ 2\ \ 8\ \ 5\ \ 7\ \ 4\ \ 6\ \ 1\ \ 9\ $, $inv(\pi) = 15$ since each of the pairs $(1, 2)$, $(1, 3)$, $(1, 4)$, $(1, 5)$, $(1, 6)$, $(1, 7)$, $(1, 8)$, $(2, 3)$, $(4, 5)$, $(4, 7)$, $(4, 8)$, $(5, 8)$, $(6, 7)$, $(6, 8)$, $(7, 8)$ is an inversion.

Let W_λ be the subset of S_n such that

$$\pi_1 < \pi_2 < \cdots < \pi_{\lambda_1}$$

$$\pi_{\lambda_1+1} < \pi_{\lambda_1+2} < \cdots < \pi_{\lambda_1+\lambda_2}$$

$$\cdots$$

$$\pi_{\lambda_1+\lambda_2+\cdots+\lambda_{k-1}+1} < \pi_{\lambda_1+\lambda_2+\cdots+\lambda_{k-1}+2} < \cdots < \pi_n$$

For example, for $\lambda = (4, 3, 3, 1)$,

$$\pi \quad = \quad 2 \quad 4 \quad 5 \quad 9 \quad 1 \quad 3 \quad 10 \quad 6 \quad 8 \quad 11 \quad 7$$

is an element of W_{4331}.

We will use the definition of W_λ for λ any combination of n, not just for λ a partition of n. Note that there is no required relationship between π_{λ_1} and π_{λ_1+1}, between $\pi_{\lambda_1+\lambda_2}$ and $\pi_{\lambda_1+\lambda_2+1}$, and so on. For any $W_\lambda = W_{\lambda_1,\lambda_2,\ldots,\lambda_k}$, define $W_{\bar{\lambda_i}} = W_{\lambda_1,\lambda_2,\ldots,\lambda_i-1,\ldots,\lambda_k}$ for $1 \le i \le k$.

Let π be a permutation in S_n. For any i in the permutation, define the *charge value of* i, $chv(i)$, recursively as follows:

$$\begin{aligned} &chv(1) = 0 \\ &chv(i) = chv(i-1) \text{ if } i \text{ is to the right of } i-1 \text{ in } \pi \\ &chv(i) = chv(i-1) + 1 \text{ if } i \text{ is to the left of } i-1 \text{ in } \pi \end{aligned}$$

Now for $\pi \in S_n$, define the *charge of* π, $ch(\pi)$, to be

$$ch(\pi) = \sum_{i=1}^{n} chv(i).$$

In the following example of a permutation $\pi = 328574619$ with $ch(\pi) = 25$, the charge values of each element are given below the permutation:

$$\pi \quad = \quad \begin{matrix} 3 & 2 & 8 & 5 & 7 & 4 & 6 & 1 & 9 \\ {\scriptstyle 2} & {\scriptstyle 1} & {\scriptstyle 5} & {\scriptstyle 3} & {\scriptstyle 4} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 0} & {\scriptstyle 5} \end{matrix}$$

The definition of the charge statistic was first given by Lascoux and Schützenberger [8].

For each element $i \in \pi$, define the *charge contribution of* i, $cc(i)$, to be zero if $i = 1$ or i lies to the right of $i-1$ in π and to be $n-i+1$ if i lies to the left of $i-1$ in π. It is easy to check that $ch(\pi) = \sum_i cc(i)$. For the previous example, the charge contribution of each element is given below that element:

$$\pi \;=\; \begin{matrix} 3 & 2 & 8 & 5 & 7 & 4 & 6 & 1 & 9 \\ {\scriptstyle 7} & {\scriptstyle 8} & {\scriptstyle 2} & {\scriptstyle 5} & {\scriptstyle 3} & {\scriptstyle 0} & {\scriptstyle 0} & {\scriptstyle 0} & {\scriptstyle 0} \end{matrix}$$

3 Charge and Inv

Many of the known results on the charge statistic are implicitly given in a number of papers or unpublished manuscripts [1] [5] [6] [7]. The goal of this section is to give explicit proofs of those results which are used in this paper as an aid to the interested reader.

Lascoux and Schützenberger [8] proved the following lemma:

Lemma 1. *For $x \in \{2, \ldots, n\}$ and $x\sigma \in S_n$, $ch(x\sigma) = ch(\sigma x) + 1$.*

This result immediately gives

Lemma 2.

$$\sum_{\pi \in S_n} q^{ch(\pi)} = (1 + q + q^2 + \cdots + q^{n-1}) \sum_{\sigma \in S_{n-1}} q^{ch(\sigma)}.$$

Proof. Let $\sigma \in S_{n-1}$, so $\sigma = \sigma_1\sigma_2\cdots\sigma_{n-1}$. Rewrite σ using the numbers $2, 3, \ldots, n$ by letting $\tilde{\sigma_i} = \sigma_i + 1$ for every i. Let $\pi = 1\tilde{\sigma_1}\tilde{\sigma_2}\cdots\widetilde{\sigma_{n-1}}$. Then $\pi \in S_n$ and $ch(\pi) = ch(\sigma)$. By Lemma 1,

$$\begin{aligned} ch(\widetilde{\sigma_{n-1}}1\tilde{\sigma_1}\tilde{\sigma_2}\cdots\widetilde{\sigma_{n-2}}) &= ch(\pi) + 1 \\ &= ch(\sigma) + 1. \end{aligned}$$

Similarly,

$$\begin{aligned} ch(\widetilde{\sigma_{n-2}}\widetilde{\sigma_{n-1}}1\tilde{\sigma_1}\cdots\widetilde{\sigma_{n-3}}) &= ch(\pi) + 2 \\ &= ch(\sigma) + 2 \\ &\cdots \\ ch(\tilde{\sigma_1}\tilde{\sigma_2}\cdots\widetilde{\sigma_{n-1}}1) &= ch(\pi) + n - 1 \\ &= ch(\sigma) + n - 1 \end{aligned}$$

Thus

$$\sum_{\pi \in S_n} q^{ch(\pi)} = (1 + q + \cdots + q^{n-1}) \sum_{\sigma \in S_{n-1}} q^{ch(\sigma)}.$$

□

It is well-known that the inversion statistic satisfies the same recurrence.

Lemma 3.
$$\sum_{\pi \in S_n} q^{inv(\pi)} = (1 + q + q^2 + \cdots + q^{n-1}) \sum_{\sigma \in S_{n-1}} q^{inv(\sigma)}.$$

Proof. For details about the inversion statistic, one can consult [3] or [4]. $\square$

The following theorem [7] follows immediately from the previous Lemmas once the initial conditions are checked.

Theorem 1.
$$\sum_{\pi \in S_n} q^{ch(\pi)} = \sum_{\pi \in S_n} q^{inv(\pi)}.$$

We now give details that the charge statistic and the inversion statistic not only have the same generating function on S_n, but they in fact have the same generating function on W_λ.

Lemma 4. *For $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_k)$ a combination of n for any integer n,*
$$\sum_{\pi \in W_{\lambda_1, \lambda_2, \ldots, \lambda_k}} q^{inv(\pi)} = \sum_{\sigma \in W_{\lambda_2, \lambda_3, \ldots, \lambda_k, \lambda_1}} q^{inv(\sigma)}.$$

Proof. Let $\pi = \pi_1 \pi_2 \ldots \pi_n \in W_{\lambda_1, \lambda_2, \ldots, \lambda_k}$. Create $\sigma = \sigma_1 \sigma_2 \ldots \sigma_k \in W_{\lambda_2, \lambda_3, \ldots, \lambda_k, \lambda_1}$ in the following manner. For $1 \leq i \leq \lambda_1$, let $\sigma_{n+1-i} = n + 1 - \pi_i$. Next, relabel the elements π_{λ_1+1} through π_n with the remaining $n - \lambda_1$ numbers, in the same relative order. For example, if
$$\pi \;=\; 2 \quad 7 \quad 11 \quad 3 \quad 6 \quad 1 \quad 10 \quad 12 \quad 15 \quad 5 \quad 8 \quad 14 \quad 4 \quad 9 \quad 13$$

in $W_{3,2,4,3,3}$, we have
$$\sigma_{15} = 16 - \pi_1 = 14$$
$$\sigma_{14} = 16 - \pi_2 = 9$$
$$\sigma_{13} = 16 - \pi_3 = 5$$

and the numbers
$$\pi_4 \quad \pi_5 \quad \cdots \quad \pi_{15} \;=\; 3 \quad 6 \quad 1 \quad 10 \quad 12 \quad 15 \quad 5 \quad 8 \quad 14 \quad 4 \quad 9 \quad 13$$

are relabeled in the same relative order using the numbers $[n] - \{5, 9, 14\}$ to give
$$\sigma_1 \quad \sigma_2 \quad \cdots \quad \sigma_{n-\lambda_1} \;=\; 2 \quad 6 \quad 1 \quad 10 \quad 11 \quad 15 \quad 4 \quad 7 \quad 13 \quad 3 \quad 8 \quad 12$$

and $\sigma \in W_{2,4,3,3,3}$. Thus
$$\sigma \;=\; 2 \quad 6 \quad 1 \quad 10 \quad 11 \quad 15 \quad 4 \quad 7 \quad 13 \quad 3 \quad 8 \quad 12 \quad 5 \quad 9 \quad 14.$$

It is easy to see that σ is unique and that one can reverse the process to take any $\sigma \in W_{\lambda_2,\lambda_3,\ldots,\lambda_k,\lambda_1}$ to a unique $\pi \in W_{\lambda_1,\lambda_2,\ldots,\lambda_k}$, so this process gives a bijection between $W_{\lambda_1,\lambda_2,\ldots,\lambda_k}$ and $W_{\lambda_2,\lambda_3,\ldots,\lambda_k,\lambda_1}$.

Now we prove that $inv(\pi) = inv(\sigma)$. Since $\pi \in W_{\lambda_1,\lambda_2,\ldots,\lambda_k}$, we have $\pi_1 < \pi_2 < \cdots < \pi_{\lambda_1}$ so there are no inversions between elements $\pi_1, \pi_2, \cdots, \pi_{\lambda_1}$. Similarly, since $\sigma_{n+1-i} = n+1-\pi_i$ we have $\sigma_{n-\lambda_1+1} < \sigma_{n-\lambda_1+2} < \cdots < \sigma_n$ so there are no inversions between elements in $\sigma_{n-\lambda_1+1}, \sigma_{n-\lambda_1+2}, \cdots, \sigma_n$. Since $\sigma_1\sigma_2\cdots\sigma_{n-\lambda_1}$ are in the same relative order as $\pi_{\lambda_1+1}\pi_{\lambda_1+2}\cdots\pi_n$, the number of inversions between elements in these two parts is the same.

Now suppose that $\pi_i = j$ for $1 \le i \le \lambda_1$. Then π_i forms inversions with $(j-1)-(i-1) = j-i$ elements in $\pi_{\lambda_1+1}\pi_{\lambda_1+2}\cdots\pi_n$ since there are $j-1$ total elements less than j and $i-1$ of them lie to the left of π_i in π. If $\pi_i = j$ then $\sigma_{n+1-i} = n+1-j$. There are $j-1$ total elements bigger than $n+1-j$ and $i-1$ of them lie to the right of σ_{n+1-j} in σ since there $i-1$ elements to the left of $\pi_i = j$ in π. This means that σ_{n+1-j}, like π_i, forms inversions with $(j-1)-(i-1) = j-i$ elements in $\sigma_1\sigma_2\cdots\sigma_{n-\lambda_1}$.

□

Lemma 5. *For $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_k)$ a combination of n for any integer n,*

$$\sum_{\pi \in W_\lambda} q^{inv(\pi)} = \left(\sum_{\sigma \in W_{\bar{\lambda_1}}} q^{inv(\sigma)}\right) + \left(q^{\lambda_1} \sum_{\sigma \in W_{\bar{\lambda_2}}} q^{inv(\sigma)}\right) + \cdots$$
$$+ \left(q^{\lambda_1+\lambda_2+\cdots+\lambda_{k-1}} \sum_{\sigma \in W_{\bar{\lambda_k}}} q^{inv(\sigma)}\right).$$

Proof. Again, for the details of results on the inversion statistic, one can consult [3] or [4]. □

Lemma 6. *For $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_k)$ a combination of n for any integer n,*

$$\sum_{\pi \in W_\lambda} q^{ch(\pi)} = \left(\sum_{\sigma \in W_{\lambda_1-1,\lambda_2,\ldots,\lambda_k}} q^{ch(\sigma)}\right) + \left(q^{\lambda_1} \sum_{\sigma \in W_{\lambda_2-1,\lambda_3,\ldots,\lambda_k,\lambda_1}} q^{ch(\sigma)}\right) + \cdots$$
$$+ \left((q^{\lambda_1+\cdots+\lambda_{k-1}}) \sum_{\sigma \in W_{\lambda_k-1,\lambda_1,\ldots,\lambda_{k-1}}} q^{ch(\sigma)}\right).$$

Proof. Let $\pi \in W_\lambda$. Suppose the 1 in π lies in block λ_i, so

$$\pi = \pi_1\pi_2\cdots\pi_{\lambda_1+\lambda_2+\cdots+\lambda_{i-1}}1\pi_{\lambda_1+\lambda_2+\cdots+\lambda_{i-1}+2}\cdots\pi_n.$$

By Lemma 1,

$$ch(\pi) = ch(1\pi_{\lambda_1+\lambda_2+\cdots+\lambda_{i-1}+2}\cdots\pi_n\pi_1\pi_2\cdots\pi_{\lambda_1+\lambda_2+\cdots+\lambda_{i-1}}) + \lambda_1 + \lambda_2 + \cdots + \lambda_{i-1}.$$

To form $\sigma \in W_{\lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_k, \lambda_1, \ldots, \lambda_{i-1}}$, we now remove the initial 1 and relabel each of the remaining π_i with $\pi_i - 1$. Since we have removed an initial 1, the charge of

$$1\pi_{\lambda_1+\lambda_2+\cdots+\lambda_{i-1}+2} \cdots \pi_n \pi_1 \pi_2 \cdots \pi_{\lambda_1+\lambda_2+\cdots+\lambda_{i-1}}$$

is equal to the charge of the newly formed σ. Thus for each $\pi \in W_\lambda$ with a 1 in the λ_i block and σ formed in this manner,

$$ch(\pi) = ch(\sigma) + (\lambda_1 + \lambda_2 + \cdots + \lambda_{i-1}).$$

which gives the desired result. □

Theorem 2. *For $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_k)$ a combination of n for any integer n,*

$$\sum_{\pi \in W_\lambda} q^{inv(\pi)} = \sum_{\pi \in W_\lambda} q^{ch(\pi)}.$$

Proof. This result follows immediately by induction from Lemmas 4, 5 and 6. □

4 An Injection from $SYT(\lambda)$ to W_λ

From Section 1, we have that $g^\lambda(q) - f^\lambda(q) = \sum_{\pi \in W_\lambda} q^{ch(\pi)} - \sum_{\pi \in SYT(\lambda)} q^{maj(T)}$ is a polynomial with non-negative coefficients. We will now define an injection h from $SYT(\lambda)$ to W_λ such that $maj(T) = ch(h(T))$. Let $T \in SYT(\lambda)$. Write down the elements in T by first reading the top row of T from right to left, then the second row of T from right to left, and so on until reaching the bottom row. Call this permutation σ. For example, if

$$T = \begin{array}{cccc} 1 & 2 & 3 & 6 \\ 4 & 8 & 9 & \\ 5 & & & \\ 7 & & & \end{array}$$

then $\sigma = 632198457$. To create $\pi \in W_\lambda$, let $\pi_i = n - \sigma_i + 1$. In the example, $\pi = 478912653$ and $\pi \in W_{4311}$. Let $h(T) = \pi$. Note that for a given T, $h(T)$ is uniquely defined. Since each row of T is strictly increasing, then the first λ_1 elements of σ are strictly decreasing, the next λ_2 elements of σ are strictly decreasing, and so on. Thus when π is formed, the first λ_i elements of π are strictly increasing, the next λ_2 elements of π are strictly increasing, and so on, so $\pi \in W_\lambda$.

Theorem 3. *For $T \in SYT(\lambda)$, $maj(T) = ch(h(T))$.*

Proof. We will prove that if $i \in D(T)$, then the charge contribution of $n - i + 1$ in $h(T)$ is equal to i. In addition, if i is not in $D(T)$, then the charge contribution of $n - i + 1$ in $h(T)$ is equal to 0.

Let $i \in D(T)$. Then i lies in a row strictly above that of $i + 1$ in T. This implies that i lies to the left of $i + 1$ in σ, and thus $n - i + 1$ lies to the left of $n - (i + 1) + 1 = n - i$

in π. By the definition of charge contribution, we find that since $n-i+1$ lies to the left of $n-i$ the charge contribution of $n-i+1$ is equal to $n-(n-i+1)-1=i$.

Suppose $i \notin D(T)$. Then i either lies in a row below $i+1$ in T or they lie in the same row, in which case i lies to the left of $i+1$. In either case, i lies to the right of $i+1$ in σ and thus $n-i+1$ lies to the right of $n-(i+1)+1=n-i$ in π. By the definition of charge contribution, we find that the charge contribution of $n-i+1$ is equal to zero.

Since $maj(T)=\sum_{\{i \in D(T)\}} i$ and $ch(\pi)=\sum_i cc(i)$, we have that $maj(T)=ch(h(T))$. $\square$

In the previous example, $D(T)=\{3,4,6\}$ so $maj(T)=13$ and $ch(h(T))=ch(478912653)$ which is also 13.

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