The Generalized Schröder Theory

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Abstract

While the standard Catalan and Schröder theories both have been extensively studied, people have only begun to investigate higher dimensional versions of the Catalan number (see, say, the 1991 paper of Hilton and Pedersen, and the 1996 paper of Garsia and Haiman). In this paper, we study a yet more general case, the higher dimensional Schröder theory. We define m-Schröder paths, find the number of such paths from $(0,0)$ to (mn,n), and obtain some other results on the m-Schröder paths and m-Schröder words. Hoping to generalize classical q-analogue results of the ordinary Catalan and Schröder numbers, such as in the works of Fürlinger and Hofbauer, Cigler, and Bonin, Shapiro and Simion, we derive a q-identity which would welcome a combinatorial interpretation. Finally, we introduce the (q,t)-m-Schröder polynomial through "m-parking functions" and relate it to the m-Shuffle Conjecture of Haglund, Haiman, Loehr, Remmel and Ulyanov.

1 Introduction

Throughout this paper we use the standard notation

$$[n] := (1-q^n)/(1-q), [n]! := [1][2]\cdots[n], \begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}$$

for the q-analogue of the integer n, the q-factorial, and the q-binomial coefficient and $(a)_n := (1-a)(1-qa)\cdots(1-q^{n-1}a)$ for the q-rising factorial. Sometimes it is necessary to write the base q explicitly as in $[n]_q$, $[n]!_q$, $\begin{bmatrix} n \\ k \end{bmatrix}_q$ and $(a;q)_n$, but we omit q in this paper since it is clear from the context. When $i+j+k=n$, $\begin{bmatrix} n \\ i,j,k \end{bmatrix} := \frac{[n]!}{[i]![j]![k]!}$ represents the q-trinomial coefficient.

Definition 1.1 *A* Dyck path *of order n is a lattice path from $(0,0)$ to (n,n) that never goes below the main diagonal $\{(i,i), 0 \le i \le n\}$, with steps $(0,1)$ (or NORTH, for brevity* N*) and $(1,0)$ (or EAST, for brevity* E*). Let $\mathcal{D}_n$ denote the set of all Dyck paths of order n.*

An example of a Dyck path of order 6 with area vector $(1,0,0,1,1,0)$ is illustrated in Figure 1. The number of Dyck paths of order n is the Catalan number, $C_n = \frac{1}{n+1}\binom{2n}{n}$.

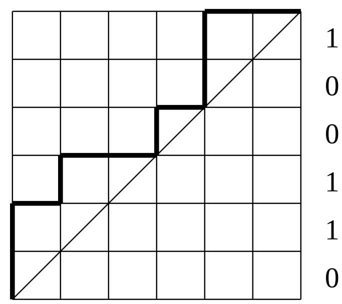

Figure 1: A Dyck path $\Pi \in \mathcal{D}_6$ with area(Π)=3.

Definition 1.2 *A* Schröder path *of order n and with d diagonal steps is a lattice path from $(0,0)$ to (n,n) that never goes below the main diagonal $\{(i,i), 0 \le i \le n\}$, with $(0,1)$ (or NORTH), $(1,0)$ (or EAST) and exactly d (1,1) (or Diagonal) steps. Let $\mathcal{S}_{n,d}$ denote the set of all Schröder paths of order n and with d diagonal steps.*

A Schröder path in $\mathcal{S}_{4,4}$ is illustrated in Figure 2. The number of Schröder paths of order n and with d diagonal steps is counted by

$$S_{n,d} = \binom{2n-d}{d} C_n = \frac{1}{n-d+1}\binom{2n-d}{d, n-d, n-d}.$$

While the standard Catalan and Schröder theories both have been extensively studied, people have only begun to investigate higher dimensional versions of the Catalan number (see [11] and [6]). In this paper, we study a yet more general case, namely the higher dimensional Schröder theory. We introduce and derive results about the m-Schröder paths and words.

2 m-Schröder Paths and m-Schröder Number

Now let's introduce the notions of generalized Dyck and Schröder paths.

Definition 2.1 *An m-*Dyck path *of order n is a lattice path from $(0,0)$ to (mn,n) which never goes below the main diagonal $\{(mi,i) : 0 \le i \le n\}$, with steps $(0,1)$ (or NORTH, for brevity* N*) and $(1,0)$ (or EAST, for brevity* E*). Let $\mathcal{D}_n^m$ denote the set of all m-Dyck paths of order n.*

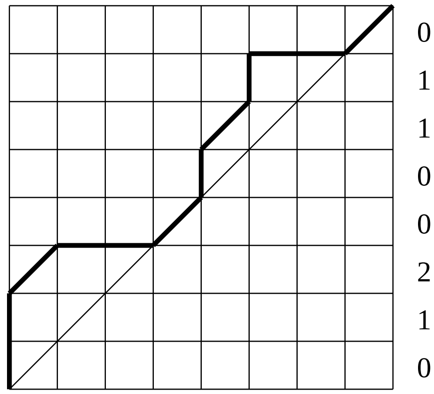

Figure 2: A Schröder path $\Pi \in S_{8,4}$.

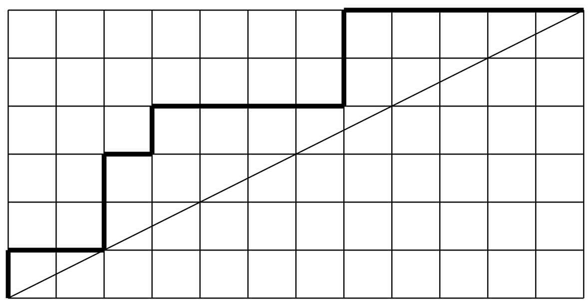

Figure 3: A 2-Dyck path in $\mathcal{D}_6^2$.

A 2-Dyck path of order 6 is illustrated in Figure 3.

As in the $m = 1$ case, given $\Pi \in \mathcal{D}_n^m$, we encode each N step by a 0 and each E step by a 1 so as to obtain a word $w(\Pi)$ of n 0's and mn 1's. This clearly provides a bijection between $\mathcal{D}_n^m$ and CW_n^m, where

$$CW_n^m = \left\{ w \in M_{n,mn} \,\middle|\, \begin{array}{c} \text{at any initial segment of } w\text{, the number of 0's times} \\ m \text{ is at least as many as the number of 1's.} \end{array} \right\}$$

We call this special subset of 01 words, CW_n^m, *Catalan words of order n and dimension m*.

It is shown in [10] (see also [11]) that the number of m-Dyck paths, denoted by C_n^m, is equal to

$$\frac{1}{mn+1}\binom{mn+n}{n},$$

which we call the m-Catalan number. In fact, Cigler [2] proved that the number of m-Dyck paths with k peaks, i.e., those with exactly k consecutive NE pairs, is the generalized Runyon number,

$$R_{n,k}^m = \frac{1}{n}\binom{n}{k}\binom{mn}{k-1}.$$

Now we turn to the more general m-Schröder theory.

Definition 2.2 *An m-*Schröder path *of order n is a lattice path from $(0,0)$ to (mn,n) which never goes below the main diagonal $\{(mi,i) : 0 \leq i \leq n\}$, with steps $(0,1)$ (or NORTH, for brevity* N*), $(1,0)$ (or EAST, for brevity* E*) and (1,1) (or Diagonal, for brevity* D*). Let $\mathcal{S}_n^m$ denote the set of all m-Schröder paths of order n, and let $S_{n,d}^m$ denote the set of all m-Schröder paths of order n and with exactly* d *diagonal steps.*

Definition 2.3 *An m-*Schröder path *of order n and with d diagonal steps is a lattice path from $(0,0)$ to (mn,n) which never goes below the main diagonal $\{(mi,i) : 0 \leq i \leq n\}$, with $(0,1)$ (or NORTH, for brevity* N*), $(1,0)$ (or EAST, for brevity* E*) and exactly* d *(1,1) (or Diagonal, for brevity* D*) steps. Let $\mathcal{S}_{n,d}^m$ denote the set of all m-Schröder paths of order n and with* d *diagonal steps.*

A 2-Schröder path of order 6 and with 2 diagonal steps is illustrated in Figure 4.

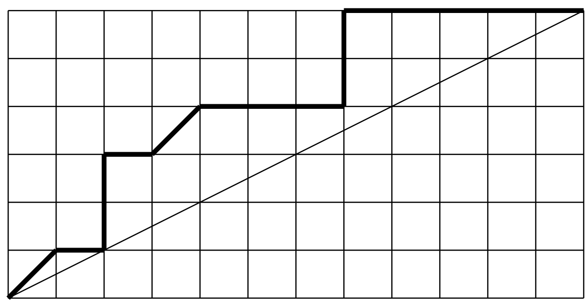

Figure 4: A 2-Schröder path in $\mathcal{S}_{6,2}^2$.

Theorem 2.1 *The number of m-Schröder paths of order n and with d diagonal steps, denoted by $S_{n,d}^m$, is equal to*

$$\frac{1}{mn-d+1}\binom{mn+n-d}{mn-d,n-d,d}.$$

Proof. For an m-Dyck path Π, let its number of peaks, or consecutive NE pairs, be denoted by $peak(\Pi)$. Notice that any m-Schröder path with d diagonal steps can be obtained uniquely by choosing d of the peaks of a uniquely decided m-Dyck path Π of the same order, and changing each of the chosen consecutive NE pair steps to a Diagonal step. Conversely, given an m-Dyck path Π of order n, choosing d of its peaks (if there are d to choose) and changing them to D steps will give a path in $\mathcal{S}_{n,d}^m$. For example, the 2-Schröder path as illustrated in Figure 4 is one of $\binom{4}{2} = 6$ paths in $\mathcal{S}_{6,4}^2$ that can be obtained from the 2-Dyck path shown in Figure 3. Hence,

$$\begin{aligned}
S^m_{n,d} &= \sum_{\Pi\in\mathcal{D}^m_n} \binom{peak(\Pi)}{d} \\
&= \sum_{k\geq d} \binom{k}{d} R^m_{n,k} \\
&= \sum_{k\geq d} \binom{k}{d} \frac{1}{n} \binom{n}{k} \binom{mn}{k-1}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\binom{n}{d}}{n} \sum_{k\geq d} \binom{n-d}{n-k} \binom{mn}{k-1} \\
&= \frac{\binom{n}{d}}{n} \binom{mn+n-d}{n-1} \\
&= \frac{1}{mn-d+1} \binom{mn+n-d}{d, n-d, mn-d}.
\end{aligned}$$

Above we used the Vandermonde Convolution (see [3, page 44]). □

As a generalization of the $m=1$ case, we name

$$S^m_n = \sum_{d=0}^{n} \frac{1}{mn-d+1} \binom{mn+n-d}{mn-d, n-d, d}$$

the *m-Schröder number.*

3 q-m-Schröder Polynomials

When Bonin, Shapiro and Simion [1] studied q-analogues of the Schröder numbers, they obtained several classical results for several single variable analogue cases. Here we generalize some of them to the m case.

Definition 3.1 *Define the m-*Narayana *polynomial $d^m_n(q)$ over the m-Schröder paths of order n to be*

$$d^m_n(q) = \sum_{\Pi\in\mathcal{S}^m_n} q^{diag(\Pi)},$$

where $diag(\Pi)$ is the number of D steps in the path Π.

Theorem 3.1 *$d^m_n(q)$ has $q=-1$ as a root.*

Proof. We use the idea of [1]. The statement is equivalent to saying that there are as many m-Schröder paths of order n with an even number of D steps as there are with an odd number of D steps. For any $\Pi \in \mathcal{S}_n^m$, there must be some first occurrence of either a consecutive NE pair of steps, or a D step. According to which occurs first, either replace the consecutive NE pair by a D, or replace the D with a consecutive NE pair. Notice that this presents a bijection between the two sets of objects we wish to show have the same cardinality. $\square$

In [5], there is a refined q-analogue identity,

$$\sum_{k\geq 1}\sum_{w\in CW_{n,k}} q^{majw} = \sum_{k\geq 1}\frac{1}{[n]}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}n\\k-1\end{bmatrix}q^{k(k-1)} = \frac{1}{[n+1]}\begin{bmatrix}2n\\n\end{bmatrix}, \tag{3.0.1}$$

where $CW_{n,k}$ is the set of Catalan words consisting of n 0's, n 1's, with k ascents (i.e. $k-1$ descents or the corresponding Dyck path has k peaks). As for the m-version, Cigler proved there are exactly

$$\frac{1}{n}\binom{n}{k}\binom{mn}{k-1}$$

m-Dyck paths with k peaks [2]. In order to generalize the results of [5], we prove the following q-identity.

Theorem 3.2

$$\sum_{k\geq d}\begin{bmatrix}k\\d\end{bmatrix}\frac{1}{[n]}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}mn\\k-1\end{bmatrix}q^{(k-d)(k-1)} = \frac{1}{[mn-d+1]}\begin{bmatrix}mn+n-d\\d,n-d,mn-d\end{bmatrix}.$$

Before we proceed to the proof of Theorem 3.2, we cite the q-Vandermonde Convolution, which may be obtained as a corollary of the q-binomial theorem.

Lemma 3.3 [7] *The q-Vandermonde Convolution.*

$$\sum_{j=0}^{h} q^{(n-j)(h-j)}\begin{bmatrix}n\\j\end{bmatrix}\begin{bmatrix}m\\h-j\end{bmatrix} = \begin{bmatrix}m+n\\h\end{bmatrix}.$$

Proof. Proof of Theorem 3.2.

$$\begin{aligned}
&\sum_{k\geq d}\begin{bmatrix}k\\d\end{bmatrix}\frac{1}{[n]}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}mn\\k-1\end{bmatrix}q^{(k-d)(k-1)}\\
&=\frac{\begin{bmatrix}n\\d\end{bmatrix}}{[n]}\sum_{k=d}^{n}\begin{bmatrix}n-d\\n-k\end{bmatrix}\begin{bmatrix}mn\\k-1\end{bmatrix}q^{(k-d)(k-1)}\\
&=\frac{\begin{bmatrix}n\\d\end{bmatrix}}{[n]}\sum_{j=0}^{n-d}\begin{bmatrix}n-d\\j\end{bmatrix}\begin{bmatrix}mn\\n-1-j\end{bmatrix}q^{(n-d-j)(n-1-j)}\\
&=\frac{\begin{bmatrix}n\\d\end{bmatrix}}{[n]}\begin{bmatrix}mn+n-d\\n-1\end{bmatrix}(q\text{-Vandermonde Convolution})\\
&=\frac{1}{[mn-d+1]}\begin{bmatrix}mn+n-d\\d,n-d,mn-d\end{bmatrix}.
\end{aligned}$$

□

Remark 3.1 *It is difficult to find a combinatorial interpretation for the left hand side of Theorem 3.2. As a matter of fact, the most straightforward generalization of (3.0.1) even fails for the 2-Dyck paths:*

$$\sum_{w\in CW_2^2}q^{majw}=1+q^2+q^3\neq\frac{[1]}{[5]}\begin{bmatrix}6\\2\end{bmatrix}=1+q^2+q^4.$$

4 (q,t)-m-Schröder Statistics and the Shuffle Conjecture

Similar to the manner of [9], for an m-Dyck path of order n, we may associate an *m-parking functions* with it by placing one of the n "cars", denoted by the integers 1 through n, in the square immediately to the right of each N step of D, with the restriction that if car i is placed immediately on top of car j, then $i > j$. Let $\mathbb{P}_n^m$ denote the collection of m-parking functions on n cars.

Definition 4.1 *Given an m-parking function, its m-*reading word *is obtained by reading from NE to SW line by line, starting from the lines farther from the m-diagonal $x = my$.*

Figure 5 illustrates an m-parking function with 231 as its m-reading word. The first line we look at is the line connecting cars 2 and 3. We read it from NE to SW so that 2 is before 3. Then the next line is the m-diagonal $x = my$ which contains car 1.

Definition 4.2 *Given an m-parking function, its* natural expansion *is defined as follows: starting from (0, 0), each N step, together with the car to its right, is duplicated m times, the car within the N step is duplicated m times and put one to each of the m N steps duplicated; leave each E step untouched.*

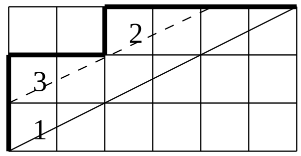

Figure 5: An m-parking function whose m-reading word is 231.

Figure 6 illustrates the natural expansion of the m-parking function shown in Figure 5. Note that the natural expansion of an m-parking function is kind of a "non-strict" standard parking function in the sense that if car placing i immediately on top of car j implies that $i \geq j$ instead of $i > j$.

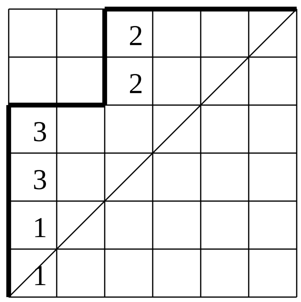

Figure 6: The natural expansion of an m-parking function.

Definition 4.3 [13, page 482, Ex. 7.93] *For two words* $u = (u_1, \ldots, u_k) \in S_k$ *and* $v = (v_1, \ldots, v_l) \in S(k+1, k+l)$, *where* $S(m+1, m+l)$ *denotes all the permuted words of* $\{k+1, \cdots, k+l\}$, $sh(u, v)$ *or* $sh((u_1, \ldots, u_k), (v_1, \ldots, v_l))$ *is the set of* shuffles *of* u *and* v, *i.e.,* $sh(u, v)$ *consists of all permutations* $w = (w_1, \ldots, w_{k+l}) \in S_{k+l}$ *such that both* u *and* v *are subsequences of* w.

If the m-reading word of an m-parking function P is a shuffle of the two words $(n-d+1, \cdots, n)$ and $(n-d, \cdots, 2, 1)$, the increasing order of $(n-d+1, \cdots, n)$ will imply that any single N segment of P contains at most 1 of $\{n-d+1, \cdots, n\}$. Furthermore, each of $\{n-d+1, \cdots, n\}$ should occupy the top spot of some N segment. Hence if we change these d top N steps all to D steps and remove the cars in the m-parking function, we will get an m-Schröder path with d diagonal steps. Conversely, given a path $\Pi \in \mathcal{S}^m_{n,d}$, we may change its d diagonal steps to d NE pairs; after that place cars $\{n-d+1, \cdots, n\}$ to the right of the d new N steps, and place cars $\{n-d, \cdots, 2, 1\}$ to the right of the other $n-d$ D steps in the uniquely right order so that the m-reading word of the m-parking function formed is a shuffle of the two words $(n-d+1, \cdots, n)$ and $(n-d, \cdots, 2, 1)$. In this way every m- Schröder corresponds to an m-parking function of the particular type. Because it is easier to manipulate when there are no D steps, we define the m-Schröder polynomial in the following way.

Definition 4.4 *The (q,t)-m-Schröder polynomial is defined as*

$$S^m_{n,d}(q,t) = \sum_{\substack{\Pi:\ \Pi \in \mathbb{P}^m_n \text{and the } m\text{-reading word of } \Pi \\ \in sh((n-d+1,\cdots,n),(n-d,\cdots,1))}} q^{dinv_m(\Pi)} t^{area(\Pi)},$$

where $dinv_m(\Pi) = dinv(\hat{\Pi})$, $\hat{\Pi}$ is the natural expansion of Π, and dinv is the obvious generalization of the statistic on parking functions introduced in [9].

The following m-Shuffle Conjecture is due to Haglund, Haiman, Loehr, Remmel and Ulyanov.

Conjecture 4.1 [8]

$$S^m_{n,d}(q,t) = < \nabla^m e_n, e_{n-d} h_d >,$$

where ∇ is a linear operator defined in terms of the modified Macdonald polynomials (for details see [8]*).*

Recently, Loehr [12] has obtained recurrences for the (q,t)-m-Catalan numbers, while Egge et. al obtained recurrences for the (q,t)-Schröder numbers [4], so an interesting open problem is whether or not there exist such recurrences for their common generalization, the (q,t)-m-Schröder numbers.

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