Packing and covering a unit equilateral triangle with equilateral triangles

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Abstract

Packing and covering are elementary but very important in combinatorial geometry, they have great practical and theoretical significance. In this paper, we discuss a problem on packing and covering a unit equilateral triangle with smaller triangles which is originated from one of Erdős' favorite problems.

Keywords: packing, minimal covering
Mathematics Subject Classification (2000): 52C15

1 Introduction

Packing and covering are elementary but very important in combinatorial geometry, they have great practical and theoretical significance. In 1932, Erdős posed one of his favorite problems on square-packing which was included in [2]: Let S be a unit square. Inscribe n squares with no common interior point. Denote by $e_1, e_2, \ldots, e_n$ the sides length of these squares. Put $f(n) = \max \sum_{i=1}^{n} e_i$. In [3], P. Erdős and Soifer gave some results of $f(n)$.

In [1], Connie Campbell and William Staton considered this problem again. Because packing and covering are usually dual to each other, we discussed a problem of a minimal square-covering in [5]. In this paper, we generalize this kind of problem to the case of using equilateral triangles to pack and cover a unit equilateral triangle, and obtain corresponding results.

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2 Packing a unit equilateral triangle

Firstly, we give the definition of the packing function:

Definition 2.1. Let T be a unit equilateral triangle. Inscribe n equilateral triangles $T_1, T_2, \ldots, T_n$ with no common interior point in such a way which satisfies: T_i has side of length t_i $(0 < t_i \leq 1)$ and is placed so that at least one of its sides is parallel to that of T.

Define $t(n) = \max \sum\limits_{i=1}^{n} t_i$.

In this part, we mainly exploit the method of [1] to get the bounds of $t(n)$ and obtain a corresponding result. Here we list some of the proofs so that the readers may better understand.

Theorem 2.2. *The following estimates are true for all positive integers n:*
(1) $t(n) \leq \sqrt{n}$.
(2) $t(n) \leq t(n+1)$.
(3) $t(n) < t(n+2)$.

Proof. (1)Let $\mathbf{s}$ be the vector $(t_1, t_2, \ldots, t_n)$, where the t_i denote the length of the sides of the equilateral triangles in the packing, and let $\mathbf{v}$ be the vector $(1, 1, \ldots, 1)$. Now $\sum\limits_{i=1}^{n} t_i \leq \|\mathbf{s}\|\|\mathbf{v}\| \leq \sum\limits_{i=1}^{n} t_i^2 n^{\frac{1}{2}} = \frac{2}{\sqrt{3}} \sum\limits_{i=1}^{n} (\frac{\sqrt{3}}{2} t_i^2) n^{\frac{1}{2}} \leq n^{\frac{1}{2}}$.

It's easy to get (2),(3) by replacing a T_i with 2 or 3 equilateral triangles with sides of length $\frac{t_i}{2}$. □

Definition 2.3. For a equilateral triangle T, dissect each of its 3 sides into n equal parts, then through these dissecting points draw parallel lines of the sides of T, so we get a packing of T by n^2 equilateral triangles with sides of length $\frac{1}{n}$. Such a configuration is called an n^2-grid. When T is a unit equilateral triangle, the packing is a standard n^2-packing.

See Figure 1 for the case $n = 3$.

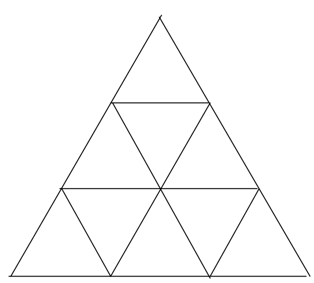

Figure 1: a 3^2-grid

Proposition 2.4. $t(k^2) = k$.

Proof. By Definition 2.3, it's easy to know that for the standard k^2-packing , $n = k^2, t_i = \frac{1}{k}$ and $\sum_{i=1}^{n} t_i = \frac{1}{k}k^2 = k$. So by the Definition of $t(n)$, $t(k^2) \geq k$ which along with Theorem 2.2(1) provides the desired equality. $\square$

Proposition 2.5. *For $k \geq 2$, $t(k^2 - 1) \geq k - \frac{1}{k}$.*

Proof. Consider the standard k^2-packing with one equilateral triangle removed. $\square$

Theorem 2.6. *If n is a positive integer such that $(n-1)$ is not a perfect square number, then $t(n) > (n-1)^{\frac{1}{2}}$.*

Proof. When $n = k^2$, by Proposition 2.4, $t(n) = \sqrt{n} > \sqrt{n-1}$.

When $n = k^2 - 1$, by Proposition 2.5, $t^2(n) \geq (k - \frac{1}{k})^2 = k^2 - 1 - 1 + \frac{1}{k^2} > n - 1$. That is $t(n) > \sqrt{n-1}$.

When $n \neq k^2$, k must lie between two perfect square numbers of different parity. That is, there is an integer k such that $k^2 < n < (k+1)^2$, $n - k^2$ and $(k+1)^2 - n$ have different parity. When neither $n-1$ nor $n+1$ is a perfect square number, consider the values of n where $k^2 + 1 < n < (k+1)^2 - 1$, there are two cases which provide the lower bound of $t(n)$ for all n on the interval $[k^2 + 2, (k+1)^2 - 2]$:

Case 1. $(k+1)^2 - n$ is odd. Say, $(k+1)^2 - n = 2a + 1 (a \geq 1)$, $k^2 < n \leq (k+1)^2 - 3$. From a standard $(k+1)^2$-packing of T, remove an $(a+1)^2$-grid and replace it with an a^2-grid packing the same area. The result is a packing of $(k+1)^2 - (a+1)^2 + a^2 = (k+1)^2 - 2a - 1 = n$ equilateral triangles, the sum of whose length is $[(k+1)^2 - (a+1)^2]\frac{1}{k+1} + a^2(\frac{a+1}{a})(\frac{1}{k+1}) = k + 1 - \frac{a+1}{k+1}$.

So $t(n) \geq k + 1 - \frac{a+1}{k+1}$, $t^2(n) \geq (k + 1 - \frac{a+1}{k+1})^2 = (k+1)^2 - 2a - 1 + (\frac{a+1}{k+1})^2 - 1 > n - 1$. That is, $t(n) > \sqrt{n-1}$.

Case 2. $n - k^2$ is odd. Say, $n - k^2 = 2a - 1 (a \geq 2)$, $k^2 + 3 \leq n < (k+1)^2$. From a standard k^2-packing of T, remove an $(a-1)^2$-grid and replace it with an a^2-grid covering the same area. The result is a packing of $k^2 - (a-1)^2 + a^2 = k^2 + 2a - 1 = n$ equilateral triangles of the unit equilateral triangle T. The sum of the length of sides is $[k^2 - (a-1)^2]\frac{1}{k} + a^2(\frac{a-1}{a})(\frac{1}{k}) = k + \frac{a-1}{k}$.

So $t(n) \geq k + \frac{a-1}{k}$, $t(n)^2 \geq (k + \frac{a-1}{k})^2 = k^2 + 2a - 1 + (\frac{a-1}{k})^2 - 1 > n - 1$. That is, $t(n) > \sqrt{n-1}$. $\square$

Similar to [1], by Theorem 2.6, we can easily get the following result.

Theorem 2.7. *If $t(n+1) = t(n)$, then n is a perfect square number.*

On the other hand, we think the following is right:

Conjecture 2.8. $t(n^2 + 1) = t(n^2)$.

3 Covering a unit equilateral triangle

Definition 3.1. Let T be a unit equilateral triangle. If n equilateral triangles $T_1, T_2, \ldots, T_n$ can cover T in such a way which satisfies:

(1) T_i has side of length $t_i (0 < t_i < 1)$ and is placed so that at least one of its sides is parallel to that of T;

(2) T_i can't be smaller, that is, there doesn't exist any $T_{i1} \subset T_i$ such that $\{T_j, j = 1, 2, \ldots, i-1, i+1, \ldots, n\} \cup \{T_{i1}\}$ can cover T. (Here we admit translation.)

We call this kind of covering a minimal covering.

In the meaning of the minimal covering, define:

$T_1(n) = \min \sum\limits_{i=1}^{n} t_i$, $T_2(n) = \max \sum\limits_{i=1}^{n} t_i$.

When $n \leq 2$, since $0 < t_i < 1$, each T_i $(i = 1, 2)$ can only cover one corner of a unit equilateral triangle, but it has three corners, so T_1, T_2 can't cover T. That is, when $n \leq 2$, $T_i(n)(i = 1, 2)$ has no meaning. So in the following, let $n \geq 3$.

3.1 The upper bound of $T_1(n)$

Theorem 3.2. *When n is even, $T_1(n) \leq 3 - \frac{4}{n}$.*

Proof. Consider a covering of a unit equilateral triangle T with a equilateral triangle T_1 which has side of length x and $n-1$ equilateral triangles $T_2, T_3, \ldots, T_n$ each of which has sides of length $1 - x$ such that $\frac{n}{2}(1 - x) = 1$, which implies $x = 1 - \frac{2}{n}$. When $n = 6$, see Figure 2 for the placement. It's easy to see this is a minimal covering. So by the definition of $T_1(n)$, $T_1(n) \leq x + (n-1)(1-x) = 3 - \frac{4}{n}$.

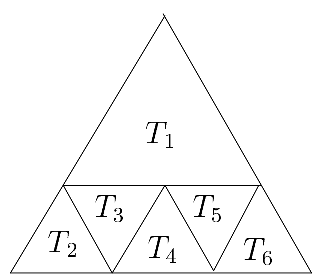

Figure 2: a unit equilateral triangle covered by six smaller equilateral triangles

□

Proposition 3.3. $T_1(3) \leq 2$.

Proof. Consider a covering of a unit equilateral triangle T with 3 equilateral triangles T_1, T_2, T_3 each of which has sides of length $\frac{2}{3}$. See Figure 3 for the placement. It's easy to see this is a minimal covering. So by the definition of $T_1(n)$, $T_1(3) \leq 3 \times \frac{2}{3} = 2$.

□

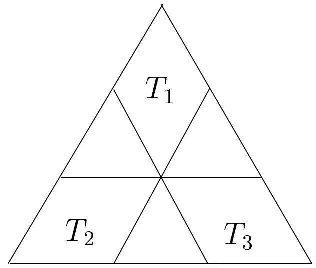

Figure 3: a unit equilateral triangle covered by 3 smaller equilateral triangles

Proposition 3.4. $T_1(5) < \frac{9}{4}$.

Proof. Consider a covering of a unit equilateral triangle T with one equilateral triangle T_1 which has side of length x, 2 equilateral triangles T_2, T_3 each of which has sides of length y and 2 equilateral triangles T_4, T_5 each of which has sides of length $1-x$, such that $y < 2(1-x)$ and $2y - x = \frac{x-(1-x)}{2}$, which implies $y = x - \frac{1}{4}$ and $\frac{1}{2} < x < \frac{3}{4}$. See Figure 4 for the placement. It's easy to see this is a minimal covering. So by the definition of $T_1(n)$, $T_1(5) \leq x + 2y + 2(1-x) = x + \frac{3}{2} < \frac{3}{4} + \frac{3}{2} = \frac{9}{4}$.

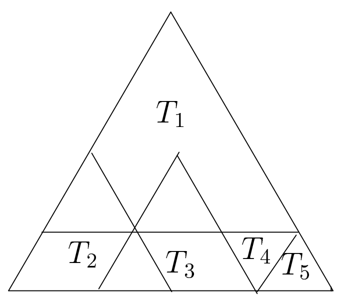

Figure 4: a unit equilateral triangle covered by 5 smaller equilateral triangles

□

Theorem 3.5. *When n is odd and $n \geq 7$, $T_1(n) \leq 4 - \frac{6}{n-3}$.*

Proof. Consider a covering of a unit equilateral triangle T with 4 equilateral triangles T_1, T_2, T_3, T_4 each of which has side of length x and $n-4$ equilateral triangles $T_5, T_6, \ldots, T_n$ each of which has sides of length $1-2x$, such that $\frac{(n-3)(1-2x)}{2} = 1$ which implies $x = \frac{1}{2} - \frac{1}{n-3}$. when $n = 7$, see Figure 5 for the placement. It's easy to see this is a minimal covering. So by the definition of $T_1(n)$, $T_1(n) \leq 4x + (n-4)(1-2x) = 4 - \frac{6}{n-3}$.

□

Here we can't give the lower bound of $T_1(n)$, but it seems obvious that the following is right:

Conjecture 3.6. $T_1(n) \geq 2$.

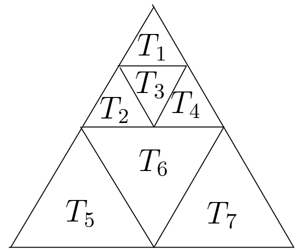

Figure 5: a unit equilateral triangle covered by seven smaller equilateral triangles

3.2 The bounds of $T_2(n)$

Proposition 3.7. $T_2(k^2) \geq k$.

Proof. It's easy to see that a standard n-packing is also a standard n-covering. By the proof of Proposition 2.4 and the definition of $T_2(n)$, the assertion holds. $\square$

Proposition 3.8. $T_2(k^2+1) \geq k$.

Proof. From a standard k^2-covering, remove a 2^2-grid and replace it with equilateral triangles $T_{i1}, T_{i2}, \ldots, T_{i5}$ covering the same area which are placed as Figure.4 such that T_{i1} is the largest equilateral triangles of $\{T_{ij} \mid j = 1, 2, \ldots, 5\}$ which implies that $t_{i1} \geq \frac{1}{k}$ and $t_{i2} = t_{i3} = \frac{2}{k} - t_{i1}$, $t_{i4} = t_{i5} = t_{i1} - \frac{1}{2k}$. The result is a covering of $k^2 - 4 + 5 = k^2 + 1$ equilateral triangles, the sum of whose length is $t = k - \frac{4}{k} + t_{i1} + 2(\frac{2}{k} - t_{i1}) + 2(t_{i1} - \frac{1}{2k}) = k - \frac{1}{k} + t_{i1} \geq k$.

Obviously, any equilateral triangle of $\{T_{ij} \mid j = 1, 2, \ldots, 5\}$ can't be smaller. This covering is a minimal covering, so we have $T_2(k^2+1) \geq k$.

$\square$

Proposition 3.9. $T_2(k^2-1) \geq k - \frac{3}{2k}$.

Proof. From a standard k^2-covering, remove a 3^2-grid and replace it with eight equilateral triangles $T_{i1}, T_{i2}, \ldots, T_{i8}$ covering the same area which are placed as Figure 6 such that T_{i1} is the largest equilateral triangles of $\{T_{ij} \mid j = 1, 2, \ldots, 8\}$ and $t_{i2} = t_{i3} = t_{i4} = t_{i5} = t_{i6} = t_{i7} = t_{i8} = \frac{3}{k} - t_{i1}$. It's obvious that $0 < t_{ij} < \frac{3}{k}(j = 1, 2, \ldots, 8)$. And $4(\frac{3}{k} - t_{i1}) = \frac{3}{k}$ which implies $t_{i1} = \frac{9}{4k}$. The result is a covering of $k^2 - 9 + 8 = k^2 - 1$ equilateral triangles, the sum of whose length is $t = k - \frac{9}{k} + t_{i1} + 7(\frac{3}{k} - t_{i1}) = k + \frac{12}{k} - 6t_{i1}$. So $t \geq k + \frac{12}{k} - 6t_{i1} = k - \frac{3}{2k}$.

Obviously, any equilateral triangles of $\{T_{ij} \mid j = 1, 2, \ldots, 8\}$ can't be smaller. So any one of the resulting $k^2 - 1$ equilateral triangles can't be smaller. This covering is a minimal covering, so $T_2(k^2-1) \geq k - \frac{3}{2k}$.

$\square$

It's easy to see that a standard n-packing is also a standard n-covering. By the proof of Theorem 2.6 and the definition of $T_2(n)$, we can get the following result in a similar way:

Theorem 3.10. *If neither $n-1$ nor $n+1$ is a perfect square number, then $T_2(n) > \sqrt{n-1}$.*

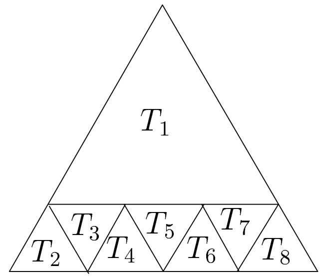

Figure 6: a 3^2-grid covered by eight equilateral triangles

To get an upper bound of $T_2(n)$, we first list the following lemma which is a known result of [4]:

Lemma 3.11. *[4] Let T be a triangle and let $\{T_i\}_{i=1}^n$ be a sequence of its positive or negative copies. If the total area of $\{T_i\}_{i=1}^n$ is greater than or equal to $4|T|$(where $|T|$ denotes the area of T), then $\{T_i\}_{i=1}^n$ permits a translative covering of T.*

Theorem 3.12. $T_2(n) \leq 4\sqrt{n}$.

Proof. Let $\{T_i\}_{i=1}^n$ be a minimal covering of the unit equilateral triangle T, and t_i denote the length of the side of $T_i (i = 1, 2, \ldots, n)$. We first prove that $\sum_{i=1}^{n} \frac{\sqrt{3}}{2} t_i^2 \leq 2\sqrt{3}$. Otherwise, if $\sum_{i=1}^{n} \frac{\sqrt{3}}{2} t_i^2 > 2\sqrt{3}$, there exists a $T_{i1} \subset T_i$, such that $t_{i1} < t_i$ and $\frac{\sqrt{3}}{2}(t_{i1}^2 + \sum_{j=1}^{i-1} t_j^2 + \sum_{j=i+1}^{n} t_j^2) \geq 2\sqrt{3}$. Notice that the area of a unit equilateral triangle is $\frac{\sqrt{3}}{2}$ and all equilateral triangle are homothetic, by Lemma 3.11, $T_1, T_2, \ldots, T_{i-1}, T_{i1}, T_{i+1}, \ldots, T_n$ can cover the unit equilateral triangle T, which contradicts the definition of a minimal covering . So $\sum_{i=1}^{n} \frac{\sqrt{3}}{2} t_i^2 \leq 2\sqrt{3}$.

Let $\mathbf{s}$ be the vector $(t_1, t_2, \ldots, t_n)$, and let $\mathbf{v}$ be the vector $(1, 1, \ldots, 1)$. Now $\sum_{i=1}^{n} t_i \leq \|\mathbf{s}\|\|\mathbf{v}\| \leq \sum_{i=1}^{n} t_i^2 n^{\frac{1}{2}} = \frac{2}{\sqrt{3}} n^{\frac{1}{2}} \sum_{i=1}^{n} \frac{\sqrt{3}}{2} t_i^2 \leq \frac{2}{\sqrt{3}} 2\sqrt{3} n^{\frac{1}{2}} = 4\sqrt{n}$. So $T_2(n) \leq 4\sqrt{n}$.

$\square$

We also have the following unsolved problem:
Problem: Improve the upper bound of $T_2(n)$.

4 The case of isosceles right triangle with legs of length 1

All the results above can be generalized to the isosceles right triangle with legs of length 1 in the same way.

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