Perfect codes in Cartesian products of 2-paths and infinite paths

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Abstract

We introduce and study a common generalization of 1-error binary perfect codes and perfect single error correcting codes in Lee metric, namely perfect codes on products of paths of length 2 and of infinite length. Both existence and nonexistence results are given.

1 Introduction

Perfect codes appeared in error correcting codes theory during the late 40's with the work of Golay and Hamming [3, 6]. They constructed perfect binary single-error correcting codes of length n where $n = 2^q - 1$ for some integer q.

Later, Golomb and Welch [4, 5] proved, for any length n, the existence of perfect single-error correcting codes in Lee metric. Such codes can be considered either as regular periodic tilings of the euclidean space $\mathbb{R}^n$ by Lee spheres of radius 1 or as periodic tilings of the grid $\mathbb{Z}^n$ by balls of radius 1. Perfect codes have also been studied on other alphabets or mixed one (see [2]).

In [1], Biggs introduced perfect codes in graphs. From this point of view, a perfect binary single-error correcting code of length k is a perfect code on the hypercube Q_k, and a tiling of the grid $\mathbb{Z}^n$ by balls of radius 1 is nothing more than a perfect code of $\mathbb{Z}^n$.

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Both Q_k and the grid $\mathbb{Z}^n$ are cartesian products of paths either of length 2 (P_2) or of infinite length (P_∞). We study the existence of perfect codes on the mixed product $P_\infty^n \Box P_2^k = \mathbb{Z}^n \Box Q_k$.

In the next section, we give the definitions we will use along this paper. Section 3 presents classical results on error correcting codes. In section 4, we construct new codes, and in section 5, we prove some inexistence results. The last section summarizes what we know and what is still open.

2 Definitions

Given $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ two graphs, the *cartesian product* $G_1 \Box G_2$ is the graph with vertex set $V_1 \times V_2$ and satisfying $(x_1, x_2)(y_1, y_2) \in E(G_1 \Box G_2)$ if and only if $x_1 y_1 \in E_1$ and $x_2 = y_2$ or $x_2 y_2 \in E_2$ and $x_1 = y_1$. We will use the notation G^n for the graph $G \Box G \Box \ldots \Box G$ (n times).

The *hypercube* of dimension k is the graph Q_k whose vertices are the words of length k over the alphabet $\{0, 1\}$, and where two vertices are adjacent if they differ in exactly one place. Notice that Q_1 is P_2 the path with 2 vertices and that $Q_{k+1} = Q_k \Box P_2$. For two vertices u and v in Q_k, we will denote by $u+v$ the vertex $((u_1+v_1), (u_2+v_2), \ldots, (u_k+v_k))$ where $+$ is the sum modulo 2.

The *infinite grid* $\mathbb{Z}^n$ is the graph whose vertices are the words of length n over the alphabet $\mathbb{Z}$ and where two vertices are adjacent if and only if they differ by 1 in exactly one place. Notice that if we denote by P_∞ the two ways infinite path, we have also $\mathbb{Z}^1 = P_\infty$ and $\mathbb{Z}^{n+1} = \mathbb{Z}^n \Box P_\infty$.

We will consider in the same idea the vertices of the cycle of length l, denoted by C_l, as the elements of $\mathbb{Z}/l\mathbb{Z}$. For two vertices u and v, we will denote by $u + v$ the vertex $(u + v) \bmod l$.

From now on, we will work on the cartesian product $\mathbb{Z}^n \Box Q_k$. This graph can also be considered as follows : the vertices are words of length $n + k$ whose n first symbols are in $\mathbb{Z}$ and the k others are in $\{0, 1\}$. Two vertices are adjacent if they differ by 1 in exactly one place. Notice that this graph is regular with degree $2n + k$. Moreover, $\mathbb{Z}^n \Box Q_0 = \mathbb{Z}^n$ and $\mathbb{Z}^0 \Box Q_k = Q_k$.

For two vertices x and y, we will denote by $d(x, y)$ the classical *distance* on graphs. For our graph $\mathbb{Z}^n \Box Q_k$,

$$d((x_1, x_2, \ldots, x_{n+k}), (y_1, y_2, \ldots, y_{n+k})) = \sum_{i=1}^{n+k} |x_i - y_i|$$

If $n = 0$, this is the *Hamming distance*, and if $k = 0$, this is the *Manhattan distance.*

For an integer r and a vertex c, we call *ball* of radius r centered on c the set of vertices v such that $d(c, v) \leq r$. In this paper, we will only consider balls of radius 1.

In a graph, a *single error correcting code* (or code for shorter) is a set of vertices such that any two code's vertices are at distance at least 3. This is equivalent to say that the balls centered on these vertices are disjoint.

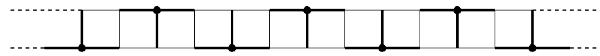

Figure 1: A perfect code on $\mathbb{Z}^1\Box Q_1$

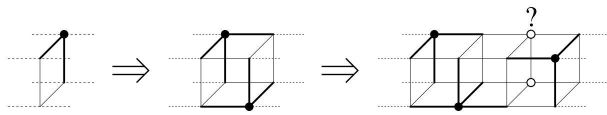

Figure 2: There exists no perfect code on $\mathbb{Z}^1\Box Q_2$

A code is said to be *perfect* if any vertex of the graph is at distance at most one of a code's vertex. It also means that any vertex belongs to a ball centered on a code's vertex and thus that these balls form a tiling of the graph.

Figure 1 shows a perfect code in $\mathbb{Z}^1\Box Q_1$. But a perfect code does not always exists as figure 2 shows.

Let G be a finite graph. On $\mathbb{Z}^n\Box G$, we will say that a code is *i-periodic* ($i \in \{1,\ldots,n\}$) if there exists a positive integer p_i such that for any vertex $x = (x_1, x_2, \ldots, x_n, v)$ (where $\forall i, x_i \in \mathbb{Z}$, $v \in V(G)$), the vertex $(x_1, x_2, \ldots, x_{i-1}, (x_i + p_i), x_{i+1}, \ldots, x_n, v)$ is in the code if and only if x is in the code. p_i is called the i-period. A code is *periodic* of period $(p_1, p_2, \ldots, p_n)$ if it is i-periodic of i-period p_i for all i.

Proposition 1 *Let S be a periodic code on $\mathbb{Z}^n\Box G$ of period $(p_1, p_2, \ldots, p_n)$. There exists some set T of words $t = (t_1, t_2, \ldots, t_n, v)$ with $\forall i \in \{1\ldots,n\}$, $0 \le t_i < p_i$ and $v \in V(G)$ such that S is the set of words*

$$\{(t_1 + \alpha_1 p_1, t_2 + \alpha_2 p_2, \ldots, t_n + \alpha_n p_n, v) | t \in T, \alpha_i \in \mathbb{Z}\}$$

3 Perfect codes deduced from known constructions

From the classical results on error correcting codes, we get:

Theorem 2 *If $n = 0$, there exists a perfect code on $\mathbb{Z}^n\Box Q_k$ if and only if there exists an integer p such that $k = 2^p - 1$.*

Examples of these perfect codes are the classical Hamming codes [6], but when $k \ge 15$, other perfect codes with the same length are known (for a survey on this topic, see [2]).

Theorem 3 *There exists a i-periodic perfect code on $\mathbb{Z}^n\Box G$ of i-period p_i if and only if a perfect code on the graph $\mathbb{Z}^{n-1}\Box C_{p_i}\Box G$ exists.*

Proof : Let S be a perfect code on $\mathbb{Z}^{n-1}\Box C_{p_i}\Box G$. Then one can easily check that the set of words

$$\{(x_1, x_2, \ldots, x_{n-1}, c + \alpha p_i, v) | (x_1, x_2, \ldots, x_{n-1}, c, v) \in S, \alpha \in \mathbb{Z}\}$$

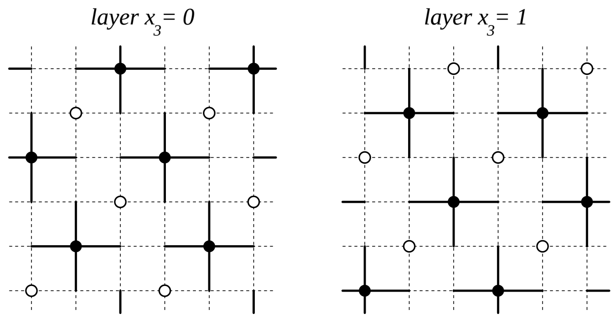

Figure 3: A perfect code on $\mathbb{Z}^2\Box Q_1$

is a perfect code of $\mathbb{Z}^n\Box G$.

On the other hand, let us consider a perfect i-periodic code S on $\mathbb{Z}^n\Box G$ of i-period p_i. Without loss of generality, we can suppose that $i = n$. The set of words

$$\{(x_1, x_2, \ldots, x_n, v)|(x_1, x_2, \ldots, x_n, v) \in S, 0 \leq x_n < p_n\}$$

is a perfect code on $\mathbb{Z}^{n-1}\Box C_{p_n}\Box G$. $\Box$

Thus, there exists a periodic perfect code on the grid $\mathbb{Z}^n$ of period $(p_1, \ldots, p_n)$ if and only if a perfect code on the graph $C_{p_1}\Box \ldots \Box C_{p_n}$ exists.

In 1968, Golomb and Welch [4] proved the existence in Lee metric of perfect one error correcting codes for any word length n over an alphabet of $2n+1$ elements. This is a perfect code on the graph C^n_{2n+1}, and this implies the following well known result.

Theorem 4 *If $k = 0$, for any $n \in \mathbb{N}$, there exists a periodic perfect code on $\mathbb{Z}^n\Box Q_k$.*

4 Constructing perfect codes

A perfect code on $\mathbb{Z}^2\Box Q_1$ exists (see figure 3). We can consider this construction as a tiling of $\mathbb{Z}^2$ with balls of radius 1 and of radius 0. The set of balls of radius 1 is obtain from the set of balls of radius 0 by a translation. With this approach, such a tiling can be generalized for any n on $\mathbb{Z}^n\Box Q_1$.

But it is a special case of a more general construction we present hereby.

Theorem 5 *There exists a perfect code on $\mathbb{Z}^n\Box Q_k$ whenever there are some $\alpha, \beta \in \mathbb{N}^*$ such that $k = 2^\alpha - 1$ and $n = \beta 2^{\alpha-1}$.*

Proof : Let $k = 2^\alpha - 1$ and $n = \beta 2^{\alpha-1}$. We will construct a perfect code on $\mathbb{Z}^n\Box Q_k$.

From theorem 2, there exists a perfect code A_0 on Q_k. We denote by $e_1, \ldots, e_k$ the k vertices of Q_k at distance 1 of $(0, \ldots, 0)$ and by A_i the set $\{c + e_i | c \in A_0\}$. By definition of a perfect code, any A_i is also a perfect code on Q_k and $A_0, A_1, \ldots, A_k$ is a partition of $V(Q_k)$. To prove the theorem, we will construct a mapping from $\mathbb{Z}^n$ that characterizes a perfect code on $\mathbb{Z}^n\Box Q_k$. Denote by $x = (x_1, x_2, \ldots, x_n)$ an element of $\mathbb{Z}^n$.

THE CASE β EVEN:

Let $\beta = 2p$. We have $n = p(k+1)$.

We define the following mapping from $\mathbb{Z}^{p(k+1)}$ to $\mathbb{Z}/(k+1)(2p+1)\mathbb{Z}$:

$$
\begin{array}{rcl}
f(x) & = & \sum_{i=0}^{k}\sum_{j=1}^{p}((2p+1)i+j)x_{ip+j} \bmod (k+1)(2p+1) \\
\text{That is } f(x) & = & x_1 + 2x_2 + \ldots + px_p \\
 & & +(2p+2)x_{p+1} + (2p+3)x_{p+2} + \ldots + (3p+1)x_{2p} \\
 & & + \ldots \\
 & & +(k(2p+1)+1)x_{kp+1} + \ldots + (k(2p+1)+p)x_{(k+1)p} \\
 & & \bmod (k+1)(2p+1)
\end{array}
$$

We claim that the set $C = \{(x,v) | \exists i \in \{0,\ldots,k\} \text{ with } f(x) = i(2p+1), v \in A_i\}$ is a perfect code of $\mathbb{Z}^n \Box Q_k$. To prove this, we will use the following lemma.

Lemma 6 *For any $x \in \mathbb{Z}^n$ and $\theta \in \{1,\ldots,(k+1)(2p+1)-1\}$, there exists exactly one neighbour y of x such that $f(y) = f(x)+\theta$ if $\theta \neq 0 \bmod 2p+1$ and no neighbour if $\theta = 0 \bmod 2p+1$.*

Proof : Let $\theta \in \{1,\ldots,(k+1)(2p+1)-1\}$ satisfy $\theta \neq 0 \bmod 2p+1$. There exist $i \in \{0,\ldots,k\}, j \in \{1,\ldots,2p\}$ such that $\theta = (2p+1)i+j$. If $j \leq p$ then $f(x_1,\ldots,x_{ip+j}+1,\ldots,x_n) = f(x)+\theta$. Else, $f(x_1,\ldots,x_{(k-i)p+(2p+1-j)}-1,\ldots,x_n) = f(x)+\theta$ (notice that $k-i \in \{0,\ldots,k\}$ and $2p+1-j \in \{1,\ldots,p\}$). We have thus considered $(k+1)2p = 2n$ distinct neighbours of x, so every neighbour of x. Therefore there are no neighbour left for the case $\theta = 0 \bmod 2p+1$. $\Box$

Suppose that C is not a code with minimum distance 3. Then there exist two distinct vertices (x,v) and $(x',v') \in C$ at distance less or equal to 2. Let i and i' be the integers such that $f(x) = i(2p+1)$ and $f(x') = i'(2p+1)$. We have $v \in A_i$ and $v' \in A_{i'}$.

1^{st} case : $x = x'$. Thus v and v' are in a same A_i and $v \neq v'$ so $d(v,v') \geq 3$: a contradiction.

2^{nd} case : $d(x,x') = 1$. From lemma 6, $f(x)-f(x') \neq 0 \bmod 2p+1$ but $f(x)-f(x') = (i-i')(2p+1)$: a contradiction.

3^{rd} case : $d(x,x') = 2$. Then $v = v'$; and since $A_0,\ldots,A_k$ is a partition of $V(Q_k)$, $i = i'$ and $f(x) = f(x')$. Let u be a common neighbour of x and x'. We have $f(u)-f(x) = f(u)-f(x')$ which is impossible by lemma 6.

We only have to prove now that this code is perfect. Let (x,v) be a vertex of $\mathbb{Z}^n \Box Q_k$. If $f(x) = i(2p+1)$ then A_i being a perfect code, there exists $v' \in A_i$ (so $(x,v') \in C$) such that $d((x,v),(x,v')) \leq 1$. Else, since $A_0,\ldots,A_k$ is a partition of $V(Q_k)$, there exists $i \in \{0,\ldots,k\}$ such that $v \in A_i$. By lemma 6, there exists a neighbour x' of x such that $f(x') = i(2p+1)$. Thus, $(x',v) \in C$.

THE CASE β ODD:

Let $\beta = 2p+1$. Notice that $\frac{k+1}{2}$ is an integer. We have $n = (2p+1)\frac{k+1}{2}$.
We define the following mapping from $\mathbb{Z}^n$ to $\mathbb{Z}/(k+1)(2p+2)\mathbb{Z}$:

$$
\begin{array}{rcl}
g(x) & = & \sum_{i=0}^{k}\sum_{j=1}^{p}((2p+2)i+j)x_{ip+j} \\
 & & + \sum_{l=1}^{\frac{k+1}{2}}(p+1)(2l-1)x_{(k+1)p+l} \bmod (k+1)(2p+2) \\
\text{That is } g(x) & = & x_1 + 2x_2 + \ldots + px_p \\
 & & +(2p+3)x_{p+1} + (2p+4)x_{p+2} + \ldots + (3p+2)x_{2p} \\
 & & + \ldots \\
 & & +(k(2p+2)+1)x_{kp+1} + \ldots + (k(2p+2)+p)x_{(k+1)p} \\
 & & +(p+1)x_{(k+1)p+1} + \ldots + k(p+1)x_{(k+1)p+\frac{k+1}{2}} \\
 & & \bmod (k+1)(2p+2)
\end{array}
$$

We claim that the set $C = \{(x,v) | \exists i \in \{0,\ldots,k\} \text{ with } g(x) = i(2p+2), v \in A_i\}$ is a perfect code of $\mathbb{Z}^n \square Q_k$. To prove this, we will use the following lemma.

Lemma 7 *For any $x \in \mathbb{Z}^n$ and $\theta \in \{1,\ldots,(k+1)(2p+2)-1\}$, there exists exactly one neighbour y of x such that $g(y) = g(x)+\theta$ if $\theta \neq 0 \bmod 2p+2$ and no neighbour if $\theta = 0 \bmod 2p+2$.*

Proof : Let $\theta \in \{1,\ldots,(k+1)(2p+2)-1\}$ satisfy $\theta \neq 0 \bmod 2p+2$. There exist $i \in \{0,\ldots,k\}, j \in \{1,\ldots,2p+1\}$ such that $\theta = (2p+2)i + j$.

- If $j < p+1$ then $g(x_1,\ldots,x_{ip+j}+1,\ldots,x_n) = g(x)+\theta$.
- If $j > p+1$ then $g(x_1,\ldots,x_{(k-i)p+(2p+2-j)}-1,\ldots,x_n) = g(x)+\theta$ (notice that $k-i \in \{0,\ldots,k\}$ and $2p+2-j \in \{1,\ldots,p\}$).
- If $j = p+1$ then $\theta = (p+1)(2i+1)$ and
  - if $i < \frac{k+1}{2}$, $g(x_1,x_2,\ldots,x_{(k+1)p+i+1}+1,\ldots,x_n) = g(x)+\theta$
  - if $i \geq \frac{k+1}{2}$, $g(x_1,x_2,\ldots,x_{(k+1)p+k-i+1}-1,\ldots,x_n) = g(x)+\theta$ (notice that $k-i+1 \in \{1,\ldots,\frac{k+1}{2}\}$).

We have thus considered $(k+1)(2p+1) = 2n$ distinct neighbours of x, so every neighbour of x. Therefore, there are no neighbour left for the case $\theta = 0 \bmod 2p+2$. □

Suppose that C is not a code with minimum distance 3. Then there exist two distinct vertices (x,v) and $(x',v') \in C$ at distance less or equal to 2. Let i and i' be the integers such that $g(x) = i(2p+2)$ and $g(x') = i'(2p+2)$. We have $v \in A_i$ and $v' \in A_{i'}$.

1^{st} case : $x = x'$. Thus v and v' are in a same A_i and $v \neq v'$ so $d(v,v') \geq 3$: a contradiction.

2^{nd} case : $d(x,x') = 1$. From lemma 7, $g(x) - g(x') \neq 0 \bmod 2p+2$ but $g(x)-g(x') = (i-i')(2p+2)$: a contradiction.

3^{rd} case : $d(x,x') = 2$. Then $v = v'$; and since $A_0, \ldots, A_k$ is a partition of $V(Q_k)$, $i = i'$ and $g(x) = g(x')$. Let u be a common neighbour of x and x'. We have $g(u) - g(x) = g(u) - g(x')$ which is impossible by lemma 7.

We only have to prove now that this code is perfect. Let (x,v) be a vertex of $\mathbb{Z}^n \Box Q_k$. If $g(x) = i(2p+2)$ then A_i being a perfect code, there exists $v' \in A_i$ (so $(x,v') \in C$) such that $d((x,v),(x,v')) \leq 1$. Else, since $A_0, \ldots, A_k$ is a partition of $V(Q_k)$, there exists $i \in \{0, \ldots, k\}$ such that $v \in A_i$. By lemma 7, there exists a neighbour x' of x such that $g(x') = i(2p+2)$. Thus, $(x',v) \in C$. □

Proposition 8 *There exists a i-periodic perfect code on $\mathbb{Z}^{n+1} \Box Q_k$ of i-period 4 if and only if a perfect code on the graph $\mathbb{Z}^n \Box Q_{k+2}$ exists.*

Proof : This proposition is an immediate consequence of theorem 3 since $Q_2 = C_4$. □

Corollary 9 *If there exists an integer p such that $2n + k = 2^p - 1$, then their exists a perfect code on $\mathbb{Z}^n \Box Q_k$.*

Proof : This is a consequence of proposition 8 and theorem 2. □

Corollary 10 *There exists a perfect code on $\mathbb{Z}^n \Box Q_k$ whenever there are some $\alpha, \beta, \gamma \in \mathbb{N}$ such that $k = 2^\alpha - 2\gamma - 1$ and $n = \beta 2^{\alpha-1} + \gamma$.*

Proof : This is a consequence of theorem 5, proposition 8, and corollary 9. □

5 Nonexistence of perfect codes

Theorem 11 *Suppose that $k \geq 2n$. Then there exists a perfect code on $\mathbb{Z}^n \Box Q_k$ if and only if there exists an integer p such that $2n + k = 2^p - 1$.*

Proof : From corollary 9, we know that a perfect code exists when n and k satisfy the condition.

For $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, let $Q_k(x)$ be the set of vertices $(u_1, \ldots, u_{n+k})$ of $\mathbb{Z}^n \Box Q_k$ such that $u_i = x_i$ for any $1 \leq i \leq n$. Suppose there exists a perfect code C on $\mathbb{Z}^n \Box Q_k$.

Since C is a perfect code, any vertex has to be in exactly one ball centered on a vertex of C. A ball centered on a vertex in $Q_k(x)$ contains $k+1$ vertices in $Q_k(x)$. A ball centered on some vertex in a $Q_k(y)$ such that $d(y,x) = 1$ contains exactly one vertex in $Q_k(x)$.

We consider γ the minimum number of vertices of the code we can find in a $Q_k(x)$. Let $x \in \mathbb{Z}^n$ satisfy $|Q_k(x) \cap C| = \gamma$. Let $(y_i)_{1 \leq i \leq 2n}$ be the vertices of $\mathbb{Z}^n$ at distance 1 from x. We define $a_i = |Q_k(y_i) \cap C| - \gamma$. Let $a = \max(a_i)$ and y be a y_i such that $a_i = a$.

Suppose $a = 0$, and thus that every a_i is null. We consider the vertices of $Q_k(x)$. $(k+1)\gamma$ of them are in balls centered on vertices of $Q_k(x)$, and $2n\gamma$ in balls centered on some vertex of $Q_k(y_i)$ for some i. Thus we have $(k+1)\gamma + 2n\gamma = 2^k$. So $k + 1 + 2n$ is a factor of 2^k, and there is some p such that $2n + k = 2^p - 1$.

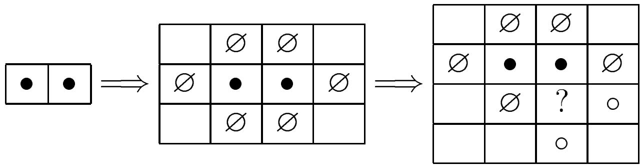

Figure 4: Nonexistence of a perfect code on $\mathbb{Z}^2 \square Q_2$

Now suppose $a > 0$. Counting the vertices of $Q_k(x)$, we get :

$$(k+1)\gamma + \sum_{i=1}^{2n} (a_i + \gamma) = 2^k \tag{1}$$

and by counting those of $Q_k(y)$:

$$(k+1)(\gamma + a) + \sum_{i=1}^{2n} (\gamma + b_i) = 2^k \tag{2}$$

where b_i are nonnegative integers defined in a way similar to the a_i's. Doing (2) − (1), we obtain $a(k+1) + \sum_{i=1}^{2n} b_i = \sum_{i=1}^{2n} a_i$. Then, since $\sum_{i=1}^{2n} b_i \geq 0$ and $\sum_{i=1}^{2n} a_i \leq 2na$, we have $a(k+1) \leq 2na$ and so $k < 2n$. $\square$

Proposition 12 *There exist no perfect code on $\mathbb{Z}^2 \square Q_2$ nor on $\mathbb{Z}^3 \square Q_2$.*

Proof : Notice that any $Q_2(x)$ may contain 1 vertex of the code (Q_2 of type '•') or no vertex (type '∅'). A Q_2 of type '•' has exactly 1 neighbour of type '•' while a type '∅' has exactly 4 neighbours of type '•'. See figure 4 for $\mathbb{Z}^2 \square Q_2$. A similar but tedious case analysis proves the nonexistence of a perfect code on $\mathbb{Z}^3 \square Q_2$. $\square$

6 Conclusion and open problems

We recapitulate our results in table 1 where $\surd$ means existence:

a by theorem 2

b by theorem 4

c by theorem 2 and proposition 8

d by theorem 5

e by theorem 5 and proposition 8

and $-$ means nonexistence by theorem 11 except for $-_f$ that are proven by proposition 12.

Clearly, any empty case fulfilling would be interesting. Furthermore, this table suggest that there are no perfect code on $\mathbb{Z}^n \square Q_k$ when k is even. By proposition 8, it would be sufficient to prove this nonexistence when $k = 2$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	$\dots$
0	$\surd$	$\surd_a$	$-$	$\surd_a$	$-$	$-$	$-$	$\surd_a$	$-$	$-$	$-$	$-$	$\dots$
1	$\surd_b$	$\surd_{c,d}$	$-$	$-$	$-$	$\surd_c$	$-$	$-$	$-$	$-$	$-$	$-$	$\dots$
2	$\surd_b$	$\surd_d$	$-_f$	$\surd_{c,d}$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$\surd_c$	$\dots$
3	$\surd_b$	$\surd_{c,d}$	$-_f$				$-$	$-$	$-$	$\surd_c$	$-$	$-$	$\dots$
4	$\surd_b$	$\surd_d$		$\surd_d$				$\surd_{c,d}$	$-$	$-$	$-$	$-$	$\dots$
5	$\surd_b$	$\surd_{c,d}$				$\surd_c$					$-$	$-$	$\dots$
6	$\surd_b$	$\surd_d$		$\surd_{c,d}$									$\dots$
7	$\surd_b$	$\surd_{c,d}$											$\dots$
8	$\surd_b$	$\surd_d$		$\surd_d$				$\surd_d$					$\dots$
9	$\surd_b$	$\surd_{c,d}$				$\surd_e$							$\dots$

Table 1: Existence of perfect codes on $\mathbb{Z}^n \square Q_k$

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