Degree Sequences of F-Free Graphs

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Abstract

Let F be a fixed graph of chromatic number $r+1$. We prove that for all large n the degree sequence of any F-free graph of order n is, in a sense, close to being dominated by the degree sequence of some r-partite graph. We present two different proofs: one goes via the Regularity Lemma and the other uses a more direct counting argument. Although the latter proof is longer, it gives better estimates and allows F to grow with n.

As an application of our theorem, we present new results on the generalization of the Turán problem introduced by Caro and Yuster [*Electronic J. Combin.* **7** (2000)].

1 Introduction

Denote by $T_r(n)$ the *Turán graph*, namely the complete r-partite graph on n vertices, with parts as equal as possible, and let $t_r(n) := e(T_r(n)) \geq (1-1/r)\binom{n}{2}$. The Erdős-Stone theorem [13] (see also Erdős and Simonovits [12]), the fundamental theorem of extremal

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graph theory, states that for an arbitrary graph F with chromatic number $\chi(F) = r + 1$ it holds that

$$\mathrm{ex}(n, F) := \max\{e(G) : v(G) = n,\ F \not\subset G\} = t_r(n) + o(n^2). \tag{1}$$

In other words, for every F-free graph G of order n there exists an r-partite graph $H = T_r(n)$ with almost as many edges as G. In this paper we consider the question whether analogous statements are true if one compares the degree sequences instead of the total number of edges.

For two graphs H and G with $V(H) = V(G)$ we say that H *dominates* G if $d_H(x) \geq d_G(x)$ for every vertex x. Erdős [9] showed that

for every K_{r+1}-free graph G, there exists an r-partite graph H such that H dominates G. (2)

In order to generalize this to arbitrary forbidden graphs F, we need a few more definitions. Given a non-increasing sequence $\mathbf{g} = (g_1, \dots, g_n)$, let $\mathcal{D}_{k,m}(\mathbf{g})$ be the sequence

$$(\underbrace{g_k - m, \dots, g_k - m}_{k \text{ times}}, g_{k+1} - m, \dots, g_n - m).$$

In other words, we replace the first k largest elements by g_k and then remove m from each element. For non-increasing sequences $\mathbf{g}$ and $\mathbf{h}$ of the same length n, we write $\mathbf{g} \preceq \mathbf{h}$ (and say that $\mathbf{h}$ *dominates* $\mathbf{g}$) if $g_i \leq h_i$ for every $i \in [n]$. We say that $\mathbf{h}$ (k,m)-*dominates* $\mathbf{g}$ if $\mathcal{D}_{k,m}(\mathbf{g}) \preceq \mathbf{h}$.

It is easy to see that if there is a permutation $\pi : [n] \to [n]$ such that $g_i \leq h_{\pi(i)}$ for every $i \in [n]$, then $\mathbf{g} \preceq \mathbf{h}$. Please also note that the notion of domination for sequences is restricted to non-increasing sequences.

Here is our main theorem. (As this will have no effect on our results, we assume that all expressions like εn are integers.)

Theorem 1 *Let F be a fixed non-empty graph of chromatic number $\chi(F) = r + 1$. For any $\varepsilon > 0$ and large $n \geq n_0(\varepsilon, F)$, the degree sequence $g_1 \geq \cdots \geq g_n$ of any F-free graph G is $(\varepsilon n, \varepsilon n)$-dominated by the degree sequence of some r-partite graph H of order n.*

Notice that Theorem 1 implies the Erdős-Stone theorem. As the Reader can see, we allow two operations on degree sequences: ignoring few vertices of high degree and decreasing each degree by a small amount. In Section 2 we will briefly discuss why Theorem 1 seems essentially best possible in the sense that both of these operations are necessary.

The paper is organized as follows. We present two different proofs of Theorem 1. The first proof, in Section 3, goes via the Regularity Lemma, and is simpler and shorter. In Section 4 we prove a more technical statement (Theorem 4) using direct counting, which immediately implies Theorem 1. Although the latter proof is more complicated, it has the big advantage that it allows for the graph F to grow with n. Before we prove Theorem 1, we discuss its various aspects in Section 2.

In Section 5 we present Theorem 5, a slight strengthening of Theorem 1. In Section 6 we present an application of Theorem 5 to the generalization of the Turán problem introduced by Caro and Yuster [5] where instead of the size $e(G) = \frac{1}{2}\sum_{x\in V(G)} d(x)$ of an F-free order-n graph G one has to maximize $e_f(G) = \sum_{x\in V(G)} f(d(x))$ for a given function f. We will prove that if a monotone function f grows 'regularly' (all precise definitions will appear in Section 6) then, asymptotically, it is enough to consider only r-partite order-n graphs, where $r := \chi(F) - 1 \geq 2$.

2 Some Remarks about Theorem 1

Let us begin by observing that both of the operations on degree sequences used in Theorem 1, namely ignoring few vertices of high degree and decreasing each degree by a small amount, are needed.

First consider the case when $r = 1$ and $F = K_{t,t}$. Note that here an r-partite graph means simply the empty graph whose degree sequence is $0, \dots, 0$. One example of a $K_{t,t}$-free graph is $K_{t-1} + \overline{K}_{n-t+1}$, which has $t-1$ vertices of degree $n-1$. Another example can be obtained by taking a random graph $G_{n,p}$, where $p = \varepsilon n^{-\frac{2}{t+1}}$, and removing an edge from each copy of $K_{t,t}$. The expected degree of a vertex is at least

$$p(n-1) - p^{t^2}\binom{n-1}{t-1}\binom{n-t}{t} = \Omega(n^{\frac{t-1}{t+1}}).$$

Using standard probabilistic tools, one can argue that with high probability every vertex has degree of this order of magnitude. Thus we can achieve either a few vertices of very high degree or the reasonably large minimum degree. Combining these constructions (and increasing t) we can have both occurrences.

On the other hand, the dependence of the degrees on t is not known in general. For some special $K_{s,t}$ there are known constructions which beat the above probabilistic argument, see e.g. [1, 2, 4, 11, 14, 16]. Observe that if a $K_{t,t}$-free order-n graph G has m vertices of degree at least d each, then $m\binom{d}{t} \leq (t-1)\binom{n}{t}$, which gives us some restrictions on m and d. Essentially, this is the only general upper bound on degrees we have.

The same, if not bigger, complications arise for $r \geq 2$. Indeed, let $F = K_{r+1}(t)$ be the *blown-up* K_{r+1} (i.e. each vertex is cloned t times). An F-free graph G can be obtained by taking a complete r-partite graph H, $V(H) = \cup_{i=1}^{r} V_i$, and adding into each part V_i an arbitrary $K_{a,a}$-free graph H_i, where $a = \lfloor \frac{t-1}{r} \rfloor + 1$. Thus, all the 'bad' things that can happen to degree sequences for $r = 1$, also occur for the general r.

Notice that we can have two parameters $\varepsilon_1, \varepsilon_2$ in Theorem 1 if the conclusion is that $\mathbf{g}$ is to be $(\varepsilon_1 n, \varepsilon_2 n)$-dominated. In Section 4 we prove the two-parameter version. It is not surprising that there is some trade-off between ε_1 and ε_2: we can decrease one at the expense of the other.

Our bounds are reasonably good when r is fixed. For example, if $\varepsilon_1, \varepsilon_2 > 0$ are fixed, then we can take $F = K_{r+1}(t)$ with $t \geq c \log n$, where $c = c(\varepsilon_1, \varepsilon_2, r) > 0$, while probabilistic constructions show that t must be $O(\log n)$. However, the dependence on

r is very bad. Chvátal and Szemerédi [7, 8] obtained the correct dependence on r in the Erdős-Stone theorem. Unfortunately, their technique does not seem to work for our problem.

3 Proof via the Regularity Lemma

In our arguments we will be encountering a situation when the domination inequality fails for some small set X of vertices. The following lemma helps us to handle such cases.

Lemma 2 *Let $r \geq 2$. Let H' be a complete r-partite graph on $[n]$ with the partition $[n] = V_1' \cup \cdots \cup V_r'$. Let $X \subset [n]$. Then there is a complete r-partite graph H on $[n]$ such that the following conditions hold.*

1. *For every $x \in X$ and $y \in \overline{X}$ we have $d_H(x) \geq d_H(y)$, where $\overline{X} := [n] \setminus X$.*
2. *For every $y \in [n]$ we have $d_H(y) \geq d_{H'}(y) - |X|$.*
3. *If $d_{H'}(y) < n/2$ for some $y \in [n]$, then $d_H(y) \geq d_{H'}(y)$.*

Proof. We iteratively modify H' as follows. As long as there are vertices $x \in X$ and $y \in \overline{X}$ such that $d_{H'}(x) < d_{H'}(y)$, repeat the following step. Of all choices of $y \in \overline{X}$, choose the one with the largest possible degree. Assume, for example, that $x \in V_1'$ and $y \in V_2'$. Clearly, we have $|V_1'| > |V_2'|$. Let I consist of those $i \in [3, r]$ such that $|V_i'| = |V_2'|$ and $V_i' \cap \overline{X} \neq \emptyset$. Move x to V_2'. Next, as long as there are $x' \in V_2' \cap X$ (possibly $x' = x$) and $y' \in V_i' \cap \overline{X}$ with $i \in I$, we move x' to V_i' and y' to V_2'.

It is routine to see that the above step ensures that $d_{H'}(x) \geq d_{H'}(z)$ for each $z \in \overline{X}$ and this property of x cannot be violated by any subsequent step. Thus we perform at most $|X|$ steps in total.

Let H be the final graph. Clearly it satisfies Condition 1. As the degree of any vertex $z \in [n]$ can drop down by at most one at each step, Condition 2 follows. Furthermore, if we initially had $d_{H'}(y) < n/2$ for some vertex y, then the part V_i' of H' containing y is strictly larger than any other part and, as it is easy to see, never increases its size. (While no new part of order larger than $n/2$ can be created.) This establishes Condition 3 and finishes the proof. ∎

Our first proof of Theorem 1 relies on the following result of Erdős, Frankl and Rödl [10, Theorem 1.5].

Theorem 3 *For every $c > 0$ and a graph F with $\chi(F) = r + 1$, there is a constant $n_0 = n_0(c, F)$ with the following property. Let G be a graph of order $n \geq n_0$ that does not contain F as a subgraph. Then G contains a set E' of less than cn^2 edges such that the subgraph G' obtained from G by deleting all edges in E' has no K_{r+1}.* ∎

Theorem 3 is proved by applying Szemerédi's Regularity Lemma so the constant $n_0 = n_0(c, F)$ given by the proof is huge, see Gowers [15].

Proof of Theorem 1. Given $\varepsilon > 0$, let $c = \varepsilon^2/8$ and let $n_0 = n_0(c, F)$ be given by Theorem 3. Given an F-free graph G of order $n \geq n_0$, let $G' \subset G$ be the K_{r+1}-free graph given by Theorem 3. Applying the theorem of Erdős as stated in (2) gives us an r-partite graph H' that dominates G'. We have $V(H') = V(G') = V(G)$.

Let $X = \{x \in V(G) : d_{G'}(x) \leq d_G(x) - \varepsilon n/2\}$. We have

$$\varepsilon^2 n^2/8 = cn^2 \geq e(G) - e(G') \geq \varepsilon n|X|/4,$$

which implies that $|X| \leq \varepsilon n/2$.

Let H be the r-partite graph obtained by applying Lemma 2 to H' and X. For every $y \in \overline{X}$, we have

$$d_H(y) \geq d_{H'}(y) - |X| \geq d_{G'}(y) - \varepsilon n/2 \geq d_G(y) - \varepsilon n.$$

As the vertices of X have the largest degrees in H and $|X| < \varepsilon n$, it follows that H is the required r-partite graph. ∎

4 Direct Proof

The following is a more technical but stronger result than Theorem 1. For a real x and a positive integer i, we define $\binom{x}{i} = x(x-1)\dots(x-i+1)/i!$.

Theorem 4 *Let $r \geq 2$. Suppose that integers m, n and $s_1 \geq s_2 \geq \dots \geq s_{r+1} \geq 1$ satisfy $s_1 \leq n/2$ and*

$$\frac{im}{2}\binom{ims_i/2n}{s_{i+1}}^i > (s_{i+1} - 1)\binom{s_i}{s_{i+1}}^i, \quad i \in [r]. \tag{3}$$

Suppose that the degree sequence $g_1 \geq \dots \geq g_n$ of an order-n graph G cannot be (s_1, m)-dominated by the degree sequence of an r-partite order-n graph H.

Then G contains $K_{r+1}(s_{r+1})$ as a subgraph.

Proof. Define $l_1 := s_1$, $a_1 := g_{l_1}$, and then, inductively for $i = 2, \dots, r$, let

$$\begin{aligned} l_i &:= \sum_{j=1}^{i-1}(n - a_j + m), \\ a_i &:= g_{l_i}. \end{aligned}$$

Finally, we let $l_{r+1} := n$.

First, we justify that the a_i's are well-defined. We trivially have $l_2 \leq \dots \leq l_r$. (Please note that we do not claim that $l_1 \leq l_2$.) Thus, it is enough to show that $l_r \leq n$. We will prove the stronger claim that

$$\sum_{j=1}^{i}(n - a_j + m) \leq n, \quad \text{for each } i \leq r, \tag{4}$$

which we will need later.

Suppose that (4) is not true. Let $i \le r$ be the smallest index such that $\sum_{j=1}^{i}(n - a_j + m) > n$. Consider the complete i-partite graph H with part sizes $v_1, \ldots, v_i$, where $v_j := n - a_j + m$, for $j \in [i-1]$, and $v_i := n - \sum_{j=1}^{i-1} v_j$.

We show that the degree sequence $h_1 \ge \cdots \ge h_n$ of the graph H (s_1, m)-dominates $\mathbf{g}$, which would be the desired contradiction. To do so, it is enough to check that for every $j \in [2, i]$ the $(v_1 + \cdots + v_{j-1} + 1)$-th component of $\mathcal{D}_{s_1,m}(\mathbf{g})$ is at most $n - v_j$, that is,

$$g_{v_1+\cdots+v_{j-1}+1} - m \le n - v_j, \tag{5}$$

and that

$$g_{s_1} - m \le h_{s_1}. \tag{6}$$

In order to prove (5) note that $v_1 + \cdots + v_{j-1} = l_j$ by definition of l_j, thus by the monotonicity of $\mathbf{g}$ we have $g_{v_1+\cdots+v_{j-1}+1} \le g_{l_j} = a_j$. We have $a_j - m = n - v_j$ for every $j \in [i-1]$ while $a_i - m < n - v_i$, which implies (5).

Let us turn to (6). Assume that $s_1 \le v_1$, for otherwise (6) follows from (5). Then $h_{s_1} = n - v_1 = g_{s_1} - m$ and (6) becomes an identity. This proves (4).

Assume that $V(G) = [n]$ with $i \in V(G)$ having degree g_i.

Initially, set $S_{1,1} = [l_1]$. Before the i-th step of our procedure, $i = 1, \ldots, r$, we have disjoint s_i-sets

$$S_{1,i} \subset [l_1],\ S_{2,i} \subset [l_2],\ \ldots,\ S_{i,i} \subset [l_i]$$

such that they span a $K_i(s_i)$-subgraph in G. By the monotonicity of $\mathbf{g}$ we know that each vertex x in $S_{j,i}$ has degree at least a_j in G. Hence, x has at least $a_j + l_{i+1} - n$ neighbors in $L_{i+1} := [l_{i+1}]$, and the number of edges between $S_i := \cup_{j=1}^{i} S_{j,i}$ and L_{i+1} is at least

$$s_i \sum_{j=1}^{i} (a_j + l_{i+1} - n) = s_i\Big(il_{i+1} - \Big(\sum_{j=1}^{i} n - a_j\Big)\Big) \ge s_i((i-1)l_{i+1} + im),$$

where we counted the edges that lie inside the intersection $S_i \cap L_{i+1}$ twice. The above estimate holds also for $i = r$ by (4). (Recall that $l_{r+1} = n$.) Let

$$Z := \Big\{ z \in L_{i+1} : |\Gamma(z) \cap S_i| \ge (i-1)s_i + \frac{ims_i}{2l_{i+1}} \Big\},$$

where $\Gamma(z)$ denotes the set of neighbors of z.

Counting the edges between S_i and L_{i+1} as seen from L_{i+1} (again counting twice those in the intersection), we obtain

$$s_i((i-1)l_{i+1} + im) \le is_i|Z| + \left((i-1)s_i + \frac{ims_i}{2l_{i+1}}\right)(l_{i+1} - |Z|).$$

This implies that

$$|Z| \ge \frac{im}{2}. \tag{7}$$

Now, every $z \in Z$ intersects each $S_{j,i}$ in at least $\frac{ims_i}{2l_{i+1}}$ points, so it covers at least $\binom{ims_i/2l_{i+1}}{s_{i+1}}^i$ copies of $K_i(s_{i+1})$. By (3) (and $l_{i+1} \le n$ and $|Z| \ge \frac{im}{2}$) we conclude that at least one such subgraph is covered at least s_{i+1} times. Let the parts of this subgraph be

$$S_{1,i+1} \subset S_{1,i}, \ \ldots, \ S_{i,i+1} \subset S_{i,i},$$

while let $S_{i+1,i+1}$ be the corresponding s_{i+1}-subset of $Z \subset [l_{i+1}]$.

This gives us the desired $K_{i+1}(s_{i+1})$ and finishes the description of the step. The theorem is proved. ∎

It is clear that in Theorem 4 it is advantageous to us to take for s_{i+1}, after m and s_i have been chosen, the largest integer satisfying (3). Thus we essentially have only two parameters: m and s_1.

It is not hard to see that Theorem 4 implies Theorem 1. In fact, if $m = \Theta(n)$ and $s_1 = \Theta(\log n)$ (and r is fixed), then we can take $s_{r+1} = \Theta(\log n)$. In general, we have some freedom in choosing s_1 and m. For example, if $F = K_{r+1}(s_{r+1})$ is fixed, then for any $m = \Theta(n)$ Theorem 4 can be satisfied for a sufficiently large constant s_1.

5 Ensuring Small Relative Errors

Here we slightly strengthen Theorem 1. Roughly speaking, we require that the additive error εn in Theorem 1 is replaced by the relative error $1+\varepsilon$. (Thus we have to be more careful about vertices of small degree.) Although the new Theorem 5 is formally stronger than Theorem 1, it can be deduced from the latter. This 'relative' version is needed for our application in Section 6.

For a scalar λ and and a sequence $\mathbf{g} = (g_1, \ldots, g_n)$, let $\lambda\,\mathbf{g}$ denote the sequence $(\lambda g_1, \ldots, \lambda g_n)$.

Theorem 5 *Let F be a fixed graph of chromatic number $\chi(F) = r+1 \ge 3$. For any $\varepsilon > 0$ there is $n_0(\varepsilon)$ such that the following holds. Let G be an arbitrary F-free graph of order $n \ge n_0(\varepsilon)$. Then there is an r-partite graph H on the same vertex set such that $\mathbf{h} \succeq (1-\varepsilon)\mathcal{D}_{\varepsilon n,0}(\mathbf{g})$, where $\mathbf{g}$ and $\mathbf{h}$ are the (non-increasing) degree sequences of G and H respectively.*

Proof. Let ε be sufficiently small. Let $\delta := \varepsilon/3$. First we prove that there is an r-partite graph H' on the same vertex set $V := V(G)$ such that $d_{H'}(x) \ge (1-\delta)\, d_G(x)$ with the exception of vertices from some set X of size at most δn.

Define $A := \{x \in V : d_G(x) \le \delta n/8\}$ and $B := V \setminus A$. The subgraph $G' := G[B]$ spanned by B is of course F-free. If $|B| \le \delta n$, then we are done: take $H' = T_r(n)$ and $X = B$.

So, assume that $|B| \to \infty$. Apply Theorem 1 to $G[B]$ with respect to the constant $c = \delta^2/8$ to obtain an r-partite H' on B. Let X consist of cn vertices of $G[B]$ with the largest degrees. (This set X, with some further additions, will be the exceptional set.)

Extend H' to a complete r-partite graph on V by arbitrarily splitting A into r almost equal parts.

If $|A| \geq \delta n/4$, then for any $x \in A$ we have $d_{H'}(x) \geq \lfloor \frac{1}{2}|A| \rfloor \geq \delta n/8 \geq d_G(x)$, i.e., we are doing fine. Otherwise, add A to X.

Let $C := \{x \in B : |\Gamma_G(x) \cap A| \geq \frac{1}{2}|A|\}$, where $\Gamma_G(x)$ denotes the set of G-neighbors of a vertex $x \in V(G)$. By counting the edges between A and C, we obtain

$$|C| \times \frac{|A|}{2} \leq |A| \times \frac{\delta n}{8},$$

that is, $|C| \leq \delta n/4$. We add C to X. Notice that any vertex of $B \setminus C$ has at least as many A-neighbors in H' as it has in G.

Every vertex $x \in B \setminus (X \cup C)$ has G-degree at least $\delta n/8$. We have,

$$|\Gamma_G(x) \cap B| - |\Gamma_{H'}(x) \cap B| \leq cn = \delta^2 n/8 \leq \delta\, d_G(x),$$

and $|\Gamma_G(x) \cap A| \leq |\Gamma_{H'}(x) \cap A|$, so $d_{H'}(x) \geq (1-\delta)\, d_G(x)$, as required. Also,

$$|X| \leq cn + \delta n/4 + \delta n/4 \leq \delta n.$$

This shows the existence of the desired graph H'.

Let the r-partite graph H be obtained by applying Lemma 2 to H' and X. By Conditions 1 and 2 the vertices of X have the largest H-degrees while the degree of any vertex dropped down by at most $|X| \leq \delta n$.

Let us compare the degrees of $x \in \overline{X}$ with respect to G and H. If $d_{H'}(x) \geq n/2$, then we have

$$d_H(x) \geq d_{H'}(x) - \delta n \geq (1-2\delta)\, d_{H'}(x) \geq (1-2\delta)\,(1-\delta)\, d_G(x) \geq (1-\varepsilon)\, d_G(x).$$

If $d_{H'}(x) < n/2$, then by Condition 3 we have $d_H(x) \geq d_{H'}(x) \geq (1-\delta)\, d_G(x)$.

Finally, the vertices of X are also 'happy' because they have the largest H-degrees while $|X| < \varepsilon n$. This completes the proof of the theorem. ∎

6 Generalized Turán Problem

Let $\mathbb{N}$ denote the set of non-negative integers and $\mathbb{R}$ the set of reals. Let $f : \mathbb{N} \to \mathbb{R}$ be an arbitrary function.

For a graph F define

$$e_f(F) := \sum_{x \in V(F)} f(d(x)),$$

where $d(x)$, as usually, denotes the degree of a vertex x. For example, for $f : x \mapsto \frac{x}{2}$ we have $e_f(F) = e(F)$; thus $e_f(F)$ can be viewed as a generalization of the size of F.

Define $\mathrm{ex}_f(n, F)$ to be the maximal value of $e_f(G)$ over all F-free graphs G of order n. This mimics the definition of the usual Turán function $\mathrm{ex}(n, F)$. The special case when

f is the *power function* $P_\mu : x \mapsto x^\mu$, with integer $\mu \geq 1$, was introduced by Caro and Yuster [5]. This paper was one of the motivations for the present research.

Let $\mathrm{ex}'_f(n, F)$ be the maximum of $e_f(H)$ over all complete $(\chi(F)-1)$-partite graphs of order n. Clearly, we have $\mathrm{ex}'_f(n, F) \leq \mathrm{ex}_f(n, F)$. Moreover, observe that since computing $\mathrm{ex}'_f(n, F)$ consists only of determining the *sizes* of a complete $(\chi(F)-1)$–partite graph H that give the optimal value for $e_f(H)$, this is more of an analytical (although possibly difficult) task than a combinatorial one. (Bollobás and Nikiforov [3] investigated this problem for the power function P_μ.)

A function $f : \mathbb{N} \to \mathbb{R}$ is called *positive* if $f(n) > 0$ for any $n \in \mathbb{N}$; f is *non-decreasing* if for any $m \leq n$ we have $f(m) \leq f(n)$. Let us call a positive non-decreasing function f *log-continuous* if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any $m, n \in \mathbb{N}$ with $n \leq m \leq (1+\delta)n$ we have

$$f(m) \leq (1+\varepsilon) f(n). \tag{8}$$

For example, P_μ is log-continuous for any $\mu > 0$ while the exponent $x \mapsto \mathrm{e}^x$ is not.

Using Erdős' result (2) it is easy to prove (see [5, 6]) that for any $n \geq 0$, $r \geq 2$ and non-decreasing $f : \mathbb{N} \to \mathbb{R}$ we have

$$\mathrm{ex}_f(n, K_{r+1}) = \mathrm{ex}'_f(n, K_{r+1}). \tag{9}$$

Caro and Yuster [5] posed the problem of computing $\mathrm{ex}_{P_\mu}(n, F)$ for an arbitrary graph F.

Here we show that if F is a fixed graph of chromatic number $r + 1 \geq 3$ and f is a positive, non-decreasing and log-continuous function, then the analog of (9) holds asymptotically.

Theorem 6 *Let F be a fixed non-bipartite graph. Let $f : \mathbb{N} \to \mathbb{R}$ be an arbitrary positive, non-decreasing, and log-continuous function. Then, as $n \to \infty$,*

$$\mathrm{ex}_f(n, F) = (1 + o(1))\, \mathrm{ex}'_f(n, F).$$

Proof. Let $r := \chi(F) - 1 \geq 2$, $c > 0$ be arbitrary, n be large, and G achieve $\mathrm{ex}_f(n, F)$.

First, let us observe that by the assumptions on f

$$\mathrm{ex}_f(n, F) \geq \mathrm{ex}'_f(n, F) \geq e_f(T_r(n)) \geq n f(\lfloor \tfrac{nr}{r+1} \rfloor) \geq \gamma n f(n), \tag{10}$$

for some constant $\gamma > 0$.

Let δ be such that $f(m) \leq (1 + c/2) f(n)$ if $m \leq (1 + 2\delta)n$. Assume that $(1-\delta)^{-1} < 1 + 2\delta$. Apply Theorem 5 to G with respect to $\varepsilon = \min(\delta, c\gamma/2)$ to obtain an r-partite graph H. Let X be the set of $\lfloor \varepsilon n \rfloor$ vertices of H of the largest degrees.

We have

$$\sum_{x \in X} \Big(f(d_G(x)) - f(d_H(x)) \Big) \leq |X| f(n) \leq \frac{c}{2}\, \mathrm{ex}_f(n, F),$$

where we used (10). By the definition of H and δ, we have

$$\sum_{x \in V \setminus X} \Big(f(d_G(x)) - f(d_H(x)) \Big) \leq \frac{c}{2} \sum_{x \in V \setminus X} f(d_H(x)) \leq \frac{c}{2}\, \mathrm{ex}_f(n, F).$$

It follows that

$$e_f(G) - e_f(H) = \sum_{x \in V} \Big(f(d_G(x)) - f(d_H(x)) \Big) \le c\,\mathrm{ex}_f(n, F), \tag{11}$$

proving the theorem as $c > 0$ was arbitrary. ∎

Remark. Taking $f : x \mapsto \log x$ we can also deal with the problem of maximizing $\prod_{x \in V(G)} d(x)$ over all F-free graphs G of order n. (However, please notice that the relative error here will not be $1 + o(1)$ but becomes such after taking the logarithm.) More generally, we can maximize $\prod_{x \in V(G)} f(d(x))$ for any non-decreasing f such that $\log(f(x))$ is positive and log-continuous.

6.1 Some Negative Examples

In Theorem 6 we do need some condition bounding the rate of growth of f. For example, if f grows so fast that $e_f(G)$ is dominated by the contribution from the vertices of degree $n-1$, then the conclusion of Theorem 6 is no longer true: for example, for $K_3(2)$ (the blown-up K_3 where each vertex of K_3 is duplicated) the value $\mathrm{ex}_f(n, K_3(2)) = (3 + o(1))\, f(n-1)$ cannot be achieved by a bipartite graph.

In fact, one can construct refuting examples of f with moderate rate of growth. For example, for any constant $c < 1$ there is a positive non-decreasing f such that

$$\frac{f(n+1)}{f(n)} \le 1 + n^{-c} \tag{12}$$

for any n and yet the conclusion of Theorem 6 does not hold for this f. Let us demonstrate the above claim.

Let $c > 0$. Choose t such that for all large n there is a $K_{t,t}$-free graph G_n of order n with all vertices having degree at least n^c each. Such t exists by the probabilistic construction of Section 2.

Let $F = K_3(2t-1)$ be a blown-up K_3. Take an arbitrary function f satisfying (12) and the additional property that there is an infinite sequence $n_1 < n_2 < \dots$ such that for any k we have

$$f(n_k + m_k) = f(n_k + m_k + 1) = f(n_k + m_k + 2) = \cdots = f(2n_k),$$

while $f(n_k) \le \frac{1}{2} f(n_k + m_k)$, where $m_k = \lceil n^c \rceil$. Such an f exists: choose the numbers n_k spaced far apart (with n_1 being sufficiently large), let $f(n+1) = f(n)$, except for $n_k \le n < n_k + m_k$ we let $f(n+1) = 2^{1/m_k} f(n)$. Note that $2^{1/m_k} < 1 + \frac{1}{m_k} < 1 + n^{-c}$ so our f does satisfy (12).

On the one hand, we have

$$\mathrm{ex}_f(2n_k, F) \ge 2\, n_k\, f(n_k + m_k). \tag{13}$$

Indeed, let G be obtained from the complete bipartite graph K_{n_k,n_k} by adding to each part the $K_{t,t}$-free graph G_{n_k} defined above. It is easy to see that $G \not\supset F$. All vertices of G have degree at least $n_k + m_k$, giving (13).

On the other hand, for any bipartite graph H of order $2n_k$ at least n_k vertices will have degree at most n_k and thus

$$e_f(H) \leq n_k f(2n_k - 1) + n_k f(n_k) \leq \frac{3}{2} n_k f(n_k + m_k).$$

We obtain by (13) that $\mathrm{ex}_f(n, F)$ cannot always be approximated by $\mathrm{ex}'_f(n, F)$.

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