Permanents of Hessenberg (0,1)-matrices

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Abstract

Let $P(m,n)$ denote the maximum permanent of an n-by-n lower Hessenberg $(0,1)$-matrix with m entries equal to 1. A "staircased" structure for some matrices achieving this maximum is obtained, and recursive formulas for computing $P(m,n)$ are given. This structure and results about permanents are used to determine the exact values of $P(m,n)$ for $n \leq m \leq 8n/3$ and for all $\mathrm{nnz}(H_n) - \mathrm{nnz}(H_{\lfloor n/2 \rfloor}) \leq m \leq \mathrm{nnz}(H_n)$, where $\mathrm{nnz}(H_n) = (n^2+3n-2)/2$ is the maximum number of ones in an n-by-n Hessenberg $(0,1)$-matrix.

1 Introduction

A *transversal* of an n-by-n $(0,1)$-matrix $A = [a_{ij}]$ is a collection of n entries of A equal to 1, no two of which are in the same row or column. The *permanent* of A, denoted per A, is the number of distinct transversals of A. Equivalently,

$$\text{per } A = \sum_{\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)},$$

where the sum is over all permutations σ of $\{1,2,\ldots,n\}$. We refer the reader to [M] for classic results, and to [CW] for a survey of recent research on permanents. A matrix A is a *lower Hessenberg matrix* if $a_{ij}=0$ whenever $j \geq i+2$. Throughout the remainder of the paper, we abbreviate lower Hessenberg to Hessenberg. Let $\mathcal{H}(m,n)$ denote the set of n-by-n Hessenberg $(0,1)$-matrices with m entries equal to 1, and let $P(m,n)$ denote the maximum permanent of a matrix in $\mathcal{H}(m,n)$.

In [BR, Ch. 7], computation of the permanent of an arbitrary rectangular matrix is considered. Additionally, upper and lower bounds for the permanent of such a $(0,1)$-matrix A are given in terms of the number of ones in each row of A, the number of ones in each column of A, or the total number of ones in A. In [SHRC, Th. 2.3, 2.5], the maximum value of the permanent of a p-by-q $(0,1)$-matrix with m entries equal to 1 for $pq-\max\{p,q\} \leq m \leq pq-2$ is given, and matrices attaining this value are determined. In [BGM, Th. 2.2], the maximum value of the permanent of an n-by-n $(0,1)$-matrix with m entries equal to 1 for $n \leq m \leq 2n$ is determined, and we observe that every matrix achieving this maximum is combinatorially equivalent to a Hessenberg matrix. In addition, the matrices attaining the maximum value of the permanent of an n-by-n $(0,1)$-matrix for $n^2-2n \leq m \leq n^2$ are determined. In this paper, we focus on Hessenberg matrices and determine the exact value of $P(m,n)$ for $n \geq 2$ and various values of m with $n \leq m \leq \frac{n^2+3n-2}{2}$.

We first state some notation and terminology (see [BR] for further details). The number of nonzero entries of the matrix A is denoted by $\mathrm{nnz}(A)$. For integers i and j with $i \leq j$, denote $\{i,i+1,\ldots,j\}$ by $\langle i,j\rangle$, with $\{i\}=\langle i\rangle$ abbreviated to i. The submatrix of A with entries from rows $\langle i_1,i_2\rangle$ and columns $\langle j_1,j_2\rangle$ is denoted by $A[\langle i_1,i_2\rangle,\langle j_1,j_2\rangle]$, with $A[\langle i_1,i_2\rangle,\langle i_1,i_2\rangle]$ abbreviated to $A[\langle i_1,i_2\rangle]$. Similarly, the submatrix of A obtained by deleting rows $\langle i_1,i_2\rangle$ and columns $\langle j_1,j_2\rangle$ is denoted by $A(\langle i_1,i_2\rangle,\langle j_1,j_2\rangle)$, with $A(\langle i_1,i_2\rangle,\langle i_1,i_2\rangle)$ abbreviated to $A(\langle i_1,i_2\rangle)$.

The matrix A is *partly decomposable* if there exist permutation matrices P and Q such that PAQ has the form

$$\begin{bmatrix} B & O \\ C & D \end{bmatrix},$$

where B and D are square (nonvacuous) matrices. Equivalently, A is partly decomposable if and only if it contains a zero submatrix with dimensions summing to n. If A is not partly decomposable, then A is *fully indecomposable*. If per $A>0$, then there exist permutation matrices P and Q, and an integer b such that PAQ has the form

$$\begin{bmatrix} A_1 & O & O & \cdots & O \\ A_{21} & A_2 & O & \cdots & O \\ \vdots & & \ddots & & \vdots \\ A_{b-1,1} & A_{b-2,2} & & A_{b-1} & O \\ A_{b1} & A_{b2} & \cdots & A_{b,b-1} & A_b \end{bmatrix}, \tag{1}$$

where the matrices $A_1, \ldots, A_b$ are fully indecomposable. The n_i-by-n_i matrices A_i are the *fully indecomposable components* of A and are unique up to permutation of rows and columns. Note that per $A = \prod_{i=1}^{b}$ per A_i. The matrix A has *total support* provided per $A(i,j) > 0$ for all i and j such that $a_{ij} = 1$; i.e., every nonzero entry of A is on some transversal.

2 Preliminary Results

In this section we develop some basic preliminary results concerning the structure and permanents of matrices in $\mathcal{H}(m,n)$. The following shows that the fully indecomposable components of a Hessenberg matrix are each permutationally equivalent to a Hessenberg matrix.

Lemma 2.1 *Let $A = [a_{ij}]$ be an n-by-n Hessenberg $(0,1)$-matrix with* per $A > 0$. *Then each fully indecomposable component of A is permutationally equivalent to a Hessenberg matrix.*

Proof. The proof is by induction on n, with the result clearly true for $n = 1$. Without loss of generality, assume that A has total support. Since per $A > 0$, column n of A contains at least one 1. Let B be the fully indecomposable component of A that intersects column n.

If there is some j such that $a_{j,j+1} = 0$, then the fully indecomposable components of A are those of $A[\langle 1,j\rangle]$ and $A[\langle j+1,n\rangle]$, and applying induction to each of these matrices yields that each fully indecomposable component of A is permutationally equivalent to a Hessenberg matrix. If $a_{nn} = 0$, then the fully indecomposable components of A are the 1-by-1 matrix $[a_{n-1,n}]$, and those of the Hessenberg matrix $A(n-1,n)$. Again the inductive hypothesis applies, and hence each fully indecomposable component of A is permutationally equivalent to a Hessenberg matrix. A similar argument handles the case that $a_{11} = 0$.

Now assume that $a_{11} = 1$, $a_{nn} = 1$ and $a_{j,j+1} = 1$ for $j = 1, 2, \ldots, n-1$. If each column of A is a column of B, then $B = A$, and clearly the fully indecomposable component (namely B) of A is Hessenberg. Otherwise, some column of A does not intersect the columns of B. Let j be the largest integer such that column j of A does not intersect the columns of B. Note $j < n$. Since B intersects columns $j+1, \ldots, n$ of A, B must contain each of the entries in positions $(j, j+1), (j+1, j+2), \ldots, (n-1, n)$ and (n,n) of A (otherwise B would be partly decomposable). This implies that B intersects rows $j, \ldots, n$. If there is some $i \geq j$ such that $a_{ij} = 1$, then the fully indecomposable component that contains a_{ij} has a row in common with B, and hence must be equal to B. But B does not intersect column j. So $a_{ij} = 0$ for $i = j, j+1, \ldots, n$. Now column j has just one 1, namely $a_{j-1,j} = 1$. Hence the 1-by-1 matrix $[a_{j-1,j}]$ is a fully indecomposable component of A. It follows that the fully indecomposable components of A are $[a_{j-1,j}]$ and the fully indecomposable components of $A(j-1,j)$. As $A(j-1,j)$ is Hessenberg, the inductive hypothesis applies. Hence each fully indecomposable component of A is permutationally equivalent to a Hessenberg matrix. ∎

A Hessenberg $(0,1)$-matrix A is *staircased* if whenever $i \geq j$ and $a_{ij} = 0$, then $a_{kj} = 0$ for $k = i+1, \ldots, n$ and $a_{il} = 0$ for $l = 1, \ldots, j-1$. Note that if A is staircased and $a_{ij} = 0$, then $a_{kl} = 0$ for all $i \leq k \leq n$ and $1 \leq l \leq j$.

Lemma 2.2 *The following hold for an n-by-n Hessenberg $(0,1)$-matrix $A = [a_{ij}]$:*

(a) If A is fully indecomposable, then $a_{11} = 1$, $a_{nn} = 1$ and $a_{i,i+1} = 1$ for $i = 1, 2, \ldots, n-1$.

(b) If A is fully indecomposable and staircased, then $a_{i+1,i} = 1$ for $i = 1, 2, \ldots, n-1$, and $a_{ii} = 1$ for $i = 1, 2, \ldots, n$.

(c) If each $a_{i,i+1} = 1$ $(i = 1, 2, \ldots, n-1)$ and k and l are integers such that $n \geq k \geq l \geq 1$, then

$$\operatorname{per} A(k,l) = \operatorname{per} A[\langle 1, l-1\rangle] \operatorname{per} A[\langle k+1, n\rangle],$$

in which a vacuous permanent with $l = 1$ or $k = n$ is set equal to 1.

Proof. If there is a j with $a_{j,j+1} = 0$, then $A[\langle 1, j\rangle, \langle j+1, n\rangle]$ is a zero submatrix of A with dimensions summing to n, and hence A is not fully indecomposable. If $a_{11} = 0$ or $a_{nn} = 0$, then A has a row or column with a single 1, and hence A is not fully indecomposable. These observations prove (a).

If there is an i such that $a_{i+1,i} = 0$, then (since A is staircased) $A[\langle i+1, n\rangle, \langle 1, i\rangle] = O$, and hence A is not fully indecomposable. Similarly, if there is an i such that $a_{ii} = 0$, then (since A is staircased) $A[\langle i, n\rangle, \langle 1, i\rangle] = O$, and hence A is not fully indecomposable. This proves (b).

Statement (c) follows by noting that $A(k,l)$ has the form

$$\begin{bmatrix} A[\langle 1, l-1\rangle] & O & O \\ * & A[\langle l, k-1\rangle, \langle l+1, k\rangle] & O \\ * & * & A[\langle k+1, n\rangle] \end{bmatrix}$$

and that $A[\langle l, k-1\rangle, \langle l+1, k\rangle]$ is a lower triangular (possibly vacuous) matrix with each of its main diagonal entries equal to 1.

■

We now show that $\mathcal{H}(m,n)$ contains a special type of matrix with maximum permanent. For a Hessenberg $(0,1)$-matrix A, an *interchangeable column pair* of A is a pair of entries (k,l) and $(k-1,l)$ with $k > l$ such that $a_{kl} = 1$ and $a_{k-1,l} = 0$. An *interchangeable row pair* of A is a pair of entries (k,l) and $(k,l+1)$ with $k > l$ such that $a_{kl} = 1$ and $a_{k,l+1} = 0$.

Theorem 2.3 *Let m and n be positive integers with $n \leq m \leq \frac{n^2+3n-2}{2}$. Then there exists a matrix $A \in \mathcal{H}(m,n)$ with permanent $P(m,n)$ such that A has the form (1), where each A_i is a fully indecomposable staircased Hessenberg matrix.*

Proof. Let $A \in \mathcal{H}(m,n)$ with per $A = P(m,n)$. By Lemma 2.1, assume that A has the form (1), where each A_i is a fully indecomposable Hessenberg matrix. We prove by induction on n that there is a matrix in $\mathcal{H}(m,n)$ with permanent $P(m,n)$ having each fully indecomposable component staircased. This is clearly true for $n = 1$.

First suppose that $b \geq 2$. Since per $A = \prod_{i=1}^{b}$ per A_i, it follows that per $A_i = P(\text{nnz}(A_i), n_i)$ for $i = 1, 2, \ldots, b$. By induction, each A_i is staircased.

Next suppose that $b = 1$, that is, A is fully indecomposable. We construct a sequence of matrices $B_r \in \mathcal{H}(m,n)$ as follows:

(a) $B_0 \longleftarrow A$

(b) $r \longleftarrow 0$

(c) While (B_r is fully indecomposable and has an interchangeable row or column pair) do:

(c1) If B_r has an interchangeable column pair, then choose such a pair (k,l), $(k-1,l)$ with l largest, and define B_{r+1} to be the matrix obtained from B_r by interchanging the 1 in position (k,l) with the 0 in position $(k-1,l)$.

(c2) Else if B_r has an interchangeable row pair, then choose such a pair (k,l), $(k,l+1)$ with k smallest, and define B_{r+1} to be the matrix obtained from B_r by interchanging the 1 in position (k,l) with the 0 in position $(k,l+1)$.

(c3) $r \longleftarrow r+1$.

Note that this algorithm terminates since B_{r+1} is either partly decomposable (in which case the algorithm is applied to the smaller fully indecomposable components) or remains fully indecomposable with fewer pairs (i,j) and (i',j') than B_r such that $j' \leq j \leq i \leq i'$, with (i',j') entry 1 and (i,j) entry 0.

Let the sequence of matrices generated by the algorithm be $B_0, B_1, B_2, \ldots, B_s$. Clearly $B_i \in \mathcal{H}(m,n)$ for $i = 1, 2, \ldots, s$. We claim that per $B_i \geq$ per B_{i-1} for $i = 1, 2, \ldots, s$. To see this let $C = B_i$ and $D = B_{i-1}$. Since D is fully indecomposable, Lemma 2.2 implies that $d_{i,i+1} = 1$ $(i = 1, 2, \ldots, n-1)$ and $d_{11} = d_{nn} = 1$. First assume that D has an interchangeable column pair. Let (k,l) and $(k-1,l)$ be the interchangeable column pair chosen to construct C. If $d_{kk} = 0$, then (by the choice of l) $a_{jk} = 0$ for $j > k$. This would imply that column k of D has just one nonzero entry, contrary to the full indecomposability of D. Hence $d_{kk} = 1$. By Lemma 2.2,

$$\text{per } D(k,l) = \text{ per } D[\langle 1, l-1\rangle] \, \text{per } D[\langle k+1, n\rangle] \tag{2}$$

and

$$\text{per } D(k-1,l) = \text{per } D[\langle 1, l-1\rangle] \, \text{per } D[\langle k, n\rangle]\,. \tag{3}$$

The first factors in the righthand sides of (2) and (3) are the same. In the second factors, note that $D[\langle k+1, n\rangle]$ is a principal submatrix of $D[\langle k, n\rangle]$. Since $d_{kk} = 1$, per $D\,[\langle k+1,n\rangle] \le$ per $D\,[\langle k,n\rangle]$. Thus, by (2) and (3), per $D\,(k,l) \le$ per $D\,(k-1,l)$. By expanding per C and per D about column l and noting that

$$\text{per } C\,(k-1,l) = \text{ per } D\,(k-1,l)\,,$$

the previous inequality gives per $D \le$ per C. As $C, D \in \mathcal{H}\,(m,n)$ and per $D = P\,(m,n)$, it follows that per $C = P\,(m,n) =$ per D. A similar argument shows that if C is obtained from D by an interchangeable row pair, then per $C =$ per D.

Thus, $B_s \in \mathcal{H}\,(m,n)$ and per $B_s = P\,(m,n)$. Either B_s is partly decomposable, or B_s is fully indecomposable and has no interchangeable pairs. In the former case, apply induction to each fully indecomposable component of B_s to arrive at a matrix in $\mathcal{H}\,(m,n)$ of maximum permanent, with each fully indecomposable component staircased. In the latter case B_s is staircased, and hence each of its fully indecomposable components (of which there is only 1) is staircased. ∎

We conclude this section with a theorem that gives a restriction on the staircased structure of each A_i of a matrix A in form (1) with maximum permanent.

Lemma 2.4 *Let A be an n-by-n fully indecomposable staircased Hessenberg $(0,1)$-matrix. Then*

$$\frac{2}{3} \ge \frac{\text{per } A(n)}{\text{per } A} \ge \frac{1}{2}.$$

Proof. Note that

$$\text{per } A = \text{per } A(n) + \text{per } A(n-1,n).$$

Since A is staircased $A(n-1,n) \le A(n)$ (entrywise). Hence per $A(n-1,n) \le$ per $A(n)$, and per $A \le 2$ per $A(n)$. This shows that per $A(n)/$per $A \ge 1/2$.

For the other inequality note that since A is a fully indecomposable staircased Hessenberg matrix, so is $A(n)$. Thus, by the above inequality, per $A(\langle n-1,n\rangle) \ge$ per $A(n)/2$. By expansion along the last row

$$\begin{aligned} \text{per } A &\ge \text{per } A(n) + \text{per } A(\langle n-1,n\rangle) \\ &\ge \text{per } A(n) + \text{per } A(n)/2 \\ &= 3 \text{ per } A(n)/2. \end{aligned}$$

It follows that $2/3 \ge$ per $A(n)/$per A. ∎

Observe that $A(n)$ can be replaced by $A(1)$ in the above proposition.

Theorem 2.5 *Let A be an n-by-n fully indecomposable staircased Hessenberg $(0,1)$-matrix. Assume that i, j, k are positive integers such that a_{ij} is the first 1 in the ith row of A, $a_{i+1,k+1}$ is the first 1 in the $(i+1)$st row of A, and $k-j \ge 2$. Let B be the matrix obtained from A by replacing its (i,j)-entry by 0 and its $(i+1,k)$-entry by 1. Then B is a Hessenberg $(0,1)$-matrix and* per $B >$ per A.

Proof. Let C be the matrix obtained from A by replacing a_{ij} by 0. Since A is fully indecomposable and staircased, so is C. Also by Lemma 2.2(c)

$$\text{per } C(i,j) = \text{ per } C[\langle 1, j-1\rangle] \text{ per } C[\langle i+1, n\rangle] \tag{4}$$

and

$$\text{per } C(i+1,k) = \text{per } C[\langle 1, k-1\rangle] \text{ per } C[\langle i+2, n\rangle]. \tag{5}$$

By Lemma 2.4, per $C[\langle i+2,n\rangle] \geq$ per $C[\langle i+1,n\rangle]/2$, and by repeated application of the preceding proposition

$$\text{per } C[\langle 1,k-1\rangle] \geq (3/2)\text{per } C[\langle 1,k-2\rangle] \geq \cdots \geq (3/2)^{k-j}\text{per } C[\langle 1,j-1\rangle].$$

Substituting these bounds into (5) and using (4) and $k-j \geq 2$,

$$\text{per } C(i+1,k) \geq (9/8)\text{per } C(i,j) > \text{per } C(i,j).$$

Since per $A =$ per $C +$ per $C(i,j)$ and per $B =$ per $C +$ per $C(i+1,k)$, it follows that per $B >$ per A. ■

Note that Theorem 2.5 implies that if $A \in \mathcal{H}(m,n)$, each fully indecomposable component of A is in staircased form and per A is maximal, then no "step" of zeros has width 3 or more; that is, $\sum_{i=1}^{r}(a_{ri} - a_{r+1,i}) \leq 2$ for $r = 1, 2, \ldots, n-1$. Note that the bound is tight as $P(11,4) = 6$ is achieved by the following matrix with a step of zeros of width 2 (i.e., $\sum_{i=1}^{3}(a_{3i} - a_{4i}) = 2$):

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

3 $P(m,n)$ for $n \leq m \leq (7n-1)/3$

Clearly $P(m,n) = 0$ for $m < n$. For $n \leq m \leq 2n$, Brualdi, Goldwasser and Michael [BGM, Theorem 2.2] show that for an n-by-n $(0,1)$-matrix with m entries equal to 1, the maximum permanent is $2^{\lfloor (m-n)/2 \rfloor}$. Additionally, they characterize the matrices achieving the maximum. The following proposition follows from their characterization by noting that each matrix achieving the maximum is combinatorially equivalent to a Hessenberg matrix. We give a self-contained proof here that makes use of the matrices being Hessenberg. Let $H_n = [h_{ij}]$ be the n-by-n Hessenberg matrix with $h_{ij} = 1$ if $j \leq i+1$. Note that per $H_n = 2^{n-1}$ and $\text{nnz}(H_n) = (n^2+3n-2)/2$.

Theorem 3.1 *For integers m and n with $2 \leq n \leq m \leq 2n$,*

$$P(m,n) = 2^{\lfloor (m-n)/2 \rfloor}.$$

Proof. Let $t = \lfloor (m-n)/2 \rfloor$, and let A be the direct sum of $t \geq 0$ matrices H_2 and $n - 2t \geq 0$ matrices H_1. Then A is n-by-n, $\mathrm{nnz}\,(A) = 4t + n - 2t = n + 2t$, and $\mathrm{per}\, A = 2^t$. If $m - n$ is even, then $\mathrm{nnz}\, A = m$, and $\mathrm{per}\, A = 2^t$. If $m-n$ is odd, then the matrix A' obtained from A by replacing the 0 in its $(n, 1)$ position by a 1 has m nonzeros and permanent 2^t. Hence, $P(m, n) \geq 2^{\lfloor (m-n)/2 \rfloor}$.

The proof that $P(m, n) \leq 2^{\lfloor (m-n)/2 \rfloor}$ is by induction on m. If $m = n$, then $P(n, n)$ is the largest permanent of an n-by-n $(0, 1)$-matrix with n entries equal to 1, and this is clearly at most $1 = 2^0 = 2^{\lfloor (m-n)/2 \rfloor}$, as desired. Assume that $m > n$, and proceed by induction. Let A be an n-by-n Hessenberg $(0, 1)$-matrix with $\mathrm{nnz}(A) = m$. By Theorem 2.3, assume that A has the form (1), where A_i is a fully indecomposable staircased n_i-by-n_i Hessenberg matrix for $i = 1, \ldots, b$. If each $n_i = 1$, then $\mathrm{per}\, A = 1 \leq 2^{\lfloor (m-n)/2 \rfloor}$. Otherwise, there exists an i such that $n_i \geq 2$. Since A_i is staircased and Hessenberg, the observation after Lemma 2.4 implies that

$$\mathrm{per}\, A_i \leq 2\, \mathrm{per}\, A_i(1). \tag{6}$$

Let j be the row of A that intersects the first row of A_i. Then (6) implies that $\mathrm{per}\, A \leq 2\, \mathrm{per}\, A(j)$. Note that $\mathrm{nnz}\,(A(j)) \leq \mathrm{nnz}(A) - 3 = m - 3$. Hence, by induction, $\mathrm{per}\, A(j) \leq 2^{\lfloor (m-3-(n-1))/2 \rfloor}$. It follows that $\mathrm{per}\, A \leq 2^{\lfloor (m-n)/2 \rfloor}$, and hence that $P(m, n) \leq 2^{\lfloor (m-n)/2 \rfloor}$, as desired. ■

Letting E_{ij} be the matrix with (i, j)-entry equal to 1 and all other entries zero, the complete results for $n = 2$ are given by the above theorem as:

$P(2, 2) = 1$, with equality for $A = H_1 \oplus H_1$;
$P(3, 2) = 1$, with equality for $A = (H_1 \oplus H_1) + E_{21}$;
$P(4, 2) = 2$, with equality for $A = H_2$.

For $n \geq 5$ and a subset of values of m with $2n + 1 \leq m \leq (7n - 1)/3$, the following recursion leads to an explicit formula for $P(m, n)$. In the next two results, we write $m = 2n + t$; thus $n \geq 3t + 1$ implies that $m \leq (7n - 1)/3$.

Theorem 3.2 *Let t and n be positive integers with $n \geq \max\{5, 3t + 1\}$. Then*

$$P(2n + t, n) = 2P(2(n-2) + t, n-2).$$

Proof. The assumptions on t and n imply that $2(n-2) + t \leq \frac{(n-2)^2 + 3(n-2) - 2}{2}$, and hence $\mathcal{H}(2(n-2) + t, n-2) \neq \emptyset$. Let $A \in \mathcal{H}(2(n-2) + t, n-2)$ with $\mathrm{per}\, A = P(2(n-2) + t, n-2)$. Then $H_2 \oplus A \in \mathcal{H}(2n + t, n)$ and has permanent $2P(2(n-2) + t, n-2)$. Hence $P(2n + t, n) \geq 2P(2(n-2) + t, n-2)$.

We now prove that $P(2n + t, n) \leq 2P(2(n-2) + t, n-2)$. By Theorem 2.3 , there is a matrix $A \in \mathcal{H}(2n + t, n)$ with permanent $P(2n + t, n)$ of the form (1) with each fully indecomposable component a staircased, Hessenberg $(0, 1)$-matrix. Let the order of A_i be n_i $(i = 1, 2, \ldots, b)$. If some $n_i = 2$, then $A_i = H_2$, and $\mathrm{per}\, A = 2\, \mathrm{per}\, A'$, where A' is the matrix obtained from A by deleting the rows and columns that intersect A_i. Since $\mathrm{nnz}(A') \leq \mathrm{nnz}(A) - 4$, $\mathrm{per}\, A' \leq P(2(n-2) + t, n-2)$, and hence $\mathrm{per}\, A \leq 2P(2(n-2) + t, n-2)$, as desired.

Suppose that $n_i \geq 3$ for $i = 1, \ldots, b$. Then

$$n = \sum_{i=1}^{b} n_i \geq 3b. \tag{7}$$

Also, by Lemma 2.2 (a) and (b), $\mathrm{nnz}(A_i) \geq 3n_i - 2$ $(i = 1, 2, \ldots, b)$. Hence

$$2n + t = \mathrm{nnz}(A) \geq \sum_{i=1}^{b} \mathrm{nnz}(A_i) \geq 3n - 2b,$$

and thus $t \geq n - 2b$. This and (7) imply that $t \geq b$ and $3t \geq n$, contradicting the hypothesis of the theorem.

Finally, suppose that some $n_i = 1$, and some $n_j \geq 3$. Since A_j is staircased, per $A_j \leq 2$ per $A_j(1)$. Hence

$$\mathrm{per}(A_i \oplus A_j) \leq \mathrm{per}(H_2 \oplus A_j(1)).$$

Let A' be the matrix obtained from A by replacing the blocks A_i and A_j by H_2 and $A_j(1)$. Then $\mathrm{nnz}(A') \leq \mathrm{nnz}(A)$, and per $A' \geq$ per A. It follows that A' can be used rather than A. But A' has a 2-by-2 fully indecomposable block. This leads back to a case already considered. Hence $P(2n + t, n) \leq 2P(2(n - 2) + t, n - 2)$. ■

Corollary 3.3 *Let t be a positive integer. There exist constants e_t and o_t such that for all $n \geq \max\{5, 3t + 1\}$*

$$P(2n + t, n) = \begin{cases} e_t 2^{n/2} & \text{if } n \text{ is even,} \\ o_t 2^{(n-1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Theorem 3.2 shows that for $n \geq \max\{5, 3t + 1\}$, the function $P(2n + t, n)$ grows by a factor of 2 each time n is increased by 2. Thus, only the initial conditions need to be determined to have an exact formula for $P(2n + t, n)$.

In particular, for $t = 1$, take the initial conditions to be $e_1 = P(9, 4)/4$ which is equal to 1, and $o_1 = P(7, 3)/2$ which is equal to 3/2. An induction argument (using Theorem 3.2) can be given to show that $P(2n+1, n) = e_1 2^{n/2}$ if n is even and $n \geq 5$, and $P(2n + 1, n) = o_1 2^{(n-1)/2}$ if n is odd and $n \geq 5$.

For $t \geq 2$, the initial conditions are obtained by setting

$$e_t = \begin{cases} P(7t, 3t)/2^{3t/2} & \text{if } t \text{ is even} \\ P(7t - 2, 3t - 1)/2^{(3t-1)/2} & \text{if } t \text{ is odd} \end{cases}$$

and

$$o_t = \begin{cases} P(7t, 3t)/2^{(3t-1)/2} & \text{if } t \text{ is odd} \\ P(7t - 2, 3t - 1)/2^{(3t-2)/2} & \text{if } t \text{ is even.} \end{cases}$$

Again, an induction argument can be used to show that the desired formula for $P(2n+t, n)$ holds for $n \geq 3t + 1$. ■

In the next section, these constants e_t and o_t are explicitly determined.

4 $P(m,n)$ for $2n+1 \le m \le 8n/3$

In this section we determine the exact values of $P(m,n)$ for $2n+1 \le m \le 8n/3$. For $n \le 2$ and m in this range, $\mathcal{H}(m,n) = \emptyset$. Thus, we take $n \ge 3$. Denote by $T_n = [t_{ij}]$ the n-by-n tridiagonal matrix with $t_{ij} = 1$ if $|i-j| \le 1$. Since per $T_1 = 1$, per $T_2 = 2$ and per $T_n =$ per $T_{n-1} +$ per T_{n-2} for $n \ge 3$, it follows that per T_n equals the n-th Fibonacci number, f_n.

We begin by establishing lower bounds on $P(m,n)$. Note that for fixed n, $P(m,n)$ is a nondecreasing function of m. For integers m and n with $2n+1 \le m \le 8n/3$, define

$$u(m,n) = \begin{cases} 2^{m/4}, & \text{if } m \equiv 0 \bmod 4, \\ 2^{(m-1)/4}, & \text{if } m \equiv 1 \bmod 4, \\ \frac{5}{4} \times 2^{(m-2)/4}, & \text{if } m \equiv 2 \bmod 4, \\ \frac{3}{2} \times 2^{(m-3)/4}, & \text{if } m \equiv 3 \bmod 4. \end{cases}$$

Proposition 4.1 *If m and n are positive integers with $2n+1 \le m \le 8n/3$, then $P(m,n) \ge u(m,n)$.*

Proof. Let $m \ge 2n+1$ and first suppose that $m \equiv 0 \bmod 4$. Let $r = (m-2n)/2$ and $s = (8n-3m)/4$. Then $r \ge 1$ and $s \ge 0$ are integers. Define A to be the direct sum of r matrices H_3 and s matrices H_2. Then $\mathrm{nnz}(A) = m$, A is n-by-n and per $A = 2^{2r+s} = 2^{m/4} = u(m,n)$. Hence $P(m,n) \ge u(m,n)$.

Second suppose that $m \equiv 1 \bmod 4$, and thus $3m \le 8n-1$. Since $m \equiv 1 \bmod 4$, $u(m,n) = u(m-1,n)$. Clearly, $P(m,n) \ge P(m-1,n)$. By the previous case $P(m-1,n) \ge u(m-1,n)$. Hence, $P(m,n) \ge u(m-1,n) = u(m,n)$.

Now suppose that $m \equiv 2 \bmod 4$, and thus $3m \le 8n-2$. Since $2n+1 \le m \le 8n/3$, it follows that $n \ge 4$ and $m \ge 10$. Set $r = (m-2-2n)/2$ and $s = (8n-3m-2)/4$. Then r and s are nonnegative integers. Define A to be the direct sum of r matrices H_3, s matrices H_2 and one T_4. Then $\mathrm{nnz}(A) = m$, A is n-by-n and per $A = 5 \times 2^{2r+s} = \frac{5}{4} \times 2^{(m-2)/4} = u(m,n)$. Hence $P(m,n) \ge u(m,n)$.

Finally suppose that $m \equiv 3 \bmod 4$, and thus $3m \le 8n-3$. Let $r = (m-1-2n)/2$ and $s = (8n-3m-3)/4$. Then r and s are nonnegative integers. Let A be the direct sum of r matrices H_3, s matrices H_2 and one T_3. Then $\mathrm{nnz}(A) = m$, A is n-by-n and per $A = 3 \times 2^{2r+s} = \frac{3}{2} \times 2^{(m-3)/4} = u(m,n)$. Hence $P(m,n) \ge u(m,n)$. ■

The main result of this section is that $P(m,n) = u(m,n)$ for $m \le 8n/3$. The proof of the main result requires several preliminary lemmas. Recall that E_{ij} is a matrix with (i,j)-entry equal to 1 and all other entries 0.

Lemma 4.2 *Let k, l and p be integers with $k \ge 2$, $l \ge 1$ and $p \ge 1$. Let B be a p-by-p Hessenberg $(0,1)$-matrix, and x a $(0,1)$-vector that is entrywise less than or equal to the* *first column of B. Let R and S be the partitioned Hessenberg matrices*

$$R = \left[\begin{array}{c|cccc|c} T_k & & O & & & O \\ \hline O & & T_l & & & E_{l1} \\ \hline & 0 & \dots & 0 & & \\ O & \vdots & \cdots & \vdots & x & B \\ & 0 & \cdots & 0 & & \end{array}\right]$$

and

$$S = \left[\begin{array}{c|c|c} T_k & E_{k1} & O \\ \hline E_{1,k-1}+E_{1k} & T_l & O \\ \hline & & \\ O & O & B \\ & & \end{array}\right].$$

Then $\operatorname{per} S \geq \operatorname{per} R$, *and the fully indecomposable components of S are $S[\langle 1, k+l\rangle]$ and those of B.*

Proof. Since every transversal of S that contains the $(k, k+1)$-entry contains either the $(k+1, k-1)$- or $(k+1, k)$-entry,

$$\begin{aligned} \operatorname{per} S &= \operatorname{per} T_k \operatorname{per} T_l \operatorname{per} B + [\operatorname{per} T_k(k,k) + \operatorname{per} T_k(k, k-1)] \operatorname{per} T_l(1) \operatorname{per} B \\ &= \operatorname{per} T_k \operatorname{per} T_l \operatorname{per} B + \operatorname{per} T_k\ f_{l-1} \operatorname{per} B. \end{aligned}$$

Since every transversal of R containing the $(k+l, k+l+1)$-entry contains a 1 of x,

$$\begin{aligned} \operatorname{per} R &= \operatorname{per} T_k \operatorname{per} T_l \operatorname{per} B + \operatorname{per} T_k \operatorname{per} T_l(l) \operatorname{per} B' \\ &= \operatorname{per} T_k \operatorname{per} T_l \operatorname{per} B + \operatorname{per} T_k\ f_{l-1} \operatorname{per} B', \end{aligned}$$

where B' is the matrix obtained from B by replacing the first column of B with x. Since x is entrywise less than or equal to the first column of B, m $\operatorname{per} B' \leq \operatorname{per} B$. Thus,

$$\operatorname{per} R \leq \operatorname{per} T_k \operatorname{per} T_l \operatorname{per} B + \operatorname{per} T_k\ f_{l-1} \operatorname{per} B = \operatorname{per} S.$$

It is easy to verify that $S[\langle 1, k+l\rangle]$ is fully indecomposable and that $S = S[\langle 1, k+l\rangle] \oplus B$. Hence the fully indecomposable components of S are $S[\langle 1, k+l\rangle]$ and the fully indecomposable components of B. ■

Considering PR^TP and PS^TP, where P is the reverse permutation matrix, the following result is obtained.

Lemma 4.3 *Let r, s and q be integers with $r \geq 1$, $s \geq 2$ and $q \geq 1$. Let C be a q-by-q Hessenberg $(0,1)$-matrix, and y^T a $(0,1)$-vector that is entrywise less than or equal to the* *last row of C. Let U and V be the partitioned Hessenberg matrices*

$$U = \left[\begin{array}{ccc|c|c} \multicolumn{3}{c|}{C} & E_{q1} & O \\ \hline \multicolumn{3}{c|}{y^T} & & \\ 0 & \cdots & 0 & & \\ \vdots & \vdots & \vdots & T_r & O \\ 0 & \cdots & 0 & & \\ \hline \multicolumn{3}{c|}{O} & O & T_s \end{array}\right]$$

and

$$V = \left[\begin{array}{c|c|c} C & O & O \\ \hline & & \\ O & T_r & E_{r1} \\ & & \\ \hline O & E_{1r} + E_{2r} & T_s \end{array}\right].$$

Then $\operatorname{per} V \geq \operatorname{per} U$, *and the fully indecomposable components of V are $V[\langle q+1, q+r+s\rangle]$ and those of C.*

Let m and n be integers with $2n+1 \leq m \leq 8n/3$. Define $\mathcal{S}(m,n)$ to be the set of all n-by-n $(0,1)$-matrices A such that $\operatorname{nnz}(A) \leq m$, $\operatorname{per} A = P(m,n)$, A has form (1) where each A_i is a fully indecomposable staircased Hessenberg matrix and each $A_{ij} = 0$ for $i \neq j$.

By Theorem 2.3, $\mathcal{S}(m,n) \neq \emptyset$. Since $P(m,n)$ is a nondecreasing function of m, if $\widehat{A}$ is an n-by-n matrix that is a direct sum of fully indecomposable staircased Hessenberg matrices with $\operatorname{nnz}(\widehat{A}) \leq m$ and $\operatorname{per} \widehat{A} \geq P(m,n)$, then $\operatorname{per} \widehat{A} = P(m,n)$ and $\widehat{A} \in \mathcal{S}(m,n)$. In particular, if $A \in \mathcal{S}(m,n)$ and some direct sum of fully indecomposable components of A has the form R described in Lemma 4.2 with $\operatorname{nnz}(x) \geq 2$, then the matrix A' obtained from A by replacing R by the matrix S of Lemma 4.2 necessarily belongs to $\mathcal{S}(m,n)$. A similar statement holds for a matrix A'' obtained from $A \in \mathcal{S}(m,n)$ by replacing a direct sum of fully indecomposable components of the form U, as in Lemma 4.3, by the matrix V of Lemma 4.3.

The next lemma shows that if $m \leq 8n/3$, then $\mathcal{S}(m,n)$ contains a matrix in one of the following four special forms.

Lemma 4.4 *For all positive integers m and n with $2n+1 \leq m \leq 8n/3$, there is a matrix $A \in \mathcal{S}(m,n)$ with at least one of the following properties:*

(a) $A_i = T_{n_i}$ *for all i;*

(b) $A_i \notin \{T_2, T_3, T_4\}$ *for all i;*

(c) $A_i = H_3$ *for at least one i;*

(d) $A_i = T_3$ *for at least one i.*

Proof. Suppose on the contrary that none of these statements hold. Then for every $A \in \mathcal{S}(m,n)$, at least one of its fully indecomposable components is not tridiagonal, at least one of its fully indecomposable components is T_2 or T_4, and none of its fully indecomposable components has order 3.

Case 1: *There is an $A \in \mathcal{S}(m,n)$ each of whose fully indecomposable components has order at least* 4.

Among all such matrices A, choose one with the minimum number of fully indecomposable components equal to T_4. By assumption there is an i with $A_i = T_4$, and a j such that $A_j \neq T_{n_j}$ and $n_j \geq 4$. Let $R = A_i \oplus A_j$, and let l be the first index such that column l of A_j does not equal column l of T_{n_j}. Then $A_i \oplus A_j$ has the form R of Lemma 4.2 with $k = 4$ and $\mathrm{nnz}(x) \geq 2$. Note that $1 \leq l \leq n_j - 2$, and thus the order of B is $p \geq 2$. Since A_j is fully indecomposable and staircased, so is B. Define S as in Lemma 4.2, and let A' be the matrix obtained from A by replacing R by S. Since $\mathrm{nnz}(x) \geq 2$, $\mathrm{nnz}(A') \leq \mathrm{nnz}(A)$. By Lemma 4.2, per $A' \geq$ per A. Hence $A' \in \mathcal{S}(m,n)$. The fully indecomposable components of S are one of order $4 + l$ and B. The choice of A requires that $p \leq 4$ (else every component of A' has order at least 4 and A' has fewer fully indecomposable components equal to T_4).

First suppose $p = 2$. Then $l = n_j - 2$ and $A_j = T_{n_j} + E_{n_j, n_j - 2}$. Let $\widehat{A}$ be obtained from A by interchanging A_i and A_j, and consider $U = A_j \oplus A_i$. Then

$$U = \left[\begin{array}{c|c|c} T_{n_j-1} & \begin{matrix}0\\ \vdots \\ 0 \\ 1\end{matrix} & O \\ \hline 0 \ \cdots \ 0\ \ 1\ \ 1 & 1 & 0\ \ 0 \\ \hline O & O & T_4 \end{array}\right],$$

Define V as in Lemma 4.3, and let $A'' \in \mathcal{S}(m,n)$ be the matrix obtained from $\widehat{A}$ by replacing U by V. The fully indecomposable components of V are T_{n_j-1} and a matrix of order 5. Because $n_j \geq 4$, none of the fully indecomposable components of A'' has order 1 or 2. Since neither (c) nor (d) holds, A'' has no fully indecomposable component of order 3. Hence, each fully indecomposable component of A'' has order at least 4. By the choice of A, A'' has at least as many fully indecomposable components equal to T_4 as A, and thus $n_j = 5$. Hence $A_i \oplus A_j = T_4 \oplus (T_5 + E_{53})$ is 9-by-9 with 24 entries equal to 1 and permanent $5 \times 10 = 50$. The matrix $H_3 \oplus H_3 \oplus H_3$ is 9-by-9 with 24 entries equal to 1 and permanent 4^3. Thus replacing $A_i \oplus A_j$ in A by $H_3 \oplus H_3 \oplus H_3$ results in a Hessenberg $(0,1)$-matrix with the same number of ones as A, but larger permanent. This is impossible, since per $A = P(m,n)$. We conclude that $p \neq 2$.

Next suppose that $p = 3$. Then $B \in \{T_3, H_3\}$, and hence A' has either T_3 or H_3 as a fully indecomposable component contrary to the assumption that neither (c) nor (d) holds.

Thus $p = 4$. The fully indecomposable components of S are one of order at least 5, and B of order 4. Thus all fully indecomposable components of A' have order at least 4.

If $B \neq T_4$, then we are led to the contradiction that A' has fewer fully indecomposable components equal to T_4 than A. Thus $B = T_4$. Since A_j is staircased and $B = T_4$, the definition of l implies that $A_j = T_{n_j} + E_{n_j-2,n_j-4}$. Now $U = A_j \oplus A_i$ has the form of U in Lemma 4.3 with $r = 3, s = 4$, $C = T_q$ and $y^T = \begin{bmatrix} 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$. Using Lemma 4.3, replace U in A by V to obtain a matrix $A''' \in \mathcal{S}(m,n)$. Arguing as with $p = 2$, by the choice of A the matrix V must have a fully indecomposable component of order 4; thus $q = 4$. Hence $n_j = 7$ and $A_j = T_7 + E_{53}$. It follows that

$$\mathrm{per}(A_i \oplus A_j) = 5 \times (21 + 4) = 125 < 126 = 6 \times 21 = \mathrm{per}((A_i + E_{31}) \oplus T_7).$$

Replacing A_i with $A_i + E_{31}$ and A_j with T_7 gives a matrix with the same number of nonzero entries as A but with a larger permanent. Therefore, Case 1 leads to a contradiction.

Case 2: *Every $A \in \mathcal{S}(m,n)$ has at least one fully indecomposable component of order less than 4.*

Among the matrices in $\mathcal{S}(m,n)$, choose A to have the minimum number of fully indecomposable components of order 1. We claim that A has no fully indecomposable component of order 1. Suppose on the contrary that some n_i equals 1. Since neither (a) nor (c) holds, there is a j such that $A_j \notin \{H_3, T_{n_j}\}$. In particular, $n_j \geq 4$. Since A_j is staircased, (6) implies that $\mathrm{per}\ A_j \leq 2\ \mathrm{per}\ A_j(1)$, and thus $\mathrm{per}(A_i \oplus A_j) \leq \mathrm{per}(H_2 \oplus A_j(1))$. Hence, replacing A_i and A_j by H_2 and $A_j(1)$, respectively, results in a matrix $A' \in \mathcal{S}(m,n)$. However, A' has one less fully indecomposable component of order 1, contrary to the choice of A. Therefore, A has no fully indecomposable component of order 1.

Among the matrices in $\mathcal{S}(m,n)$ with no fully indecomposable component of order 1, now choose A to have the minimum number of fully indecomposable components of order 2. We claim that A has no fully indecomposable component of order 2. Suppose on the contrary that n_i equals 2. Since neither (a) nor (c) holds, there is a fully indecomposable component A_j of order at least 4 that is not T_{n_j}. Let $R = A_i \oplus A_j$, and let l be the first index such that column l of A_j does not equal column l of T_{n_j}. Then $A_i \oplus A_j$ has the form of R in Lemma 4.2 with $k = 2$ and $\mathrm{nnz}(x) \geq 2$. Using Lemma 4.2, replace R by S to obtain a matrix $A' \in \mathcal{S}(m,n)$. The choice of A requires that some fully indecomposable component of S has order 2. Since the fully indecomposable components of S are B and a matrix of order $2 + l$, B must have order 2. Thus $A_j = T_{n_j} + E_{n_j,n_j-2}$. Let $\widehat{A}$ be obtained from A by interchanging A_i and A_j, and consider $U = A_j \oplus A_i$. Then

$$U = \left[\begin{array}{ccccc|c|c} & & & & & 0 & \\ & & & & & \vdots & \\ & & T_{n_j-1} & & & 0 & O \\ & & & & & 1 & \\ \hline 0 & \cdots & 0 & 1 & 1 & 1 & 0\ \ 0 \\ \hline & & O & & & O & T_2 \end{array}\right],$$

and the matrix V in Lemma 4.3 is $T_{n_j-1} \oplus H_3$. Applying Lemma 4.3, replace U in $\widehat{A}$ by V to obtain a matrix $A'' \in \mathcal{S}(m,n)$. But A'' has H_3 as a fully indecomposable component, contrary to our assumption that (c) does not hold. Thus, we are led to a contradiction, and conclude that there is an $A \in \mathcal{S}(m,n)$ with no fully indecomposable components of orders 1 or 2. Since no fully indecomposable component of a matrix in $\mathcal{S}(m,n)$ has order 3, Case 2 leads to a contradiction.

Both Cases 1 and 2 lead to a contradiction, thus our original supposition that none of (a)-(d) hold is false. ∎

In the next lemma, the bound $P(m,n) \leq u(m,n)$, with $u(m,n)$ as defined at the beginning of this section, is obtained in the case that $\mathcal{S}(m,n)$ contains a matrix of a special type.

Lemma 4.5 *Let m and n be positive integers with $2n+1 \leq m \leq 8n/3$. Suppose that there exists $A \in \mathcal{S}(m,n)$ such that $A_i = H_{n_i}$ for all i. Then*

$$\text{per } A \leq u(m,n).$$

Proof. Note that $\frac{\text{nnz}(H_k)}{k} > 8/3$ for $k \geq 4$, $\frac{\text{nnz}(H_3)}{3} = 8/3$ and $\frac{\text{nnz}(H_k)}{k} < 8/3$ for $k = 1,2$. Since $\text{nnz}(A) \leq 8n/3$, it follows that either $n_i = 3$ for all i or $n_i \leq 2$ for at least one i.

Note that $\text{per }(H_a \oplus H_b) = \text{per }(H_{a+1} \oplus H_{b-1})$ for all a,b with $b \geq 2$. In particular, since $\text{nnz}(H_1 \oplus H_b) \geq \text{nnz}(H_2 \oplus H_{b-1})$ for all $b \geq 2$, we can replace each occurence of $H_1 \oplus H_b$ in A by $H_2 \oplus H_{b-1}$. Similarly, we can replace each $H_2 \oplus H_b$ in A by $H_3 \oplus H_{b-1}$ for all $b \geq 3$. Therefore, without loss of generality we may assume that one of the following holds: (a) all fully indecomposable components of A are matrices H_3, (b) A has a fully indecomposable component of order 2, and all other fully indecomposable components have orders 2 or 3, (c) A has a fully indecomposable component of order 1, and all other fully indecomposable components have orders 1 or 2.

First suppose that (a) holds; say A is the direct sum of r matrices H_3. Then $n = 3r$, $\text{nnz}(A) = 8r = 8n/3$ and $\text{per } A = 4^r = 2^{2r}$. It follows that $m = 8n/3$, $m \equiv 0 \bmod 4$ and that $\text{per } A = 2^{m/4} = u(m,n)$, as desired.

Next suppose that (b) holds; say that A is the direct sum of $s \geq 1$ matrices H_2 and $r \geq 0$ matrices H_3. Then $n = 2s + 3r$, $\text{nnz}(A) = 4s + 8r$ and $\text{per } A = 2^{s+2r} = 2^{\text{nnz}(A)/4}$. Since A has the maximum permanent of matrices in $\mathcal{S}(m,n)$, by Proposition 4.1 $m \in \{\text{nnz}(A), \text{nnz}(A)+1\}$. Thus, since $\text{nnz}(A)$ is a multiple of 4, $m \equiv 0 \bmod 4$ or $m \equiv 1 \bmod 4$. If $m \equiv 0 \bmod 4$, then $m = \text{nnz}(A)$ and $\text{per } A = 2^{m/4} = u(m,n)$, as desired. If $m \equiv 1 \bmod 4$, then $m = \text{nnz}(A) + 1$ and $\text{per } A = 2^{(m-1)/4} = u(m,n)$, as desired.

Finally, suppose that (c) holds; say that A is the direct sum of $l \geq 1$ matrices H_1 and $s \geq 0$ matrices H_2. Then $n = l + 2s$, $\text{nnz}(A) = l + 4s$ and $\text{per } A = 2^s = 2^{(\text{nnz}(A)-n)/2}$. Since A has the maximum permanent of matrices in $\mathcal{S}(m,n)$, by Proposition 4.1 $m \in \{\text{nnz}(A), \text{nnz}(A)+1\}$. Thus $m \leq l + 4s + 1 \leq 2n$, so (c) cannot occur. ∎

Next we use a result from the literature on arbitrary $(0,1)$-matrices to characterize equality for a bound on $\text{per } A$ for $A \in \mathcal{H}(m,n)$. This lemma is useful in proving that $P(m,n) \leq u(m,n)$ in the case that $\mathcal{S}(m,n)$ contains a matrix each of whose fully indecomposable components is tridiagonal. In the next proof, the n-by-n "cycle matrix" $C_n = [c_{ij}]$ has $c_{i,i+1} = 1$ for $1 \leq i \leq n-1$, $c_{n1} = 1$ and all other $c_{ij} = 0$.

Lemma 4.6 *Let $A \in \mathcal{H}(m,n)$ be fully indecomposable and staircased. Then*

$$\text{per } A \le 2^{m-2n} + 1, \tag{8}$$

with equality if and only if $A \in \{T_2, T_3, T_4\}$.

Proof. By [BR, Theorem 7.4.14] and the characterization by Foregger, (8) holds for an arbitrary $(0,1)$-matrix with equality if and only if $n \ge 2$ and there exist permutation matrices P and Q and a positive integer p such that PAQ has the form

$$\begin{bmatrix} B_1 & O & \cdots & O & E_1 \\ E_2 & B_2 & \cdots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & B_{p-1} & O \\ O & O & \cdots & E_p & B_p \end{bmatrix},$$

where B_i is n_i-by-n_i, $B_i = I + C_{n_i}$ if $n_i \ge 2$, $B_i = [1]$ if $n_i = 1$, and $\text{nnz}(E_i) = 1$ $(i = 1, 2, \ldots, p)$.

Since A is staircased and fully indecomposable, $3n - 2 \le m$ by Lemma 2.2 (a) and (b). Let q be the number of n_i that are equal to 1. Then $3n - 2 \le m = 2n + p - q$, giving $n + q \le p + 2$. Also $2p - q \le \sum_{i=1}^{p} n_i = n$. Adding these two inequalities gives $p \le 2$ and thus $n + q \le 4$.

The proof is completed by making the following observations. If $n = 4$, then necessarily $q = 0$, $p = 2$ and $A = T_4$. If $n = 2$, then since A is fully indecomposable Hessenberg, $A = T_2$. If $n = 3$, then since A is fully indecomposable Hessenberg, $A = H_3$ or $A = T_3$. It is easy to verify that equality does not hold in (8) when $A = H_3$. ■

Using the above lemmas, we now determine an upper bound on $P(m,n)$.

Theorem 4.7 *If m and n are positive integers with $2n+1 \le m \le 8n/3$, then $P(m,n) \le u(m,n)$.*

Proof. The proof is by induction on n. It is easy to verify that $P(7,3) = 3$ and $P(8,3) = 4$, thus for $n = 3$, $P(m,n) = u(m,n)$.

Assume that $n \ge 4$. It suffices to show that for some (and hence every) matrix $A \in \mathcal{S}(m,n)$, we have per $A \le u(m,n)$. Note, by Lemma 4.4, we can assume that at least one of (a)-(d) holds. Also, by Lemma 4.5, if there is an $A \in \mathcal{S}(m,n)$ each of whose fully indecomposable components is an H_{n_i}, then $P(m,n) \le u(m,n)$. Henceforth we assume that every matrix in $\mathcal{S}(m,n)$ has at least one fully indecomposable component A_i with $A_i \ne H_{n_i}$. In particular, this implies that $\text{nnz}(A) = m$ for each $A \in \mathcal{S}(m,n)$, since if $\text{nnz}(A) < m$, then a 0 in this fully indecomposable component could be changed to 1, increasing the permanent.

Case 1: *Statement (a) of Lemma 4.4 holds.*

Then there is an $A \in \mathcal{S}(m,n)$ so that $A_i = T_{n_i}$ for all i. Each row of the chart below gives a Hessenberg $(0,1)$-matrix W and a Hessenberg $(0,1)$-matrix X such that W and X have the same order, $\mathrm{nnz}(W) \geq \mathrm{nnz}(X)$ and $\operatorname{per} W < \operatorname{per} X$. For $p \geq 6$, the matrix X_p is the matrix obtained from T_p by replacing the ones in positions $(2,3)$ and $(3,2)$ by zeros, and the zeros in positions $(5,3)$ and $(6,4)$ by ones. Then $\mathrm{nnz}(X_p) = \mathrm{nnz}(T_p)$, and on setting $f_0 = 1$ it follows that

$$\begin{aligned}
\operatorname{per} X_p &= 2(f_{p-2} + f_{p-5} + f_{p-6}) \\
&= 2(f_{p-2} + f_{p-4}) \\
&> 2f_{p-2} + f_{p-4} + f_{p-5} \\
&= 2f_{p-2} + f_{p-3} = f_p = \operatorname{per} T_p.
\end{aligned}$$

W	per W	X	per X	Constraints
$T_1 \oplus T_p$	f_p	$T_2 \oplus T_{p-1}$	$2f_{p-1}$	$p \geq 3$
T_p	f_p	X_p	$2f_{p-2} + 2f_{p-4}$	$p \geq 6$
$T_5 \oplus T_5$	64	$H_3 \oplus H_3 \oplus T_4$	80	
$T_5 \oplus T_4$	40	$H_3 \oplus H_3 \oplus T_3$	48	
$T_5 \oplus T_3$	24	$H_3 \oplus H_3 \oplus H_2$	32	
$T_4 \oplus T_4$	25	$H_3 \oplus H_3 \oplus H_2$	32	
$T_4 \oplus T_3$	15	$H_3 \oplus H_2 \oplus H_2$	16	
$T_3 \oplus T_3$	9	$H_2 \oplus T_4$	10	

Suppose that there is a direct sum of a subset of the fully indecomposable components of A that is equal to a W occuring in the chart. Then W can be replaced in A by X to obtain a Hessenberg $(0,1)$-matrix A' of order n with $\mathrm{nnz}(A') \leq \mathrm{nnz}(A)$ and $\operatorname{per} A' > \operatorname{per} A$, contradicting the fact that $\operatorname{per} A = P(m,n)$. Hence no subset of the fully indecomposable components of A has the form of a W in the chart. Hence, $n_i \leq 5$ for all i, there is at most one i with $n_i \in \{3,4,5\}$, and if there is an i with $n_i = 1$ then all remaining n_j are at most 2. Since $m \geq 2n+1$, there is at least one i with $n_i \geq 3$. Hence each $n_i \in \{2,3,4,5\}$, and at least one n_i does not equal 2 since $m \geq 2n+1$.

First suppose some $n_i = 3$. Then A is the direct sum of T_3 and $k \geq 1$ matrices T_2. It follows that $n = 2k+3$, $m = 4k+7 = 2n+1$, $m \equiv 3 \bmod 4$ and $\operatorname{per} A = 2^k \times 3 = \frac{3}{2} \times 2^{(m-3)/4} = u(m,n)$.

Next suppose some $n_i = 4$. Then A is the direct sum of T_4 and $k \geq 0$ matrices T_2. It follows that $n = 2k+4$, $m = 4k+10 = 2n+2$, $m \equiv 2 \bmod 4$ and $\operatorname{per} A = 2^k \times 5 = \frac{5}{4} \times 2^{(m-2)/4} = u(m,n)$.

Finally suppose some $n_i = 5$. Then A is the direct sum of T_5 and $k \geq 0$ matrices T_2. It follows that $n = 2k+5$, $m = 4k+13 = 2n+3$, $m \equiv 1 \bmod 4$ and $\operatorname{per} A = 2^k \times 8 = 2^{(m-1)/4} = u(m,n)$.

Thus the result holds in this case.

Case 2: *Statement (b) of Lemma 4.4 holds.*

Then there is an $A \in \mathcal{S}(m,n)$ none of whose fully indecomposable components belong to $\{T_2, T_3, T_4\}$. By Lemma 4.6,

$$\text{per } A = \prod_{i=1}^{b} \text{per } A_i \le \prod_{i=1}^{b} 2^{\text{nnz}(A_i)-2n_i} = 2^{\text{nnz}(A)-2n} = 2^{m-2n}.$$

Let v be the unique integer $\in \{0,1,2,3\}$ such that $m \equiv v \bmod 4$. Since $m \le 8n/3$, it follows that $3m \le 8n - v$. If $v = 0$, then $m - 2n \le m - (3/4)m = m/4$, and hence $2^{m-2n} \le 2^{m/4} = u(m,n)$. If $v = 1$, then $m - 2n \le m - (3m+1)/4 = (m-1)/4$, and hence $2^{m-2n} \le 2^{(m-1)/4} = u(m,n)$. If $v = 2$, then $m - 2n \le m - (3m+2)/4 = (m-2)/4$, and hence $2^{m-2n} \le 2^{(m-2)/4} < (5/4)2^{(m-2)/4} = u(m,n)$. If $v = 3$, then $m - 2n \le m - (3m+3)/4 = (m-3)/4$, and hence $2^{m-2n} \le 2^{(m-3)/4} < (3/2)2^{(m-3)/4} = u(m,n)$.

Therefore the result holds for each m.

Case 3: *Statement (c) of Lemma 4.4 holds.*

Then there is $A \in \mathcal{S}(m,n)$ and an i such that $A_i = H_3$. Let A' be the matrix obtained from A by deleting the rows and columns that intersect A_i. Then $\text{nnz}(A') = m - 8$, A' is $(n-3)$-by-$(n-3)$ and per $A = 4 \times$ per A'. Since $m \le 8n/3$, $\text{nnz}(A') = m - 8 \le 8(n-3)/3$.

First suppose that $m - 8 \ge 2(n-3) + 1$. Then by induction and the fact that $m \equiv m - 8 \bmod 4$, per $A' \le u(m-8, n-3) = u(m,n)/4$. As per $A = 4 \times$ per A', the desired upper bounds now follow.

Now suppose that $m - 8 \le 2(n-3)$. As $m \ge 2n+1$, it follows that $m = 2n+1$ or $m = 2n+2$. First consider $m = 2n+1$. Then $\text{nnz}(A') = 2(n-3) - 1 < 2(n-3)$. Hence, by Theorem 3.1, per $A' \le 2^{\lfloor (n-4)/2 \rfloor} = 2^{\lfloor (m-9)/4 \rfloor}$. Thus, since m is odd,

$$\begin{aligned} \text{per } A \le 2^{\lfloor (m-1)/4 \rfloor} &= \begin{cases} 2^{(m-1)/4}, & \text{if } m \equiv 1 \bmod 4 \\ 2^{(m-3)/4}, & \text{if } m \equiv 3 \bmod 4, \end{cases} \\ &\le u(m,n). \end{aligned}$$

On the other hand, if $m = 2n+2$, then $\text{nnz}(A') = 2(n-3)$. Hence, by Theorem 3.1, per $A' \le 2^{\lfloor (n-3)/2 \rfloor} = 2^{\lfloor (m-8)/4 \rfloor}$. Thus, since m is even,

$$\begin{aligned} \text{per } A \le 2^{\lfloor m/4 \rfloor} &= \begin{cases} 2^{m/4} & m \equiv 0 \bmod 4 \\ 2^{(m-2)/4} & m \equiv 2 \bmod 4, \end{cases} \\ &\le u(m,n). \end{aligned}$$

Case 4: *Statement (d) of Lemma 4.4 holds.*

Then there is an $A \in \mathcal{S}(m,n)$ and an i with $A_i = T_3$. If $A_j = T_{n_j}$ for all j, then Case 1 applies, and if some $A_j = H_3$, then Case 3 applies and per $A \le u(m,n)$. Otherwise, there is a j such that A_j has a line with at least four entries equal to 1. By Laplace expansion of per A_j along such a line, per A_j is a sum of at least 4 permanents of matrices of the form $A_j(r,s)$. Thus, for some (r,s) such that the (r,s)-entry of A_j is 1, per $A_j(r,s) \le$ per $A_j/4$. Let A' be the matrix obtained from A by changing the $(3,1)$-entry of A_i to

1 and the (r,s)-entry of A_j to 0. Note that per A_i = per $T_3 = 3 = (3/4)$per H_3, and per A_j − per $A_j(r,s) \geq 3$ per $A_j/4$. Thus $\mathrm{nnz}(A') = \mathrm{nnz}(A)$, and

$$\begin{aligned} \text{per } A' &= \text{per } H_3 \times [\text{per } A_j - \text{per } A_j(r,s)] \times \prod_{t\neq i,j} \text{per } A_t \\ &\geq 4 \text{ per } T_3/3 \times 3 \text{ per } A_j/4 \times \prod_{t\neq i,j} \text{per } A_t \\ &= \text{per } A_i \times \text{per } A_j \times \prod_{t\neq i,j} \text{per } A_t \\ &= \text{per } A. \end{aligned}$$

It follows that per $A' = P(m,n)$, and hence $A' \in \mathcal{S}(m,n)$. Since A' has H_3 as a fully indecomposable component, Case 3 applies.

Thus, in each case there exists a matrix in $\mathcal{S}(m,n)$ with permanent having $u(m,n)$ as an upper bound. ■

Combining Proposition 4.1 and Theorem 4.7 gives the main result of this section, with $u(m,n)$ as defined at the beginning of this section.

Corollary 4.8 *If m and n are positive integers with $2n+1 \leq m \leq 8n/3$, then $P(m,n) = u(m,n)$.*

5 Dense Hessenberg matrices

In this section we determine the exact values of $P(m,n)$ for all values of m such that $m = \mathrm{nnz}(H_n) - z$ with $0 \leq z \leq \frac{k^2+3k-2}{2}$ and $k = \lfloor n/2 \rfloor$. We begin with a result that gives the permanent of certain order $n-1$ submatrices of H_n.

Lemma 5.1 *For $j \leq i$,*

$$\text{per } H_n(i,j) = \begin{cases} 1 & \text{if } i = n \text{ and } j = 1, \\ 2^{j-2} & \text{if } i = n \text{ and } j \geq 2, \\ 2^{n-i-1} & \text{if } n-1 \geq i \geq 1 \text{ and } j = 1, \\ 2^{n-i+j-3} & \text{if } n-1 \geq i \geq 1 \text{ and } j \geq 2. \end{cases}$$

In addition, for $n-1 \geq i \geq 1$, per $H_n(i,i+1) = 2^{n-2}$.

Proof. For $j \leq i$, the formula for per $H_n(i,j)$ follows from Lemma 2.2(c). Since $H_n(i,i+1) = H_{n-1}$, per $H_n(i,i+1) = 2^{n-2}$. ■

Define $M_n = [m_n(i,j)]$ to be the n-by-n matrix with (i,j)-entry equal to per $H_n(i,j)$ if $j \leq i+1$ and 0 otherwise. Note that $m_n(i,j)$ is equal to the number of transversals of

H_n that contain the (i,j)-entry. For example,

$$M_6 = \begin{bmatrix} 16 & 16 & 0 & 0 & 0 & 0 \\ 8 & 8 & 16 & 0 & 0 & 0 \\ 4 & 4 & 8 & 16 & 0 & 0 \\ 2 & 2 & 4 & 8 & 16 & 0 \\ 1 & 1 & 2 & 4 & 8 & 16 \\ 1 & 1 & 2 & 4 & 8 & 16 \end{bmatrix}.$$

Lemma 5.2 *Let $A = [a_{ij}]$ be an n-by-n Hessenberg $(0,1)$-matrix. Then* per $A \geq 2^{n-1} - \sum_{\{(i,j):j\leq i+1, a_{ij}=0\}} m_n(i,j)$.

Proof. If $a_{ij} = 0$, then the $m_n(i,j)$ transversals of H_n that contain the (i,j)-entry are not transversals of A. Since some transversals may be counted more than once, there are at most

$$\sum_{\{(i,j):j\leq i+1, a_{ij}=0\}} m_n(i,j)$$

transversals of H_n that are not transversals of A. The result now follows. ■

It is well known that every permutation can be expressed as the composition of disjoint cycles. In the following theorem, we identify a transversal τ with its permutation. Thus, by the cycles of τ we mean the cycles of the corresponding permutation.

Given a nonnegative integer z with $z \leq \text{nnz}(H_n)$, define σ_{nz} to be the sum of the z smallest nonzero entries of M_n. For example, $\sigma_{60} = 0$, $\sigma_{61} = 1$, $\sigma_{62} = 2$, $\sigma_{63} = 3$, $\sigma_{64} = 4$, $\sigma_{65} = 6$ and $\sigma_{66} = 8$. The following theorem determines $P(m,n)$ for n even and $m \geq \frac{3n^2+6n}{8}$, i.e., for a dense Hessenberg matrix of even order.

Theorem 5.3 *Let $n = 2k \geq 4$ be an even positive integer, and let z be an integer with $0 \leq z \leq \frac{k^2+3k-2}{2}$. Then $P(nnz(H_n) - z, n) = 2^{n-1} - \sigma_{nz}$.*

Proof. When $n = 2k = 4$, it is easily verified that:

$P(9,4) = 4$, with equality for $A = (H_2 \oplus H_2) + E_{32}$;
$P(10,4) = 5$, with equality for $A = T_4$;
$P(11,4) = 6$, with equality for $A = H_4 - (E_{41} + E_{42})$;
$P(12,4) = 7$, with equality for $A = H_4 - E_{41}$;
$P(13,4) = 8$, with equality for $A = H_4$.

Note that $P(9,4)$ is not achieved by any fully indecomposable matrix. In the remainder of the proof, assume that $n \geq 6$.

Let $\mathcal{I}$ be the set of all pairs of integers (i,j) with $n \geq i > j \geq 1$ and $(i,j) \neq (k+1,k)$. For each $(i,j) \in \mathcal{I}$, let $\mathcal{S}_{ij}$ denote a fixed set of transversals of H_n such that the sets $\mathcal{S}_{ij}$ are mutually disjoint and each element of $\mathcal{S}_{ij}$ contains the (i,j)-entry. With regard to any such sets $\mathcal{S}_{ij}$, an upper bound is now obtained (see (9) below) for the permanent of an n-by-n Hessenberg $(0,1)$-matrix $A = [a_{ij}]$ for which $a_{i,i+1} = 1$ $(i = 1, 2, \ldots, n-1)$, $a_{ii} = 1$

$(i = 1, 2, \ldots, n)$ and $a_{k+1,k} = 1$ (and all other a_{ij} with $i > j$ are 0 or 1). Note that each element in the set $\cup_{\{(i,j)\in\mathcal{I} \text{ and } a_{ij}=0\}}\mathcal{S}_{ij}$ is a transversal of H_n that is not in A. Since the sets $\mathcal{S}_{ij}$ are mutually disjoint, there are at least

$$\sum_{\{(i,j)\in\mathcal{I} \text{ and } a_{ij}=0\}} |\mathcal{S}_{ij}|$$

transversals of H_n that are not transversals of A. Hence,

$$\text{per } A \le 2^{n-1} - \sum_{\{(i,j)\in\mathcal{I} \text{ and } a_{ij}=0\}} |\mathcal{S}_{ij}|. \tag{9}$$

We construct a family of such sets $\mathcal{S}_{ij}$ as follows. There are 4 types of pairs (i, j):

Type A: $i \ge k+1$, $k \ge j$ and $(i, j) \ne (k+1, k)$;

Type B: $i > j \ge k+1$;

Type C: $k-1 \ge i > j$;

Type D: $k = i > j$.

For (i, j) of type A, let $\mathcal{S}_{ij}$ consist of all transversals of the matrix

$$\begin{bmatrix} H_{j-1} & O & O \\ O & C_{i-j+1} & O \\ O & O & H_{n-i} \end{bmatrix},$$

where C_{i-j+1} is the cycle matrix defined in Section 4 and H_{j-1} (H_{n-i}) is vacuous if $j = 1$ $(i = n)$. We make the following observations if (i, j) is of Type A:

(A1) If $\tau \in \mathcal{S}_{ij}$, then the cycle of τ that contains k also contains $k+1$, but is not the 2-cycle $(k, k+1)$;

(A2) Each transversal of H_n (and thus of every $\mathcal{S}_{ij}$) contains at most one cycle that has an entry in $\langle 1, k\rangle$ and an entry in $\langle k+1, n\rangle$;

(A3) By (A1) and (A2), the sets $\mathcal{S}_{ij}$ of Type A are mutually disjoint;

(A4) If (i, j) is of type A, then

$$|\mathcal{S}_{ij}| = \text{per } H_{j-1}\text{per } H_{n-i}$$

$$= \begin{cases} 1 & \text{if } i = n \text{ and } j = 1, \\ 2^{j-2} & \text{if } i = n \text{ and } k \ge j \ge 2, \\ 2^{n-i-1} & \text{if } n-1 \ge i \ge k+1 \text{ and } j = 1, \\ 2^{n-i+j-3} & \text{if } n-1 \ge i \ge k+1 \text{ and } k \ge j \ge 2, (i, j) \ne (k+1, k). \end{cases}$$

For (i,j) of type B, let $\mathcal{S}_{ij}$ be the set of all transversals of the matrix

$$\begin{bmatrix} H_{k-1} & O & O & O \\ 0 & I_{j-k} & O & O \\ O & O & C_{i-j+1} & O \\ O & O & O & I_{n-i} \end{bmatrix},$$

where H_{k-1} (I_{n-i}) is vacuous if $k=1$ ($i=n$). We make the following observations if (i,j) is of type B:

(B1) If $\tau \in \mathcal{S}_{ij}$, then τ contains the 1-cycle (k,k);

(B2) $(i,j,\ldots,i-1)$ is the unique cycle of τ of length at least 2 with all of its elements in $\langle k+1,n\rangle$;

(B3) By (B2), the sets $\mathcal{S}_{ij}$ of type B are disjoint, and by (B1) and (A1), any set $\mathcal{S}_{ij}$ of type A and any set $\mathcal{S}_{i'j'}$ of type B are disjoint;

(B4) $|\mathcal{S}_{ij}| = 2^{k-2}$.

For (i,j) of type C, let $\mathcal{S}_{ij}$ be the set of all transversals of the matrix

$$\begin{bmatrix} I_{j-1} & O & O & O & O \\ O & C_{i-j+1} & O & O & O \\ O & O & I_{k-1-i} & O & O \\ O & O & O & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & O \\ O & O & O & O & H_{k-1} \end{bmatrix},$$

where I_{j-1} ($I_{k-1-i}; H_{k-1}$) is vacuous if $j=1$ ($i=k-1; k=1$). We make the following observations if (i,j) is of Type C:

(C1) If $\tau \in \mathcal{S}_{ij}$, then the cycle of τ that contains k is the 2-cycle $(k,k+1)$;

(C2) If $\tau \in \mathcal{S}_{ij}$, then the unique cycle of length at least 2 with all of its elements in $\langle 1,k-1\rangle$ is $(i,j,\ldots,i-1)$;

(C3) By (C2), the sets $\mathcal{S}_{ij}$ of types C are disjoint. By (A1), (B1) and (C1), any set $\mathcal{S}_{ij}$ of type C and any set $\mathcal{S}_{i'j'}$ of type A or B are disjoint;

(C4) $|\mathcal{S}_{ij}| = 2^{k-2}$.

Finally, for (i,j) of type D, let $\mathcal{S}_{ij}$ be the set of all transversals of the matrix

$$\begin{bmatrix} I_{j-1} & O & O \\ O & C_{k-j+1} & O \\ O & O & H_k \end{bmatrix},$$

where I_{j-1} is vacuous if $j=1$. We make the following observations if (i,j) is of type D:

(D1) If $\tau \in \mathcal{S}_{ij}$, then the cycle of τ that contains k has length at least 2 and each of its entries is in $\langle 1, k\rangle$.

(D2) $(k, j, j+1, \ldots, k-1)$ is the unique cycle of τ of length at least 2 with all of its elements in $\langle 1, k\rangle$;

(D3) By (D2), the sets $\mathcal{S}_{ij}$ of type D are disjoint. By (A1), (B1), (C1) and (D1), any set $\mathcal{S}_{ij}$ of type D and any set $\mathcal{S}_{i'j'}$ of type A, B or C are disjoint;

(D4) $|\mathcal{S}_{ij}| = 2^{k-1}$.

Let S_n be the n-by-n matrix with (i,j)-entry equal to $|\mathcal{S}_{ij}|$ when $\mathcal{S}_{ij}$ is defined, and 0 otherwise. For example, for $n = 6$

$$S_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 2 & 0 & 0 \\ 1 & 1 & 2 & 2 & 2 & 0 \end{bmatrix}.$$

Fix z to be an integer with $0 \le z \le \frac{k^2+3k-2}{2}$, and let A be an n-by-n Hessenberg $(0,1)$-matrix with z entries on or below the superdiagonal equal to 0 such that per $A = P(\mathrm{nnz}(H_n) - z, n)$. By Theorem 2.3, we may assume that each fully indecomposable component of A is Hessenberg and staircased.

We claim that A is fully indecomposable. Suppose to the contrary that A is not fully indecomposable; then without loss of generality, A is in the form (1) with $b \ge 2$ and per $A \le 2^{n-b}$. Let H' be the matrix obtained from $H_k \oplus H_k$ by replacing the $(k+1,k)$ and $(k,k+1)$ entries by ones. Then per $A \le 2^{n-2} = \mathrm{per}(H_k \oplus H_k) <$ per $H' \le P(k^2+3k, n) \le P(\mathrm{nnz}(H_n) - z, n)$, where the last inequality follows as $k \ge 3$. This contradicts the assumption that per $A = P(\mathrm{nnz}(H_n) - z, n)$, so A is fully indecomposable.

By Lemma 2.2(b), $A \ge T_n$ (entrywise). Thus if $i > j$ and $a_{ij} = 0$, then $(i,j) \in \mathcal{I}$. Hence by (9),

$$\text{per } A \le 2^{n-1} - \sum_{\{(i,j)\in\mathcal{I} \text{ and } a_{ij}=0\}} |\mathcal{S}_{ij}| \le 2^{n-1} - s_{nz},$$

where s_{nz} is the sum of the z smallest nonzero entries of S_n. Since $z \le (k^2+3k-2)/2$, by Lemma 5.1 and (A4) above, the z smallest nonzero entries of S_n and M_n agree. Hence, $s_{nz} = \sigma_{nz}$ and $P(\mathrm{nnz}(H_n) - z, n) \le 2^{n-1} - \sigma_{nz}$. Since, by Lemma 5.2, $P(\mathrm{nnz}(H_n) - z, n) \ge 2^{n-1} - \sigma_{nz}$, it follows that $P(\mathrm{nnz}(H_n) - z, n) = 2^{n-1} - \sigma_{nz}$. ∎

A corresponding result that determines $P(m,n)$ for n odd and $m \ge \frac{3n^2+8n+5}{8}$ is now derived. Let $n = 2k+1 \ge 3$ be an odd positive integer, and z an integer with $0 \le z \le \frac{k^2+3k-2}{2}$. Modifying (as described below) the proof of Theorem 5.3 gives the values of $P(\mathrm{nnz}(H_n) - z, n)$.

Define the types A, B, C and D as in the proof of Theorem 5.3, and the sets $\mathcal{S}_{ij}$ as before when (i, j) is of type A or B. For (i, j) of type C, we now define $\mathcal{S}_{ij}$ to be the set of all transversals of

$$\begin{bmatrix} I_{j-1} & O & O & O & O \\ O & C_{i-j+1} & O & O & O \\ O & O & I_{k-1-i} & O & O \\ O & O & O & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & O \\ O & O & O & O & H_k \end{bmatrix},$$

where I_{j-1} (I_{k-1-i}) is vacuous if $j = 1$ ($i = k-1$). For (i, j) of type D, we now define $\mathcal{S}_{ij}$ to be the set of all transversals of

$$\begin{bmatrix} I_{j-1} & O & O \\ O & C_{k-j+1} & O \\ O & O & H_{k+1} \end{bmatrix},$$

where I_{j-1} is vacuous if $j = 1$. Note that statements (A1)–(A4), (B1)–(B4), (C1)–(C3), (D1)–(D3) hold verbatim. Statement (C4) becomes $|\mathcal{S}_{ij}| = 2^{k-1}$ and statement (D4) becomes $|\mathcal{S}_{ij}| = 2^k$.

In modifying the penultimate paragraph of the proof of Theorem 5.3, let H' be obtained from $H_k \oplus H_{k+1}$ by replacing the $(k+1, k)$ and $(k, k+1)$ entries by ones. Then per $A \leq 2^{n-2} =$ per $(H_k \oplus H_{k+1}) <$ per $H' \leq P(k^2 + 4k + 2, n) \leq P(\text{nnz}(H_n) - z, n)$, where the last inequality follows if $k \geq 1$. These modifications to the proof of Theorem 5.3 give the following result.

Theorem 5.4 *Let $n = 2k+1 \geq 3$ be an odd positive integer, and let z be an integer with $0 \leq z \leq \frac{k^2+3k-2}{2}$. Then $P(nnz(H_n) - z, n) = 2^{n-1} - \sigma_{nz}$.*

6 Concluding Remarks

For $n = 3$ and $3 \leq m \leq 8$, our values of $P(m, n)$ are the same as the values given in [BGM] for the maximum permanent of an arbitrary 3-by-3 $(0,1)$-matrix with m entries equal to 1. However, for $n = 4$ and $m = 10$, this larger class can attain a maximum permanent of 6 [BGM, Table 1] given by $H_1 \oplus J_3$, whereas $P(10, 4) = 5$.

Results from previous sections give $P(m, 2)$, $P(m, 3)$ and $P(m, 4)$ for all possible values of m. For $n = 5$, theorems from Sections 3 and 4 give $P(m, 5)$ for $5 \leq m \leq 13$, whereas values of $P(m, 5)$ for $m \geq 15$ are determined from Theorem 5.4. The value of $P(14, 5)$ does not follow immediately from our theorems. However, we can use previous results on the staircase structure to determine the value of $P(14, 5)$. If $A \in \mathcal{H}(14, 5)$ is partly decomposable, then per $A \leq 2^{4-1} = 8$. If $A \in \mathcal{H}(14, 5)$ is fully indecomposable, then the diagonal, super- and sub-diagonal entries are all equal to 1 (accounting for 13 ones) and $A = T_5 + E_{ij}$, with $(i, j) \in \{(3, 1), (4, 2), (5, 3)\}$. Such a matrix A has permanent equal to 9 or 10, thus $P(14, 5) = 10$, with the maximum attained by $A = T_5 + E_{31}$. In conclusion, we note in general that values of $P(m, n)$ for $8n/3 < m < (n^2 + 3n - 2 - (k^2 + 3k - 2))/2$, where $k = \lfloor n/2 \rfloor$, remain to be determined.

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