On an identity for the cycle indices of rooted tree automorphism groups

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Submitted: Jul 25, 2006; Accepted: Sep 15, 2006; Published: Sep 22, 2006
Mathematics Subject Classifications: 05A15,05A19,05C30

Abstract

This note deals with a formula due to G. Labelle for the summed cycle indices of all rooted trees, which resembles the well-known formula for the cycle index of the symmetric group in some way. An elementary proof is provided as well as some immediate corollaries and applications, in particular a new application to the enumeration of k-decomposable trees. A tree is called k-decomposable in this context if it has a spanning forest whose components are all of size k.

1 Introduction

Pólya's enumeration method is widely used for graph enumeration problems – we refer to [6] and the references therein for instance. For the application of this method, information on the cycle indices of certain groups is needed – mostly, these are comparatively simple examples, such as the cyclic group, the dihedral group or the symmetric group. A very well-known formula gives the cycle index of the symmetric group S_n (we adopt the notation from [6] here):

$$Z(S_n) = \sum_{j_1+2j_2+\ldots+nj_n=n} \prod_{k=1}^{n} \frac{s_k^{j_k}}{k^{j_k} j_k!}. \tag{1}$$

One has

$$\sum_{n=0}^{\infty} Z(S_n) t^n = \exp \sum_{k=1}^{\infty} \frac{s_k}{k} t^k,$$

an identity which is of importance in various tree counting problems (cf. again [6]).

*The author is supported by project S9611 of the Austrian Science Foundation FWF

In the past, several tree counting problems related to the automorphism groups of trees have been investigated. We state, for instance, the enumeration of identity trees (see [7]), and the question of determining the average size of the automorphism group in certain classes of trees (see [9, 10]).

Therefore, it is not surprising that so-called *cycle index series* or *indicatrix series* [2, 8] are of interest in enumeration problems. Given a combinatorial species F, the indicatrix series is given by

$$Z_F(s_1, s_2, \ldots) = \sum_{c_1+2c_2+3c_3+\ldots<\infty} f_{c_1,c_2,c_3,\ldots} \frac{s_1^{c_1} s_2^{c_2} s_3^{c_3} \ldots}{1^{c_1} c_1! 2^{c_2} c_2! 3^{c_3} c_3! \ldots},$$

where $f_{c_1,c_2,c_3,\ldots}$ denotes the number of F-structures on $n = c_1 + 2c_2 + 3c_3 + \ldots$ points which are invariant under the action of any (given) permutation σ of these n points with cycle type $(c_1, c_2, \ldots)$ (i.e. exactly c_k cycles of length k). See for instance [2, 6, 8] and the references therein for more information on cycle index series. Equivalently, it can be defined via

$$Z_F(s_1, s_2, \ldots) = \sum_{n\geq 0} \frac{1}{n!} \left(\sum_{\sigma \in S_n} \operatorname{fix} F[\sigma] x_1^{\sigma_1} x_2^{\sigma_2} x_3^{\sigma_3} \ldots \right),$$

where $\operatorname{fix} F[\sigma]$ is the number of F-structures for which the permutation σ is an automorphism and $(\sigma_1, \sigma_2, \ldots)$ is the cycle type of σ [2].

In this note, we deal with the special family $\mathcal{T}$ of rooted trees. Yet another reformulation shows that the cycle index series is also

$$\sum_{T\in\mathcal{T}} Z(\mathrm{Aut}(T)),$$

where $Z(\mathrm{Aut}(T))$ is the cycle index of the automorphism group of T. The following formula for the cycle index series is due to G. Labelle [8, Corollary A2]:

Theorem 1 *The cycle index series for rooted trees is given by*

$$Z_{\mathcal{T}}(s_1, s_2, \ldots) = \sum_{c_1>0} \sum_{c_2,c_3,\ldots\geq 0} \frac{c_1^{c_1-1} s_1^{c_1}}{c_1!} \prod_{i>1} \frac{1}{c_i! i^{c_i}} \left(\sum_{j|i} j c_j \right)^{c_i - 1} \left(\sum_{j|i, j\neq i} j c_j \right) s_i^{c_i}.$$

Note that the expression resembles (1), though it is somewhat longer. This result seems to be not too well-known, but it certainly deserves attention. In [8], Labelle proves it in a more general setting, using a multidimensional version of Lagrange's inversion formula due to Good [4]. On the other hand, Constantineau and J. Labelle provide a combinatorial proof in [3].

First of all, we will give a simple proof (though, of course, less general than Labelle's) for this formula, for which only the classical single-variable form of Lagrange inversion will be necessary; then, some immediate corrolaries are stated. Finally, the use of the cycle index series is demonstrated by applying the formula to the enumeration of weighted trees and k-decomposable trees.

2 Proof of the main theorem

By the recursive structure of rooted trees and the multiplicative properties of the cycle index, it is not difficult to see that $Z = Z_{\mathcal{T}}(s_1, s_2, \ldots)$ satisfies the relation

$$Z = s_1 \exp\left(\sum_{m\geq 1} \frac{1}{m} Z_m\right),$$

which is given, for instance, in a paper of Robinson [12, p. 344] and the book of Bergeron et al. [2, p. 167]. Here, Z_m is obtained from Z by replacing every s_i with s_{mi}. Now, we prove the following by induction on k:

$$Z = \sum_{\substack{c_1,\ldots,c_k\geq 0\\ c_1>0}} \frac{c_1^{c_1-1} s_1^{c_1}}{c_1!} \prod_{i=2}^{k} \frac{1}{c_i! i^{c_i}} \left(\sum_{j|i} j c_j\right)^{c_i-1} \left(\sum_{j|i, j\neq i} j c_j\right) s_i^{c_i}$$
$$\exp\left(\sum_{m>k} \frac{1}{m} \left(\sum_{d|m, d\leq k} d c_d\right) Z_m\right)$$

in the ring of formal power series. Then, for finite k, the coefficient of $s_1^{c_1} \ldots s_k^{c_k}$ follows at once, since $\sum_{m>k} \frac{1}{m} \left(\sum_{d|m,d\leq k} d c_d\right) Z_m$ doesn't contain the variables $s_1, \ldots, s_k$.

First note that, by Lagrange's inversion formula (cf. [5, 6]), we have

$$w = \sum_{c\geq 1} \frac{c^{c-1}}{c!} x^c$$

and

$$\exp(aw) = \sum_{c\geq 0} \frac{a(c+a)^{c-1}}{c!} x^c$$

if $w = xe^w$. This yields

$$Z = s_1 \exp\left(Z + \sum_{m\geq 2} \frac{1}{m} Z_m\right) = \sum_{c_1\geq 1} \frac{c_1^{c_1-1}}{c_1!} s_1^{c_1} \exp\left(\sum_{m\geq 2} \frac{c_1}{m} Z_m\right),$$

which is exactly the desired formula for $k = 1$. For the induction step, we note that

$$Z_l = s_l \exp\left(\sum_{m\geq 1} \frac{1}{m} Z_{ml}\right)$$

and thus, by the induction hypothesis,

$$Z = \sum_{\substack{c_1,\dots,c_{k-1}\geq 0\\ c_1>0}} \frac{c_1^{c_1-1}s_1^{c_1}}{c_1!}\prod_{i=2}^{k-1}\frac{1}{c_i!i^{c_i}}\left(\sum_{j|i} jc_j\right)^{c_i-1}\left(\sum_{j|i,j\neq i} jc_j\right)s_i^{c_i}$$

$$\exp\left(\frac{1}{k}\left(\sum_{d|k,d\neq k} dc_d\right)Z_k + \sum_{m>k}\frac{1}{m}\left(\sum_{d|m,d<k} dc_d\right)Z_m\right)$$

$$= \sum_{\substack{c_1,\dots,c_{k-1}\geq 0\\ c_1>0}} \frac{c_1^{c_1-1}s_1^{c_1}}{c_1!}\prod_{i=2}^{k-1}\frac{1}{c_i!i^{c_i}}\left(\sum_{j|i} jc_j\right)^{c_i-1}\left(\sum_{j|i,j\neq i} jc_j\right)s_i^{c_i}$$

$$\sum_{c_k\geq 0}\frac{1}{c_k!\cdot k}\left(\sum_{j|k,j\neq k} jc_j\right)\left(c_k+\frac{1}{k}\sum_{j|k,j\neq k} jc_j\right)^{c_k-1} s_k^{c_k}$$

$$\exp\left(\sum_{l>1}\frac{kc_k}{kl}Z_{kl}\right)\exp\left(\sum_{m>k}\frac{1}{m}\left(\sum_{d|m,d<k} dc_d\right)Z_m\right)$$

$$= \sum_{\substack{c_1,\dots,c_k\geq 0\\ c_1>0}} \frac{c_1^{c_1-1}s_1^{c_1}}{c_1!}\prod_{i=2}^{k}\frac{1}{c_i!i^{c_i}}\left(\sum_{j|i} jc_j\right)^{c_i-1}\left(\sum_{j|i,j\neq i} jc_j\right)s_i^{c_i}$$

$$\exp\left(\sum_{m>k}\frac{1}{m}\left(\sum_{d|m,d\leq k} dc_d\right)Z_m\right).$$

This finishes the induction. ∎

Corollary 2 *The number $t_n = |\mathcal{T}_n|$ of rooted trees on n vertices is given by*

$$t_n = \sum_{\substack{c_1+2c_2+\ldots=n\\ c_1>0}} \frac{c_1^{c_1-1}}{c_1!}\prod_{i>1}\frac{1}{c_i!i^{c_i}}\left(\sum_{j|i} jc_j\right)^{c_i-1}\left(\sum_{j|i,j\neq i} jc_j\right).$$

Proof: Simply set $s_1 = s_2 = \ldots = 1$ in the identity

$$\sum_{T\in\mathcal{T}_n} Z(\mathrm{Aut}(T)) = \sum_{\substack{c_1+2c_2+\ldots=n\\ c_1>0}} \frac{c_1^{c_1-1}s_1^{c_1}}{c_1!}\prod_{i>1}\frac{1}{c_i!i^{c_i}}\left(\sum_{j|i} jc_j\right)^{c_i-1}\left(\sum_{j|i,j\neq i} jc_j\right)s_i^{c_i}.$$

∎

As a second corollary, we obtain Cayley's formula for the number of rooted labeled trees.

Corollary 3 *The number of rooted labeled trees on n vertices is given by n^{n-1}.*

Proof: Note that the coefficient of s_1^n in the cycle index of a rooted tree T on n vertices is precisely $|\operatorname{Aut}(T)|^{-1}$. Thus, we have

$$\sum_{T \in \mathcal{T}_n} |\operatorname{Aut}(T)|^{-1} = \frac{n^{n-1}}{n!}.$$

But $\frac{n!}{|\operatorname{Aut} T|}$ is exactly the number of different labelings of T, which finishes the proof. ■

3 Further applications

Theorem 1 can also be applied to a general class of enumeration problems: let a set $\mathcal{B}$ of combinatorial objects with an additive weight be given, and let $B(z)$ be its counting series. Now, if we want to enumerate trees on n vertices, where an element of $\mathcal{B}$ is assigned to every vertex of the tree, the counting series is given by

$$\sum_{\substack{c_1+2c_2+\ldots=n \\ c_1>0}} \frac{c_1^{c_1-1}}{c_1!} B(z)^{c_1} \prod_{i>1} \frac{1}{c_i! i^{c_i}} \left(\sum_{j|i} j c_j \right)^{c_i-1} \left(\sum_{j|i, j \neq i} j c_j \right) B(z^i)^{c_i}.$$

The coefficient of z equals the total weight. For example, the counting series for rooted weighted trees on n vertices (i.e. each vertex is assigned a positive integer weight, cf. Harary and Prins [7]) is given by

$$W(z) = \sum_{\substack{c_1+2c_2+\ldots=n \\ c_1>0}} \frac{c_1^{c_1-1}}{c_1!} \left(\frac{z}{1-z} \right)^{c_1} \prod_{i>1} \frac{1}{c_i! i^{c_i}} \left(\sum_{j|i} j c_j \right)^{c_i-1} \left(\sum_{j|i, j \neq i} j c_j \right) \left(\frac{z^i}{1-z^i} \right)^{c_i}.$$

The first few instances are

- $n = 1$: $W(z) = \frac{z}{1-z} = z + z^2 + z^3 + \ldots$,
- $n = 2$: $W(z) = \frac{z^2}{(1-z)^2} = z^2 + 2z^3 + 3z^4 + \ldots$,
- $n = 3$: $W(z) = \frac{z^3(2+z)}{(1-z)^2(1-z^2)} = 2z^3 + 5z^4 + 10z^5 + \ldots$.

Finally, we are going to consider a new application of Theorem 1. This example deals with the decomposability of trees: we call a tree k-decomposable (a special case of the general concept of λ-decomposability, see [1, 16]) if it has a spanning forest whose components are all of size k. It has been shown by Zelinka [17] that such a decomposition, if it exists, is always unique. The special case $k = 2$, which has already been investigated by Moon [11] and Simion [13, 14], corresponds to perfect matchings. Now, let $D(x)$ denote the generating function for the number of k-decomposable rooted trees. Since a decomposable rooted tree is made up from a rooted tree on k vertices (the component containing the root) and collections of k-decomposable rooted trees attached to each of these k vertices, we obtain the following functional equation for k-decomposable trees:

$$D(x) = \sum_{\substack{c_1+2c_2+\ldots=k \\ c_1>0}} \frac{c_1^{c_1-1}}{c_1!} E(x)^{c_1} \prod_{i>1} \frac{1}{c_i! i^{c_i}} \left(\sum_{j|i} j c_j \right)^{c_i-1} \left(\sum_{j|i, j\neq i} j c_j \right) E(x^i)^{c_i},$$

where $E(x) = x \exp\left(\sum_{m\geq 1} \frac{1}{m} D(x^m)\right)$. For $k = 2$, we obtain

$$D(x) = x^2 \exp\left(\sum_{m\geq 1} \frac{2}{m} D(x^m) \right),$$

giving the known counting series for trees with a perfect matching (Sloane's A000151 [15], see also [11, 13, 14]):

$$D(x) = x^2 + 2x^4 + 7x^6 + 26x^8 + 107x^{10} + 458x^{12} + \ldots$$

For $k = 3$, to give a new example, we have

$$D(x) = \frac{3x^3}{2} \exp\left(\sum_{m\geq 1} \frac{3}{m} D(x^m) \right) + \frac{x^3}{2} \exp\left(\sum_{m\geq 1} \frac{1}{m} \left(D(x^m) + D(x^{2m}) \right) \right),$$

yielding

$$D(x) = 2x^3 + 10x^6 + 84x^9 + 788x^{12} + \ldots$$

Of course, it is possible to calculate the counting series of k-decomposable rooted trees for arbitrary k in this way. The functional equation can also be used to obtain information about the asymptotic behavior (cf. [6, 16]).

Acknowledgment

The author is highly indebted to an anonymous referee for providing him with a lot of valuable information, in particular references [2, 3, 4, 8, 12].

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