Parameter Augmentation for Two Formulas

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Abstract

In this paper, by using the q-exponential operator technique on the q-integral form of the Sears transformation formula and a Gasper q-integral formula, we obtain their generalizations.

1 Notation

In this paper, we follow the notation and terminology in ([4]). For a real or complex number q ($|q| < 1$). let

$$(\lambda)_\infty = (\lambda; q)_\infty = \prod_{n=1}^{\infty}(1 - aq^{n-1}); \tag{1.1}$$

and let $(\lambda : q)_\mu$ be defined by

$$(\lambda)_\mu = (\lambda; q)_\mu = \frac{(\lambda; q)_\infty}{(\lambda q^\mu; q)_\infty}$$

for arbitrary parameters λ and μ, so that

$$(\lambda)_n = (\lambda; q)_n = \{^{1,}_{(1-\lambda)(1-\lambda q)\ldots(1-\lambda q^{n-1}),} \qquad ^{n=0}_{(n\in N=1,2,3,\cdots)}$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k (q)_{n-k}}$$

Further, recall the definition of basic hypergeometric series,

$${}_s\phi_{s-1}\left[\begin{matrix}\alpha_1, \cdots, & \alpha_s \\ \beta_1, \cdots, & \beta_{s-1}\end{matrix} \,\middle|\, q; z\right] := \sum_{n=0}^{\infty} \frac{(\alpha_1, \cdots \alpha_s)_n}{(q, \beta_1, \cdots, \beta_{s-1})_n} z^n. \tag{1.2}$$

Here, we will frequently use the Cauchy identity and its special case ([4])

$$\frac{(ax;q)_\infty}{(x;q)_\infty}=\sum_{n=0}^{\infty}\frac{(a;q)_n x^n}{(q;q)_n} \tag{1.3}$$

$$\frac{1}{(x;q)_\infty}=\sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n} \tag{1.4}$$

$$(-x;q)_\infty=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}x^n}{(q;q)_n} \tag{1.5}$$

2 The exponential operator $T(bD_q)$

The usual q-differential operator, or q-derivative, is defined by

$$D_q\{f(a)\}=\frac{f(a)-f(aq)}{a} \tag{2.1}$$

By convention, D_q^0 is understood as the identity.
The Leibniz rule for D_q is the following identity, which is a variation of the q-binomial theorem ([1])

$$D_q^n\{f(a)g(a)\}=\sum_{k=0}^{n}q^{k(k-n)}\begin{bmatrix} n \\ k \end{bmatrix}D_q^k\{f(a)\}D_q^{n-k}\{g(q^k a)\} \tag{2.2}$$

In ([3]), Chen and Liu construct a q-exponential operator based on this, denoted T:

$$T(bD_q)=\sum_{n=0}^{\infty}\frac{(bD_q)^n}{(q;q)_n} \tag{2.3}$$

For $T(bd_q)$, there hold the following operator identities.

$$T(bD_q)\{\frac{1}{(at;q)_\infty}\}=\frac{1}{(at,bt;q)_\infty} \tag{2.4}$$

$$T(bD_q)\{\frac{1}{(as,at;q)_\infty}\}=\frac{(abst;q)_\infty}{(as,at,bs,bt;q)_\infty} \tag{2.5}$$

3 A generalization of the q-integral form of the sears transformation

In this section, we consider the following formula ([3, Theorem 6.2])

$$\int_c^d \frac{(qt/c,qt/d,abcdet;q)_\infty}{(at,bt,et;q)_\infty}d_qt=\frac{d(1-q)(q,dq/c,c/d,abcd,bcde,acde;q)_\infty}{(ac,ad,bc,bd,ce,de;q)_\infty} \tag{3.1}$$

Chen and Liu showed it can be derived from the Andrews-Askey integral by the q-exponential operator techniques. Here, again using the q-exponential operator technique on it, we obtain a generalization of this identity. We have

Theorem 3.1. *we have*

$$\int_c^d \frac{(qt/c, qt/d, abcdft, bcdeft; q)_\infty}{(at, bt, et, ft; q)_\infty} \times {}_3\phi_2 \left[\begin{matrix} bt, & ft, & bcdf \\ & abcdft, & bcdeft \end{matrix} \middle| q; acde \right] d_q t$$

$$= \frac{d(1-q)(q, dq/c, c/d, abcd, bcde, bcdf, cdef, acdf; q)_\infty}{(ac, ad, bc, bd, ce, de, cf, df; q)_\infty} \tag{3.2}$$

Proof: Dividing both sides of (3.1) by $(abcd, acde; q)_\infty$. we obtain

$$\int_c^d \frac{(qt/c, qt/d, abcdet; q)_\infty}{(at, bt, et, abcd, acde; q)_\infty} d_q t = \frac{d(1-q)(q, dq/c, c/d, bcde; q)_\infty}{(ac, ad, bc, bd, ce, de; q)_\infty}$$

Taking the action $T(fD_q)$ on both sides of the above identity, we have

$$\int_c^d \frac{(qt/c, qt/d; q)_\infty}{(bt, et; q)_\infty} T(fD_q)\{\frac{(abcdet; q)_\infty}{(at, abcd, acde;)_\infty}\} d_q t$$
$$= \frac{d(1-q)(q, dq/c, c/d, bcde; q)_\infty}{(bc, bd, ce, de; q)_\infty} T(fD_q)\{\frac{1}{(ac, ad; q)_\infty}\}$$

By the Leibniz formula, it follows that

$$\begin{aligned}
&T(fD_q)\{\frac{(abcdet; q)_\infty}{(at, abcd, acde; q)_\infty}\} \\
&= \sum_{n=0}^\infty \frac{(bt; q)_n (cde)^n}{(q; q)_n} \sum_{k=0}^\infty \frac{f^k}{(q; q)_k} D_q^k \{\frac{a^n}{(at, abcd; q)_\infty}\} \\
&= \sum_{n=0}^\infty \frac{(bt; q)_n (cde)^n}{(q; q)_n} \sum_{k=0}^\infty \frac{f^k}{(q; q)_k} \sum_{j=0}^k q^{j(j-k)} \begin{bmatrix} k \\ j \end{bmatrix} D_q^j \{\frac{1}{(at, abcd; q)_\infty}\} D_q^{k-j} (aq^j)^n \\
&= \sum_{n=0}^\infty \frac{(bt; q)_n (cde)^n}{(q; q)_n} \sum_{j=0}^\infty \frac{(fD_q)^j}{(q; q)_j} \{\frac{1}{(at, abcd; q)_\infty}\} \sum_{m=0}^n q^{j(n-m)} a^{n-m} \begin{bmatrix} n \\ m \end{bmatrix} f^m \\
&= \sum_{n=0}^\infty \frac{(bt; q)_n (cde)^n}{(q; q)_n} \sum_{m=0}^n a^{n-m} \begin{bmatrix} n \\ m \end{bmatrix} f^m T(fq^{n-m} D_q \{\frac{1}{(at, abcd; q)_\infty}\} \\
&= \sum_{m=0}^\infty \frac{(fcde)^m}{(q; q)_m} \sum_{k=0}^\infty \frac{(bt; q)_{k+m}}{(q; q)_k} (acde)^k \frac{(abcdftq^k; q)_\infty}{(at, abcd, ftq^k, bcdfq^k; q)_\infty}\} \\
&= \frac{(abcdft; q)_\infty}{(at, abcd, ft, bcdf; q)_\infty} \sum_{k=0}^\infty \frac{(ft, bcdf, bt; q)_k}{(q, abcdft; q)_k} (acde)^k \sum_{m=0}^\infty \frac{(q^k bt; q)_m}{(q; q)_m} (fcde)^m \\
&= \frac{(abcdft, bcdeft; q)_\infty}{(at, abcd, ft, bcdf, cdef; q)_\infty} {}_3\phi_2 \left[\begin{matrix} bt, & ft, & bcdf \\ & abcdft, & bcdeft \end{matrix} \middle| q; acde \right]
\end{aligned} \tag{3.3}$$

and

$$T(fD_q)\{\frac{1}{(ac,ad;q)_\infty}\} = \frac{(acdf;q)_\infty}{(ac,ad,cf,df;q)_\infty} \tag{3.4}$$

Combining (3.3) and (3.4), we get Theorem 1.

4 A generalization of Gasper's Formula

We observe the following integral formula which was discovered by Gasper ([5]), In ([3]), Chen and Liu had proved it from the Asky-Roy intergral in one step of parameter augmentation.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, abcdfe^{i\theta};q)_\infty}{(ae^{i\theta}, be^{i\theta}, fe^{i\theta}, ce^{-i\theta}, de^{-i\theta};q)_\infty}d\theta$$

$$= \frac{(\rho c/d, dq/\rho c, \rho, q/\rho, abcd, bcdf, acdf;q)_\infty}{(q, ac, ad, bc, bd, cf, df;q)_\infty} \tag{4.1}$$

where $\max|a|, |b|, |c|, |d| < 1$, $cd\rho \neq 0$.
In this paper, we obtain the following Theorem by again using the q-exponential operator technique on it.

Theorem 4.1. *we have*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, abcdfge^{i\theta}, bcdfge^{i\theta};q)_\infty}{(ae^{i\theta}, be^{i\theta}, fe^{i\theta}, ge^{i\theta}, ce^{-i\theta}, de^{-i\theta};q)_\infty} \times {}_3\phi_2\left[\begin{matrix} fe^{i\theta}, & ge^{i\theta}, & gcdf \\ & acdfge^{i\theta}, & bcdfge^{i\theta}\end{matrix}\,\middle|\, q; abcd\right] d\theta$$

$$= \frac{(\rho c/d, dq/\rho c, \rho, q/\rho, acdf, acdg, bcdf, bcdg, cdfg;q)_\infty}{(q, ac, ad, bc, bd, cf, df, cg, dg;q)_\infty} \tag{4.2}$$

Proof: Dividing both sides of (4.1) by $(abcd, acdf;q)_\infty$, and taking the action of $T(gD_q)$ on both sides of it, we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho;q)_\infty}{(be^{i\theta}, fe^{i\theta}, ce^{-i\theta}, de^{-i\theta};q)_\infty}T(gD_q)\{\frac{(abcdfe^{i\theta};q)_\infty}{(ae^{i\theta}, abcd, acdf;q)_\infty}\}d\theta$$

$$= \frac{(\rho c/d, dq/\rho c, \rho, q/\rho)_\infty}{(q, bc, bd, cf, df;q)_\infty}T(gD_q)\{\frac{1}{(ac,ad;q)_\infty}\}$$

By the Leibniz formula, it follows that

$$
\begin{aligned}
& T(gD_q)\{\frac{(abcdfe^{i\theta};q)_\infty}{(ae^{i\theta},abcd,acdf;q)_\infty}\} \\
&= \sum_{n=0}^{\infty}\frac{(fe^{i\theta};q)_n(bcd)^n}{(q;q)_n}\sum_{k=0}^{\infty}\frac{g^k}{(q;q)_k}D_q^k\{\frac{a^n}{(ae^{i\theta},acdf;q)_\infty}\} \\
&= \sum_{n=0}^{\infty}\frac{(fe^{i\theta};q)_n(bcd)^n}{(q;q)_n}\sum_{k=0}^{\infty}\frac{g^k}{(q;q)_k}\sum_{j=0}^{k}q^{j(j-k)}\begin{bmatrix} k \\ j \end{bmatrix}D_q^j\{\frac{1}{(ae^{i\theta},acdf;q)_\infty}\}D_q^{k-j}(aq^j)^n \\
&= \sum_{n=0}^{\infty}\frac{(fe^{i\theta};q)_n(bcd)^n}{(q;q)_n}\sum_{j=0}^{\infty}\frac{(gD_q)^j}{(q;q)_j}\{\frac{1}{(ae^{i\theta},acdf;q)_\infty}\}\sum_{m=0}^{n}q^{j(n-m)}a^{n-m}\begin{bmatrix} n \\ m \end{bmatrix}g^m \\
&= \sum_{n=0}^{\infty}\frac{(fe^{i\theta};q)_n(bcd)^n}{(q;q)_n}\sum_{m=0}^{n}a^{n-m}\begin{bmatrix} n \\ m \end{bmatrix}g^m T(gq^{n-m}D_q\{\frac{1}{(ae^{i\theta},acdf;q)_\infty}\} \\
&= \sum_{m=0}^{\infty}\frac{(gbcd)^m}{(q;q)_m}\sum_{k=0}^{\infty}\frac{(fe^{i\theta};q)_{k+m}}{(q;q)_k}(abcd)^k\frac{(acdfge^{i\theta}q^k;q)_\infty}{(ae^{i\theta},acdf,ge^{i\theta}q^k,gcdfq^k;q)_\infty}\} \\
&= \frac{(abcdfge^{i\theta};q)_\infty}{(ae^{i\theta},acdf,ge^{i\theta},gcdf;q)_\infty}\sum_{k=0}^{\infty}\frac{(ge^{i\theta},gcdf,fe^{i\theta};q)_k}{(q,acdfge^{i\theta};q)_k}(abcd)^k\sum_{m=0}^{\infty}\frac{(q^kfe^{i\theta};q)_m}{(q;q)_m}(gbcd)^m \\
&= \frac{(acdfge^{i\theta},bcdfge^{i\theta};q)_\infty}{(ae^{i\theta},acdf,ge^{i\theta},gbcd,gcdf;q)_\infty}{}_3\phi_2\left[\begin{matrix} ge^{i\theta}, & fe^{i\theta}, & gcdf \\ & acdfge^{i\theta}, & bcdfge^{i\theta} \end{matrix}\,\middle|\, q;abcd\right]
\end{aligned} \tag{4.3}
$$

and

$$
T(gD_q)\{\frac{1}{(ac,ad;)_\infty}\} = \frac{(acdg;q)_\infty}{(ac,ad,cg,dg;q)_\infty} \tag{4.4}
$$

Combining (4.3) and (4.4), we get Theorem 2.

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