Chains, Subwords, and Fillings: Strong Equivalence of Three Definitions of the Bruhat Order

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Abstract

Let S_n be the group of permutations of $[n] = \{1, \ldots, n\}$. The Bruhat order on S_n is a partial order relation, for which there are several equivalent definitions. Three well-known conditions are based on ascending chains, subwords, and comparison of matrices, respectively. We express the last using fillings of tableaux, and prove that the three equivalent conditions are satisfied *in the same number of ways*.

1 Preliminaries

Let S_n be the group of permutations of $[n] = \{1, \ldots, n\}$. The Bruhat order on S_n is a partial order relation that appears frequently in various contexts, and for which there are several equivalent definitions. In this section we recall three of them and introduce some reformulations of these definitions. For more about the Bruhat order, including details and proofs of the equivalence of Definitions 1, 2, and 3, see [BB], [Fu], or [Hu].

1.1 Chains

For $1 \leqslant i < j \leqslant n$, let $(i, j) \in S_n$ be the transposition $i \leftrightarrow j$. We say that $v \prec (i, j)v$ if and only if the values i and j are not inverted in v.

Definition 1. The Bruhat order on S_n is the transitive closure of $\prec$.

In other words, $v \preccurlyeq w$ if and only if there exists a chain

$$v = v_0 \xrightarrow{(i_1,j_1)} v_1 \xrightarrow{(i_2,j_2)} v_2 \longrightarrow \cdots \xrightarrow{(i_m,j_m)} v_m = w \ , \tag{1}$$

such that, for all $k = 1, \ldots, m$, we have $v_{k-1} \prec (i_k, j_k)v_{k-1} = v_k$. (To allow reflexivity $v \preccurlyeq v$, we allow chains with no edges). Then $w_0 = (n\ n{-}1\ \ldots\ 1)$ is the unique maximum in the Bruhat order, and $v \preccurlyeq w$ if and only if $ww_0 \preccurlyeq vw_0$.

Definition. We say that the ascending chain (1) is a *relevant chain* if $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_m$.

Example 1. There are twenty-two ascending chains from (2134) to (4231), but only two of them are relevant:

$$(2134) \xrightarrow{(1,4)} (2431) \xrightarrow{(2,4)} (4231)$$
$$(2134) \xrightarrow{(1,3)} (2314) \xrightarrow{(1,4)} (2341) \xrightarrow{(2,3)} (3241) \xrightarrow{(3,4)} (4231)$$

Notation. Let $\mathcal{C}(v, w)$ be the set of relevant chains from v to w.

Proposition 1. Let v and w be permutations in S_n. Then $v \preccurlyeq w$ if and only if $\mathcal{C}(v, w) \neq \emptyset$.

Proof. It is clear that if $\mathcal{C}(v, w) \neq \emptyset$, then $v \preccurlyeq w$. The other implication (if $v \preccurlyeq w$, then there exists a relevant chain from v to w) will follow from the main result of this paper, Theorem 1. $\square$

1.2 Subwords

For $1 \leqslant i \leqslant n{-}1$, let $s_i \in S_n$ be the transposition $(i, i{+}1)$; by convention, s_0 is the identity. A word is an array $a = [i_1, i_2, \ldots, i_m]$ with entries (letters) from $\{0, 1, \ldots, n-1\}$. The length of the word a is m, the number of letters. To each word we attach the permutation $s_a = s_{i_1} s_{i_2} \ldots s_{i_m}$. (If the word a is empty, then s_a is the identity.) A subword of a word a is a word $a' = [\epsilon_1 i_1, \epsilon_2 i_2, \ldots, \epsilon_m i_m]$, with $\epsilon_k \in \{0, 1\}$ for all $k = 1, \ldots, m$.

Definition. Let $w \in S_n$. A *reduced word* for w is a word of minimal length with corresponding permutation w.

A canonical construction of a reduced word for w is

$$a(w) = [a_{n-1}, \ldots, a_2, a_1] \ ,$$

such that, for all $k = 1, .., n-1$,

- a_k is a (possibly empty) sequence of increasing consecutive letters, and
- $s_{a_k} s_{a_{k-1}} \ldots s_{a_1}$ and w have the values $1, \ldots, k$ in the same positions.

The reduced word $a(w)$ corresponds to a special factorization of w as a product of (possibly trivial) cycles. If $a_k = [k, \ldots, j_k - 1]$ is a nonempty sequence of increasing consecutive letters, with $1 \leqslant k < j_k \leqslant n$, then s_{a_k} is the cycle $k \to k+1 \to \cdots \to j_k \to k$ in S_n, and we denote this cycle by c_{k,j_k}. If $a_k = []$ is empty, then the corresponding permutation is the identity, as is, by convention, the trivial cycle $c_{k,k}$. The reduced word $a(w)$ corresponds to the decomposition

$$w = c_{n-1,j_{n-1}} \cdots c_{2,j_2} c_{1,j_1} \ .$$

Example. If $w = (4231) \in S_4$, then

$$\begin{array}{ll} a_1 = [1,2,3] & s_{a_1}(1234) = s_1 s_2 s_3(1234) = (234\mathbf{1}) = c_{1,4} \\ a_2 = [2] & s_{a_2}s_{a_1}(1234) = s_2(2341) = (\mathbf{3}2\mathbf{41}) = c_{2,3}c_{1,4} \\ a_3 = [3] & s_{a_3}s_{a_2}s_{a_1}(1234) = s_3(3241) = (4\mathbf{231}) = c_{3,4}c_{2,3}c_{1,4}\ , \end{array}$$

hence $a(w) = [3,2,1,2,3]$, corresponding to the factorization $(4231) = c_{3,4}c_{2,3}c_{1,4}$. When we want to emphasize the components a_3, a_2, and a_1, we write the reduced word $a(w)$ either as $[a_3, a_2, a_1] = [[3],[2],[1,2,3]]$, or as

$$a(w) = \begin{array}{lll} 3 & & \\ 2 & & \\ 1 & 2 & 3 \end{array} = \begin{array}{l} a_3 \\ a_2 \\ a_1 \end{array} ,$$

and we read it from top to bottom and from left to right.

Notation. Let $\mathcal{S}(v,w)$ be the set of all subwords of $a(w)$ that are words (not necessarily reduced, even after deleting the zeros) for v.

Example 2. If $v = (2134)$ and $w = (4231)$, there are exactly two subwords of the reduced word $a(w) = [3,2,1,2,3]$ that are words for $v = (2134)$:

$$\mathcal{S}((2134),(4231)) = \{[3,0,1,0,3],[0,0,1,0,0]\} = \left\{ \begin{array}{lll} 3 & & \\ 0 & & \\ 1 & 0 & 3 \end{array} , \begin{array}{lll} 0 & & \\ 0 & & \\ 1 & 0 & 0 \end{array} \right\} .$$

A second definition of the Bruhat order, equivalent with Definition 1, is given in terms of subwords. While the definition below is valid for any reduced word of w, we will formulate it in terms of the canonical word $a(w)$.

Definition 2. Let v and w be permutations in S_n. We say that $v \preccurlyeq w$ in the Bruhat order if and only if there exists a subword of the reduced word $a(w)$ whose corresponding permutation is v. In other words, $v \preccurlyeq w$ if and only if $\mathcal{S}(v,w) \neq \emptyset$.

1.3 Fillings

Let v be a permutation in S_n. The *associated tableau* $T(v)$ is a tableau that has n boxes on the first column and $v(k)$ boxes on row k, for all $k = 1,..,n$. For every $p, q \in [n]$, we define

$$r_v(p,q) = \#\{i \leqslant p \mid v(i) \leq q\}\ ,$$

the number of rows of $T(v)$ contained in the top-left rectangle with p rows and q columns.

Example. If $v = (2134) \in S_4$, then

$$T(v) = \begin{array}{l} \square\square \\ \square \\ \square\square\square \\ \square\square\square\square \end{array} \qquad \text{and} \qquad r_v = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

A third definition of the Bruhat order, equivalent with Definitions 1 and 2, is given in terms of the arrays r.

Definition 3. Let $v, w \in S_n$. We say that $v \preccurlyeq w$ in the Bruhat order if and only if

$$r_v(p,q) \geqslant r_w(p,q), \qquad \text{for all } 1 \leqslant p,q \leqslant n \,. \tag{2}$$

For every $u \in S_n$ and $k \in [n]$ we have $r_u(n,k) = r_u(k,n) = k$, hence $v \preccurlyeq w$ if and only if condition (2) is satisfied for all $1 \leqslant p,q \leqslant n-1$.

Definition. A filling of the tableau $T(v)$ is a labeling of the boxes of $T(v)$ such that

1. The first box on the k^{th} row is labeled with k, for all $k = 1,..,n$;
2. In each row, the labels are weakly decreasing;
3. In each column, the labels are distinct.

The *standard filling* of $T(v)$ is a labeling of the boxes of $T(v)$ such that all boxes on row k are labeled by k.

Example. If $w = (4231) \in S_4$, then

$$T(w) = \begin{array}{l} \square\square\square\square \\ \square\square \\ \square\square\square \\ \square \end{array} \qquad \text{with standard filling} \qquad \begin{array}{llll} 1 & 1 & 1 & 1 \\ 2 & 2 & & \\ 3 & 3 & 3 & \\ 4 & & & \end{array} \quad .$$

Definition. Let $v, w \in S_n$. A *$w-$filling of* $T(v)$ is a filling of $T(v)$ with the entries of the standard filling of $T(w)$.

Example 3. There are exactly two $(4231)-$fillings of $T(2134)$:

$$\begin{array}{llll} 1 & 1 & & \\ 2 & & & \\ 3 & 3 & 3 & \\ 4 & 2 & 1 & 1 \end{array} \qquad \text{and} \qquad \begin{array}{llll} 1 & 1 & & \\ 2 & & & \\ 3 & 2 & 1 & \\ 4 & 3 & 3 & 1 \end{array}$$

Notation. Let $\mathcal{F}(v,w)$ be the set of $w-$fillings of $T(v)$.

Proposition 2. Let v and w be permutations in S_n. Then $v \preccurlyeq w$ if and only if $\mathcal{F}(v,w) \neq \emptyset$.

Proof. Let v and w be permutations in S_n for which $\mathcal{F}(v,w) \neq \emptyset$. For $p,q \in [n-1]$, there are $p - r_v(p,q)$ boxes on the first p entries of column $q+1$ of $T(v)$, and these boxes are labeled by entries coming from the first p entries of column $q+1$ of the standard filling of $T(w)$. Therefore $p - r_w(p,q) \geqslant p - r_v(p,q)$, which implies that $v \preccurlyeq w$. The other implication (if $v \preccurlyeq w$, then there exists a $w-$filling of $T(v)$) will follow from the main result of this paper, Theorem 1. $\square$

2 The Main Result

Let v and w be permutations in S_n. The main result of this paper is an algorithmic construction of bijections among $\mathcal{C}(v,w)$, $\mathcal{S}(v,w)$, and $\mathcal{F}(v,w)$.

Theorem 1. If $v, w \in S_n$, then $\mathcal{C}(v,w)$, $\mathcal{S}(v,w)$, and $\mathcal{F}(v,w)$ have the same number of elements.

By Definition 2, $v \preccurlyeq w$ if and only if $\mathcal{S}(v,w)$ is nonempty. Therefore, if $v \preccurlyeq w$, then $\mathcal{C}(v,w)$ and $\mathcal{F}(v,w)$ are nonempty, and this finishes the proofs of Propositions 1 and 2. To summarize:

Corollary. Let $v, w \in S_n$. The following conditions are equivalent:

1. $v \preccurlyeq w$;
2. $\mathcal{S}(v,w) \neq \emptyset$;
3. $\mathcal{C}(v,w) \neq \emptyset$;
4. $\mathcal{F}(v,w) \neq \emptyset$.

The last three conditions are *strongly* equivalent: the sets $\mathcal{S}(v,w)$, $\mathcal{C}(v,w)$, and $\mathcal{F}(v,w)$ are not only simultaneously nonempty, but in fact have the same number of elements for all pairs (v,w).

Before showing the algorithmic constructions that prove Theorem 1, we say a few words about the significance of this result for the computation of generators in the equivariant cohomology ring of flag varieties. A more detailed presentation will be given in a forthcoming paper.

Let $M = \mathcal{F}l_n(\mathbb{C})$ be the variety of complete flags in $\mathbb{C}^n$. A generic linear action of the compact torus T^n on $\mathbb{C}^n$ induces an effective action of a subtorus $T = T^{n-1}$ on M, and the fixed point set M^T corresponds bijectively to S_n. An equivariant cohomology class is determined by its restriction to the fixed point set, and for each $v \in M^T$, there exists a canonical class τ_v, such that $\tau_v(w) = 0$ if $v \not\preccurlyeq w$. When $v \preccurlyeq w$, $\tau_v(w)$ can be computed by two different methods.

The first method, specific to flag varieties, uses (left) divided difference operators ([Kn]). If w_0 is the longest permutation in S_n, then the divided difference method gives a formula for $\tau_v(w)$ as a sum (of rational expressions) over $\mathcal{S}(ww_0, vw_0)$. The second method applies to a more general class of Hamiltonian $T-$spaces, and uses normalized Morse interpolation ([Za]). The value $\tau_v(w)$ is expressed as a sum (of rational expressions) over a set of ascending chains from v to w, and modulo multiplication by w_0, this set corresponds bijectively to $\mathcal{C}(ww_0, vw_0)$. The construction of a bijection between $\mathcal{S}(ww_0, vw_0)$ and $\mathcal{C}(ww_0, vw_0)$ is a first step in relating the two approaches. In a separate paper we will complete the reconciliation, by showing that the rational expressions are the same for a chain and for the corresponding subword, and we will discuss partial flag varieties, where the relationship is somehow more complicated.

2.1 Construction of $\Phi\colon \mathcal{C}(v,w) \to \mathcal{S}(v,w)$

For every chain $\gamma \in \mathcal{C}(v,w)$, we construct a subword $\Phi(\gamma) \in \mathcal{S}(v,w)$ by starting with the reduced word $a(w)$ and using the transpositions provided by the chain γ to delete letters from $a(w)$. The construction of $\Phi(\gamma)$ is based on the DELETE algorithm described below, and each step of the algorithm is justified by next lemma.

Lemma 1. Let $w \in S_n$ and $a(w) = [a_{n-1}, \ldots, a_2, a_1]$ be the canonical reduced word for w. Let $a' = [a'_{n-1}, \ldots, a'_i, a_{i-1}, \ldots, a_1]$ be a subword of $a(w)$, such that a'_k is a subword of a_k for every $k = i, \ldots, n-1$. Let $w' = s_{a'}$ be the permutation associated to a', and $w'' = (i,j)w'$. If $w'' \prec w'$, then there exists a unique word a'' for w'' such that

- $a'' = [a'_{n-1}, \ldots, a'_{i+1}, a''_i, a_{i-1}, \ldots, a_1]$, and
- a''_i is a subword of a'_i, obtained by deleting one letter from the leftmost consecutive subsequence of a'_i.

Proof. The uniqueness of a'' is clear, and the main idea behind the construction of a'' is to try to move the transposition (i,j), conjugated, to the other side of a', one cycle at a time.

Claim 1: $(i,j)s_{a'_{n-1}} \cdots s_{a'_{i+1}} = s_{a'_{n-1}} \cdots s_{a'_{i+1}}(i,\ell)$ for some transposition (i,ℓ).

This follows from the fact that the conjugation of any transposition is also a transposition. More precisely, if $\sigma \in S_n$, then $\sigma(i,j)\sigma^{-1}$ is the transposition that swaps $\sigma(i)$ and $\sigma(j)$. In our case $\sigma = (s_{a'_{n-1}} \cdots s_{a'_{i+1}})^{-1}$, and since all the nonzero letters in $a'_{n-1}, \ldots, a'_{i+1}$ are strictly greater than i, we have $\sigma(h) = h$ for $h \leqslant i$. Then $\sigma(i) = i$, and $\ell = \sigma(j) = (s_{a'_{n-1}} \cdots s_{a'_{i+1}})^{-1}(j) > i$.

Claim 2: If $(i,j)w' \prec w'$, then the first consecutive subsequence in a'_i starts with i.

By applying $s_{a_{i-1}} \cdots s_{a_1}$, we do not create any inversion (as values) of the form (i,h), for any $h > i$. If the first letter in a'_i is not i, then the remaining transpositions in a' only operate with values strictly above i, and therefore cannot produce an inversion of the form (i,h), for $h > i$. But (i,j) is such an inversion in w', hence a'_i must start with i. Let $k = s_{a'_i}^{-1}(i)$. Then the first consecutive subsequence in a'_i is $[i, i+1, \ldots, k-1]$.

Claim 3: $\ell \leqslant k$.

Since (i,j) is an inversion in w', we have $(w')^{-1}(i) > (w')^{-1}(j)$, hence

$$(s_{a_{i-1}} \cdots s_{a_1})^{-1}(s_{a'_{n-1}} \cdots s_{a'_i})^{-1}(i) > (s_{a_{i-1}} \cdots s_{a_1})^{-1}(s_{a'_{n-1}} \cdots s_{a'_i})^{-1}(j)\ .$$

But both $(s_{a'_{n-1}} \cdots s_{a'_{i+1}} s_{a'_i})^{-1}(i)$ and $(s_{a'_{n-1}} \cdots s_{a'_{i+1}} s_{a'_i})^{-1}(j)$ are greater than or equal to i, and $s_{a_{i-1}} \cdots s_{a_1}$ does not change the relative order of values greater than or equal to i. Therefore

$$i \leqslant s_{a'_i}^{-1}(s_{a'_{n-1}} \ldots s_{a'_{i+1}})^{-1}(j) < s_{a'_i}^{-1}(s_{a'_{n-1}} \ldots s_{a'_{i+1}})^{-1}(i) = s_{a'_i}^{-1}(i) = k\ .$$

But $(s_{a'_{n-1}} \cdots s_{a'_{i+1}})^{-1}(j) = \ell$, hence $i \leqslant s_{a'_i}^{-1}(\ell) < k$. Since the set $\{i, \ldots, k\}$ is invariant under $s_{a'_i}$ and $s_{a'_i}(k) = i$, it follows that $i < \ell \leqslant k$.

Claim 4: $(i,\ell)c_{i,k} = c_{i,\ell-1}c_{\ell,k}$.

This follows from a simple computation and, in terms of reduced words, is written as

$$[i,i{+}1,\ldots,\ell{-}1,\ldots,i{+}1,i][i,i{+}1,\ldots,k{-}1] = [i,i{+}1,\ldots,\ell{-}2,0,\ell,\ldots,k{-}1].$$

The simple transpositions $i,i{+}1,\ldots,\ell{-}1$ delete the first letters of the second word, but then $i,i{+}1,\ldots,\ell{-}2$ are added back.

Therefore a'' is obtained from a' by deleting the letter $\ell{-}1$ from a'_i. $\square$

The unique subword a'' is obtained starting from a' and using (i,j) to delete a letter from a', and we write that as

$$a'' = \mathrm{DELETE}(a',(i,j)) \,.$$

We are now ready to define $\Phi\colon \mathcal{C}(v,w) \to \mathcal{S}(v,w)$. Let $\gamma \in \mathcal{C}(v,w)$ be the relevant chain

$$v = v_0 \xrightarrow{(i_1,j_1)} v_1 \xrightarrow{(i_2,j_2)} v_2 \longrightarrow \cdots \xrightarrow{(i_m,j_m)} v_m = w \,.$$

Based on Lemma 1, we construct inductively a sequence $b_m, b_{m-1}, \ldots, b_1, b_0$ by:

- $b_m = a(w)$, and
- $b_{k-1} = \mathrm{DELETE}(b_k,(i_k,j_k))$ for $k = m, m{-}1, \ldots, 1$.

Note that $b_k \in \mathcal{S}(v_k,w)$ for all $k = m, m{-}1, \ldots, 1, 0$. We define

$$\Phi(\gamma) = b_0 \in \mathcal{S}(v_0,w) = \mathcal{S}(v,w) \,.$$

Before proving that Φ is a bijection, we show how it works in a particular example.

Example. Let $v = (2134)$, $w = (4231)$, and $\gamma \in \mathcal{C}(2134, 4231)$, given by

$$(2134) \xrightarrow{(1,4)} (2431) \xrightarrow{(2,4)} (4231) \,.$$

Then $a(w) = [[3],[2],[1,2,3]]$, $[2,3,2]$ is a reduced word for the transposition $(2,4)$, and $[1,2,3,2,1]$ is a reduced word for $(1,4)$. The DELETE algorithm works as follows:

$$\begin{array}{c} b_2 = [3,2,1,2,3] \\ (2431) \xrightarrow{(2,4)} (4231) \end{array} \qquad \begin{array}{rcccc} 2,3,2 & \to & 3 & \to & 2 \\ 2 & \to & \boxed{2} & & \\ & & 1,2,3 & & \end{array} \qquad b_1 = [3,0,1,2,3]$$

$$\begin{array}{c} b_1 = [3,0,1,2,3] \\ (2134) \xrightarrow{(1,4)} (2431) \end{array} \qquad \begin{array}{rclcl} 1,2,3,2,1 & \to & 3 & \to & 1,2,1 \\ 1,2,1 & \to & 0 & \to & 1,2,1 \\ 1,2,1 & \to & 1,\boxed{2},3 & & \end{array} \qquad b_0 = [3,0,1,0,3]$$

$$\Phi(\gamma) = [3,0,1,0,3] = [[3],[0],[1,0,3]] \,.$$

The notation $2,3,2 \to 3 \to 2$ means $[2,3,2][3] = [3][2]$, that is, $(2,4)$ is moved to the other side of $[3]$ as $(2,3)$. The boxed letters are the letters deleted at each step.

2.2 The inverse of Φ

To prove that Φ is bijective, it suffices to construct an inverse ("subword-to-chain") map $\Phi^{-1}\colon \mathcal{S}(v,w) \to \mathcal{C}(v,w)$. Since Φ has been constructed using the DELETE algorithm, it is enough to show how one can reverse the algorithm, and trace back the sequence of permutations that deleted the letters of the word $a(w) = [a_{n-1}, \dots, a_1]$. A key remark is that when we apply the DELETE algorithm following (in reverse order) the edges of a relevant chain, we delete the letters from top to bottom, and from right to left. So for the inverse procedure, we insert the letters from bottom to top, and from left to right.

Here is how this works for $v = (2143)$, $w = (4231)$ and the subword $u' = [[0],[0],[1,0,0]]$ of $a(w) = [[3],[2],[1,2,3]]$ (see Example 2). The last deleted letter is the 2 in the last list. At that point, the preceding subword must have been $[0,0,1,2,0]$, and to delete the 2, the transposition that acted on the last row must have been $(1,3)$, with word $[1,2,1]$. Tracing it back, we see that the original transposition must have been $(1,3)$, hence the first edge of the chain is $(2134) \longrightarrow (2314)$. So the reverse process goes as follows:

$$\begin{array}{rclcl}
1,2,1 & \to & 0 & \to & 1,2,1 \\
1,2,1 & \to & 0 & \to & 1,2,1 \\
1,2,1 & \to & 1,\boxed{2},0 & \to & \dashv
\end{array}
\qquad (2134) \xrightarrow{(1,3)} (2314)$$

$$\begin{array}{rclcl}
1,2,3,2,1 & \to & 0 & \to & 1,2,3,2,1 \\
1,2,3,2,1 & \to & 0 & \to & 1,2,3,2,1 \\
1,2,3,2,1 & \to & 1,2,\boxed{3} & \to & \dashv
\end{array}
\qquad (2314) \xrightarrow{(1,4)} (2341)$$

$$\begin{array}{rclcl}
2 & \to & 0 & \to & 2 \\
2 & \to & \boxed{2} & \to & \dashv \\
 & & 1,2,3 & &
\end{array}
\qquad (2341) \xrightarrow{(2,3)} (3241)$$

$$\begin{array}{rclcl}
3 & \to & \boxed{3} & \to & \dashv \\
 & \to & 2 & \to & \\
 & & 1,2,3 & &
\end{array}
\qquad (3241) \xrightarrow{(3,4)} (4231)$$

(The boxed letters are the letters that we push back into the subword.) Then the relevant chain $\gamma = \Phi^{-1}(a)$ corresponding to $a = [0,0,1,0,0]$ is

$$(2134) \xrightarrow{(1,3)} (2314) \xrightarrow{(1,4)} (2341) \xrightarrow{(2,3)} (3241) \xrightarrow{(3,4)} (4231)\ .$$

To prove the reverse procedure works in general, it suffices to prove the following lemma.

Lemma. Let $w \in S_n$ and let $a(w) = [a_{n-1}, \dots, a_1]$ be the special reduced word for w. Let $a'' = [a''_{n-1}, \dots, a''_i, a_{i-1}, \dots, a_1]$ be a subword of $a(w)$, such that a''_k is a subword of a_k for every $k = i, \dots, n-1$, and such that a''_i has at least one letter deleted from a_i. Let $\ell - 1$ be the leftmost deleted letter in a''_i, hence $a''_i = [i, \dots, \ell-2, 0, \dots\ ]$. Let $a' = [a''_{n-1}, \dots, a''_{i+1}, a'_i, a_{i-1}, \dots, a_1]$ be the subword of $a(w)$ obtained by un-deleting the letter $\ell-1$ from a''_i, and let $w'' = s_{a''}$ and $w' = s_{a'}$ be the permutations corresponding to a'' and a'. Then $w' = (i,j)w''$, $w' \succ w''$, and

$$a'' = \text{DELETE}(a', (i,j)) \ .$$

Proof. It is not hard to see that $s_{a''_i} = (i,\ell)s_{a'_i}$, hence

$$w'' = (s_{a''_{n-1}} \cdots s_{a''_{i+1}})s_{a''_i}(s_{a_{i-1}} \cdots s_{a_1}) = (s_{a''_{n-1}} \cdots s_{a''_{i+1}})(i,\ell)s_{a'_i}(s_{a_{i-1}} \cdots s_{a_1})$$

Let $\sigma = s_{a''_{n-1}} \cdots s_{a''_{i+1}}$. Then $\sigma(i,\ell)\sigma^{-1}$ is the transposition that swaps $\sigma(i)$ and $\sigma(\ell)$. Since σ fixes all values less than or equal to i, it follows that $\sigma(i) = i$ and $\sigma(\ell) > i$. If $j = \sigma(\ell)$, then $\sigma(i,\ell) = (i,j)\sigma$, which implies

$$w'' = \sigma(i,\ell)s_{a'_i}(s_{a_{i-1}} \cdots s_{a_1}) = (i,j)\sigma s_{a'_i}(s_{a_{i-1}} \cdots s_{a_1}) = (i,j)w' \ .$$

The first deleted letter in a''_i is $\ell-1$, so $(s_{a''_i})^{-1}(i) = \ell-1$ and $(s_{a''_i})^{-1}(\ell) \geqslant \ell$. Therefore

$$i \leqslant \ell-1 = (s_{a''_i})^{-1}\sigma^{-1}(i) < \ell \leqslant (s_{a''_i})^{-1}(\ell) = (s_{a''_i})^{-1}\sigma^{-1}(j) \ .$$

But $(s_{a_{i-1}} \cdots s_{a_1})^{-1}$ does not change the relative order of entries above $i-1$, hence

$$(w'')^{-1}(i) = (s_{a_{i-1}} \cdots s_{a_1})^{-1}(s_{a''_i})^{-1}\sigma^{-1}(i) < s_{a_1})^{-1}(s_{a''_i})^{-1}\sigma^{-1}(j) = (w'')^{-1}(j) \ .$$

Therefore (i,j) is not an inversion (as values) in w'', and $w'' \prec (i,j)w'' = w'$. It is clear that a'' is obtained from a' by deleting one letter with the help of the transposition (i,j). □

Let $a \in \mathcal{S}(v,w)$. Applying the reverse procedure for every deleted letter of a, moving from bottom to top, and from left to right, we recover the relevant chain γ that produced the subword. This proves that Φ has an inverse, so it is a bijection.

2.3 Construction of $\Psi\colon \mathcal{C}(v,w) \to \mathcal{F}(v,w)$

Let $v, w \in S_n$, and $\gamma \in \mathcal{C}(v,w)$. We start with the standard filling of $T(w)$, and, using the transpositions provided by the chain γ, change it to a $w-$filling of $T(v)$. The construction of $\Psi(\gamma)$ is based on the SLIDE algorithm described below, and each step of the algorithm is justified by Lemma 2.

Lemma 2. Let $w \in S_n$ and f_w be the standard filling of the associated tableau $T(w)$. Let $u \in S_n$ and let f_u be a $w-$filling of $T(u)$, such that

- f_u and f_w match completely on the first i columns, and
- on column $i+1$, f_u and f_w match on boxes strictly above row $u^{-1}(i)$.

Let $\sigma = (i,j)u$. The associated tableau $T(\sigma)$ is obtained from $T(u)$ by moving (sliding) the last $j-i$ boxes, from the row $u^{-1}(j)$ of $T(u)$ to the end of the row $u^{-1}(i)$. Let f_σ be the labeling of $T(\sigma)$ obtained from f_u by moving the f_u-labels together with the boxes. If $\sigma \prec u$, then

- f_σ is a $w-$filling of $T(\sigma)$;
- f_σ and f_w match completely on the first i columns;
- on column $i+1$, f_σ and f_w match on boxes strictly above row $\sigma^{-1}(i)$.

Proof. The only problem that might prevent f_σ from being a $w-$filling of $T(\sigma)$ is a violation of the nondecreasing on rows condition, and this could only happen at the end of row $\sigma^{-1}(j) = u^{-1}(i)$. However, if $\sigma = (i,j)u \prec u$, then (i,j) is an inversion (as values) in u, hence $u^{-1}(i) > u^{-1}(j)$. Therefore the boxes are moved downwards, and the second hypothesis on f_u implies that

$$f_u[u^{-1}(j), i+1] = f_u[u^{-1}(j), i] \ ,$$

so we break between boxes with the same label. At the end of the row $\sigma^{-1}(j)$ of $T(\sigma)$ we have

$$\begin{aligned} f_\sigma[\sigma^{-1}(j), i+1] = f_\sigma[u^{-1}(i), i+1] = f_u[u^{-1}(j), i+1] = f_u[u^{-1}(j), i] = \\ = u^{-1}(j) < u^{-1}(i) = f_u[u^{-1}(i), i] = f_\sigma[\sigma^{-1}(j), i] \ , \end{aligned}$$

so f_σ is a $w-$filling of $T(\sigma)$.

The $w-$filling f_σ matches completely with f_w on the first i columns, because f_σ and f_u match on the first i columns, and so do f_u and f_w. Moreover, on column $i+1$, we haven't changed anything above the row $\sigma^{-1}(i) = u^{-1}(j)$, and that row is above the row $u^{-1}(i)$. □

The filling f_σ is obtained starting from f_u and using (i,j) to identify the sliding move, and we write that as

$$f_\sigma = \mathrm{SLIDE}(f_u, (i,j)) \ .$$

We are now ready to define $\Psi\colon \mathcal{C}(v,w) \to \mathcal{F}(v,w)$. Let $\gamma \in \mathcal{C}(v,w)$ be the relevant chain

$$v = v_0 \xrightarrow{(i_1,j_1)} v_1 \xrightarrow{(i_2,j_2)} v_2 \longrightarrow \cdots \xrightarrow{(i_m,j_m)} v_m = w \ .$$

Based on Lemma 2, we construct inductively a sequence $f_m, f_{m-1}, \ldots, f_1, f_0$ by:

- $f_m = f_w$, the standard filling of $T(w)$, and
- $f_{k-1} = \mathrm{SLIDE}(f_k, (i_k, j_k))$ for $k = m, m-1, \ldots, 1$.

Note that since $i_m \geqslant i_{m-1} \geqslant \cdots \geqslant i_2 \geqslant i_1$, the triple $(u, f_u, (i,j)) = (v_k, f_k, (i_k, j_k))$ satisfies the hypotheses of Lemma 2 for every $k = m, \ldots, 1$, hence the sequence $(f_k)_k$ is legitimately defined. Moreover, $f_k \in \mathcal{F}(v_k, w)$ for all $k = m, m-1, \ldots, 1, 0$, and we define

$$\Psi(\gamma) = f_0 \in \mathcal{F}(v_0, w) = \mathcal{F}(v, w) \ .$$

Before proving that Ψ is a bijection, we show how it works in a particular example.

Example. Let $v = (2134)$, $w = (4231)$, and $\gamma \in \mathcal{C}(2134, 4231)$ given by

$$(2134) \xrightarrow{(1,3)} (2314) \xrightarrow{(1,4)} (2341) \xrightarrow{(2,3)} (3241) \xrightarrow{(3,4)} (4231)\ .$$

The SLIDE algorithm works as follows, with the labels to be moved at each step in bold:

$$\begin{array}{ccccccccc}
(4231) & \xleftarrow{(3,4)} & (3241) & \xleftarrow{(2,3)} & (2341) & \xleftarrow{(1,4)} & (2314) & \xleftarrow{(1,3)} & (2134)\\
\begin{array}{llll}1&1&1&\mathbf{1}\\2&2&&\\3&3&3&\\4&&&\end{array} & \to & \begin{array}{llll}1&1&\mathbf{1}&\\2&2&&\\3&3&3&1\\4&&&\end{array} & \to & \begin{array}{llll}1&1&&\\2&2&1&\\3&\mathbf{3}&\mathbf{3}&\mathbf{1}\\4&&&\end{array} & \to & \begin{array}{llll}1&1&&\\2&\mathbf{2}&\mathbf{1}&\\3&&&\\4&3&3&1\end{array} & \to & \begin{array}{llll}1&1&&\\2&&&\\3&2&1&\\4&3&3&1\end{array}
\end{array}.$$

Therefore

$$\Psi(\gamma) = \begin{array}{llll}1&1&&\\2&&&\\3&2&1&\\4&3&3&1\end{array}.$$

2.4 The inverse of Ψ

To prove that Ψ is bijective, it suffices to construct an inverse ("filling-to-chain") map, $\Psi^{-1}\colon \mathcal{F}(v, w) \to \mathcal{C}(v, w)$. Since Ψ is based on the SLIDE algorithm, it suffices to show how one can reverse each step. Here is how it works in a particular case (see Examples 1 and 3):

$$\begin{array}{ccccc}
\begin{array}{llll}1&1&&\\2&&&\\3&3&3&\\4&2&1&1\end{array} & \leftarrow & \begin{array}{llll}1&1&&\\2&2&1&1\\3&3&3&\\4&&&\end{array} & \leftarrow & \begin{array}{llll}1&1&1&1\\2&2&&\\3&3&3&\\4&&&\end{array}\\
(2134) & \xrightarrow{(1,4)} & (2431) & \xrightarrow{(2,4)} & (4231)
\end{array}$$

To prove the reverse procedure works in general, it suffices to prove the following lemma.

Lemma. Let $\sigma, w \in S_n$ such that $\sigma \neq w$, let f_w be the standard filling of $T(w)$, and let f_σ be a $w-$filling of $T(\sigma)$. Let $1 \leqslant i < n$ be defined by the condition that f_σ and f_w match on columns $1, \ldots, i$, but differ on column $i+1$. Let $1 \leqslant j \leqslant n$ be defined by $f_\sigma[\sigma^{-1}(j), i+1] = \sigma^{-1}(i)$, let $u = (i, j)\sigma$, and let f_u be the labeling of $T(u)$ obtained by moving the last $j-i$ labels on row $u^{-1}(i) = \sigma^{-1}(j)$ to the end of row $u^{-1}(j) = \sigma^{-1}(i)$. Then $f_\sigma = \mathrm{SLIDE}(f_u, (i, j))$.

Proof. The proof is based on the following sequence of claims.

Claim 1: $1 \leqslant i < n$, so i is well-defined.

The first columns of f_σ and f_w do match, so $1 \leqslant i$, and $i \neq n$, since $\sigma \neq w$.

Claim 2: $\sigma^{-1}(i)$ does appear on column $i+1$ of f_σ, so j is well-defined.

Otherwise, the entries on column $i+1$ would be the entries on column i, less the entry $\sigma^{-1}(i)$, and the weakly decreasing condition would imply that columns $i+1$ of f_σ and f_w match.

Claim 3: $i<j$ and (i,j) is not an inversion (as values) in σ, hence $\sigma \prec (i,j)\sigma = u$.

This follows from the fact that j is the length of the row $\sigma^{-1}(j)$, hence $j>i$. Moreover, the weakly decreasing condition implies $\sigma^{-1}(j) = f_\sigma[\sigma^{-1}(j), i] \geqslant f_\sigma[\sigma^{-1}(j), i+1] = \sigma^{-1}(i)$.

Claim 4: f_u is a $w-$filling of $T(u)$.

The only problem might occur between columns i and $i+1$ on row $u^{-1}(j)$. But

$$f_u[u^{-1}(j), i+1] = f_\sigma[\sigma^{-1}(j), i+1] = \sigma^{-1}(i) = f_w[\sigma^{-1}(i), i] = f_\sigma[\sigma^{-1}(i), i] = f_u[u^{-1}(j), i] \ ,$$

so the labels are weakly decreasing on row $u^{-1}(j)$.

Claim 5: $(u, f_u, (i,j))$ satisfy the hypotheses of Lemma 2.

First, f_u and f_w match on columns $1, \ldots, i$, since f_σ and f_w do. Let N be the number of elements of the set

$$\{\sigma^{-1}(k) \mid \sigma^{-1}(k) < \sigma^{-1}(j) \text{ and } k \geqslant i\} = \{w^{-1}(k) \mid w^{-1}(k) < w^{-1}(j) \text{ and } k \geqslant i\} \ .$$

These are the smallest N labels on column i of f_σ and f_w. On the same rows, there are $N-1$ boxes on the column $i+1$ of $T(\sigma)$, with the box for row $\sigma^{-1}(i)$ missing. The labels of these $N-1$ boxes are taken from the set $\{w^{-1}(k) \mid k > i\}$, and there are at most $N-1$ such labels less than $w^{-1}(j) = \sigma^{-1}(j)$. Since every label on the column $i+1$ of f_σ is less than or equal to the corresponding label on column i, that implies that f_w and f_σ match on column $i+1$ strictly above row $\sigma^{-1}(j)$, and hence f_w and f_u match on column $i+1$ strictly above row $u^{-1}(i)$.

Claim 6: $f_\sigma = \mathrm{SLIDE}(f_u, (i,j))$.

This is clear from the re-construction of f_u. $\square$

The number of matches of f_u and f_w on column $i+1$ is strictly greater than the number of matches on column $i+1$ between f_σ and f_w. Therefore the re-construction algorithm is finite: for every $w-$filling $f_v \in \mathcal{F}(v,w)$ of the associated tableau $T(v)$, by repeating this procedure, we will get back to the standard filling of $T(w)$. The transpositions (i,j) give a relevant chain $\gamma \in \mathcal{C}(v,w)$, and $\Psi(\gamma) = f_v$. Therefore Ψ has an inverse, hence it is a bijection.

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