Permutations, cycles and the pattern 2–13

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Abstract

We count the number of occurrences of restricted patterns of length 3 in permutations with respect to length *and* the number of cycles. The main tool is a bijection between permutations in standard cycle form and weighted Motzkin paths.

1 Introduction

Let $\mathcal{S}_n$ denote the set of permutations of $[n] = \{1, 2, \ldots, n\}$. A *pattern* in a permutation $\pi \in \mathcal{S}_n$ is a permutation $\sigma \in \mathcal{S}_k$ and an occurrence of σ as a subword of π: There should exist $i_1 < \cdots < i_k$ such that $\sigma = R(\pi(i_1) \cdots \pi(i_k))$, where R is the reduction operator that maps the smallest element of the subword to 1, the second smallest to 2, and so on.

For example, an occurrence of the pattern 3–2–1 in $\pi \in \mathcal{S}_n$ means that there exist $1 \leq i < j < k \leq n$ such that $\pi(i) > \pi(j) > \pi(k)$.

We further consider *restricted* patterns, introduced by Babson and Steingrímsson, [1]. The restriction is that two specified adjacent elements in the pattern *must be adjacent* in the permutation as well. The position of the restriction in the pattern is indicated by *an absence* of a dash (–). Thus, an occurrence of the pattern 3–21 in $\pi \in \mathcal{S}_n$ means that there exist $1 \leq i < j < n$ such that $\pi(i) > \pi(j) > \pi(j+1)$.

Here we are mainly interested in patterns of the type 2–13. We remark that it is shown by Claesson, [3], that the occurrences of 2–13 are equidistributed with the occurrences of the pattern 2–31, as well as with 13–2 and with 31–2. The number of permutations with k occurrences of 2–13 where given by Claesson and Mansour, [4], for $k \leq 3$ and for $k \leq 8$ by Parviainen, [6].

The starting point of [6] and this paper is a generating function related to the solution of a certain much studied Markov chain, the asymmetric exclusion process, [2]. This function, of 4 variables, is the continued fraction

$$F(q,x,y,t) = \cfrac{1}{1 - t([1]_q^x + [1]_q^y) - \cfrac{t^2[1]_q[2]_q^{x,y}}{1 - t([2]_q^x + [2]_q^y) - \cfrac{t^2[2]_q[3]_q^{x,y}}{1 - t([3]_q^x + [3]_q^y)\cdots}}},$$

where

$$\begin{aligned}
[h]_q &= 1 + q + \cdots + q^{h-2} + q^{h-1}, \\
[h]_q^x &= 1 + q + \cdots + q^{h-2} + xq^{h-1}, \\
[h]_q^y &= 1 + q + \cdots + q^{h-2} + yq^{h-1}, \\
[h]_q^{x,y} &= 1 + q + \cdots + q^{h-3} + (x + y - xy)q^{h-2} + xyq^{h-1}.
\end{aligned}$$

It was shown in [4] and [6] that $F(q,1,1,t)$ counts the number of permutations with k occurrences of the pattern 2–13. The main goal of this paper is to study $F(q,x,1,t)$ and give a combinatorial interpretation of the coefficients. It turns out that the variable x is connected to the cycle structure of permutations.

2 Introducing cycles

First consider $F(1,x,1,t)$, and expand in t:

$$F(1,x,1,t) = 1 + (1+x)t + (2 + 3x + x^2)t^2 + (6 + 11x + 6x^2 + x^3)t^3 + O(t^4).$$

These coefficients certainly looks like the unsigned Stirling numbers of the first kind. Thus $F(1,x,1,t)$ should count the number of permutations with respect to length and number of cycles. This will indeed follow from the main theorem.

As $F(q,1,1,t)$ counts the number of occurrences of the pattern 2–13 and $F(1,x,1,t)$ the number of cycles, $F(q,x,1,t)$ should give (some kind of) bivariate statistic of occurrences of 2–13 and cycle distribution.

2.1 Cyclic occurrence of patterns

The *standard cycle form* of a permutation $\pi \in \mathcal{S}_n$ is the permutation written in cycle form, with cycles starting with the smallest element, and cycles ordered in decreasing order with respect to their minimal elements. Let $C(\pi)$ denote the standard cycle form of a permutation π.

Example 1. If $\pi = 47613852$, then $C(\pi) = (275368)(14)$.

Definition 1. Let π be a permutation of $[n]$, with standard cycle form

$$C(\pi) = (c_1^1 c_2^1 \cdots c_{i_1}^1)(c_1^2 c_2^2 \cdots c_{i_2}^2) \cdots (c_1^k c_2^k \cdots c_{i_k}^k),$$

and let $\sigma = AB$ be a permutation of $[m]$, $m \le n$. The pattern A–B occurs *cyclically* in π if it occurs in one of the following senses

Between cycles: If A–B occurs in the permutation

$$\hat{\pi} = c_1^1 c_2^1 \cdots c_{i_1}^1 c_1^2 c_2^2 \cdots c_{i_2}^2 \cdots c_1^k c_2^k \cdots c_{i_k}^k$$

and there exist $a < b$ such that A occurs in $c_1^a \cdots c_{i_a}^a$ and B occurs in $c_1^b \cdots c_{i_b}^b$, we say that A–B occurs *between cycles in* π.

Within cycles: Let $\tilde{\pi} = c_1^a \cdots c_{i_a}^a$. If A–B occurs in $\tilde{\pi}$ we say that A–B occurs *within cycle* a *in* π.

Example 2. If $C(\pi) = (275368)(14)$ there are 2 occurrences of 2–13 *between* cycles, 2–14 and 3–14, and 2 occurrences of 2–13 *within* cycles, 7–68 and 5–36.

Let $\Phi_{i,j}(n)$ denote the number of permutations of length n, with i cyclic occurrences of 2–13 and j cycles.

Theorem 1. *The function $F(q, x, 1, t)$ is the (ordinary) generating function for $\Phi_{i,j}(n)$:*

$$\Phi_{i,j}(n) = [q^i x^j t^n] F(q, x, 1, t).$$

3 Proof of Theorem 1

We will use the fact [5, Theorem 1] that $F(q, x, 1, t)$ is the generating function for weighted bi-coloured Motzkin paths.

Definition 2. A *Motzkin path* of length n is a sequence of vertices $p = (v_0, v_1, \ldots, v_n)$, with $v_i \in \mathbb{N}^2$ (where $\mathbb{N} = \{0, 1, \ldots\}$), with steps $v_{i+1} - v_i \in \{(1,1), (1,-1), (1,0)\}$ and $v_0 = (0,0)$ and $v_n = (n, 0)$.

A *bicoloured Motzkin path* is a Motzkin path in which each east, $(1,0)$, step is labelled by one of two colours.

From now on *all Motzkin paths considered will be bi-coloured.*

Let N (S) denote a north, $(1,1)$, step (resp., south, $(1,-1)$, step), and E and F the two different coloured east steps. Further, let N_h, S_h, E_h, F_h denote the weight of a N, S, E, F step, respectively, that starts at height h. The weight of a Motzkin path is the product of the steps weights.

If the weights are given by

$$N_h = [h+2]_q^x, S_h = [h]_q, E_h = [h+1]_q^x \text{ and } F_h = [h+1]_q, \tag{1}$$

it follows immediately from [5, Theorem 1] that $[q^i x^j t^n] F(q, x, 1, t)$ is the number of Motzkin paths of length n with weight $q^i x^j$. Let $\mathcal{M}_n$ denote the set of weighted Motzkin paths of length n with step weights given by (1).

To establish Theorem 1 we will use a bijection between sets of permutations and weighted Motzkin paths of length n.

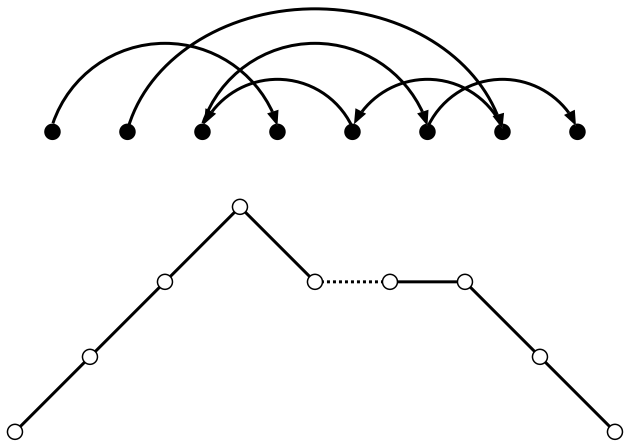

Figure 1: On top, the arc representation of $C(\pi) = (275368)(14)$. The node weights are, in order, $x, xq, q, 1, q, 1, q, 1$. The shape pairs are, in order, $(\emptyset, \ \rightarrow), (\emptyset, \ \rightarrow), (\emptyset, \ \rightleftharpoons)$, $(\rightarrow, \ \emptyset), (\leftarrow, \ \leftarrow), (\rightarrow, \ \rightarrow), (\rightleftharpoons, \ \emptyset), (\rightarrow, \ \emptyset)$. At bottom, the image Motzkin path.

3.1 The arc representation

We use a graphical representation of permutations to aid in the description of the mapping. For permutation $\pi \in \mathcal{S}_n$ with standard cycle form

$$C(\pi) = (c_1^1 c_2^1 \cdots c_{i_1}^1)(c_1^2 c_2^2 \cdots c_{i_2}^2) \cdots (c_1^k c_2^k \cdots c_{i_k}^k),$$

make n nodes in a line, representing the elements 1 to n. For $s = 1, \ldots k$ and $t = 1, \ldots, i_s - 1$ draw an arc from node c_t^s to node c_{t+1}^s. If cycle s is of size 1 draw a loop from c_1^s to itself. See Figure 1 for an example.

Associate each node with a left and a right shape. The left (right) shape is connections to nodes on the left (right) side with the node. There are 4 possibilities for the left shape of a node k:

- No arcs with is right endpoint at k — the shape is $\emptyset$.
- No arc leaving k to the left, but an arc entering k from the right — the shape is $\rightarrow$.
- An arc leaving k to the left, but no arc entering k from the right — the shape is $\leftarrow$.
- Arcs both leaving to the left and entering from the left — the shape is $\rightleftharpoons$.

Similary, the possible right shapes are $\{\emptyset, \rightarrow, \leftarrow, \rightleftharpoons\}$. See Figure 1 for an example.

3.1.1 Weights in the arc representation

We now give each element, or node in the arc representation, a weight $x^a q^{b+w}$, in such a way that the product of a permutation's elements weights is $x^k q^m$, where k is the number of cycles in π and m is the number of cyclic occurrences of 2–13.

Imagine the arcs being drawn in sequence, in the order $c_1^1 \to c_2^1, c_2^1 \to c_3^1, \ldots, c_{i_k-1}^k \to c_{i_k}^k$. If $i_s = 1$ for a cycle s, we draw the loop $c_1^s \to c_1^s$. In this drawing procedure we say that a node is visited once we have drawn an arc starting or ending at that node, whichever occurs first.

Give node k weight $x^a q^{b+w}$, where

- a is 1 if the left right shape pair of the node is $(\emptyset, \to)$ and a is 0 otherwise (element k is the first in the cycle),
- b is the number of times an arc *belonging to a different cycle* that is drawn *after* the node is visited passes over the node *from left to right* (element k plays the role of "2" in b occurrences of 2–13 *between* cycles),
- w is the number of times an arc *belonging to the same cycle* that is drawn *after* the node is visited passes over the node *from left to right* (element k plays the role of "2" in w occurrences of 2–13 *within* cycles).

See Figure 1 for an example.

3.2 One surjection and two bijections

First we define a mapping Ψ from $\mathcal{S}_n$ to Motzkin paths of length n, and prove that it is a surjection.

Definition 3. If $\pi \in \mathcal{S}_n$ have left shapes $\{l_1, \ldots, l_n\}$ and right shapes $\{r_1, \ldots, r_n\}$, let step k in $\Psi(\pi)$ be s_k, where s_k is given by the following table (where "–" denotes pairs of shapes that do not appear). Further, give step k the same weight as node k.

$l_k \backslash r_k$	$\emptyset$	$\to$	$\leftarrow$	$\rightleftharpoons$
$\emptyset$	E	N	F	N
$\to$	S	E	–	–
$\leftarrow$	–	–	F	–
$\rightleftharpoons$	S	–	–	–

Lemma 2. *The the mapping Ψ is a surjection from the set of permutations to the set of Motzkin paths with no F steps at level 0.*

Proof. To show that the image is a Motzkin path, the conditions for a Motzkin path must be verified. Namely, that the number of N steps up to step m are always less than or equal to the number of S steps, and that equality holds after the last step.

As the shape pairs $(\leftarrow, \leftarrow)$, $(\to, \to)$, $(\emptyset, \emptyset)$ and $(\emptyset, \leftarrow)$ map to E and F steps, it may be assumed that these shapes do not occur.

Now, in a valid arc diagram, for every node k with shape pair $(\to, \emptyset)$ or $(\rightleftharpoons, \emptyset)$ there is a corresponding node with shape pair either $(\emptyset, \to)$ or $(\emptyset, \rightleftharpoons)$ (the starting point of the arc that ends at k). Therefore the number of $(\emptyset, \to)$ and $(\emptyset, \rightleftharpoons)$ shape pairs up to and including node k must be greater than or equal to the number of $(\rightarrow, \emptyset)$ and $(\rightleftharpoons, \emptyset)$ shape pairs. Further, these counts must agree for $k = n$. This is exactly what is needed.

To show that Ψ is a surjection, consider any Motzkin path p with no F steps at level 0. We will build an arc diagram a that maps to p.

For each F step in p we can associate a unique pair of N and S steps. (The rightmost N step to the left of the F step ending at the F step's level is paired with the leftmost S step to the right of the F starting at the F step's level.) Let n, f, s denote the positions of the N, F, S steps, respectively. In the arc diagram a, draw an arc from node n to node s and one from node s to node f.

For the remaining N and S steps, fix one pairing of these, and draw arcs from the nodes corresponding to the N steps, to the associated nodes corresponding to the S steps.

For every E step in p, draw an loop at the corresponding node.

Clearly, a represents a permutation, and $\Psi(a) = p$ as desired. □

The next step is to show that Ψ defines a bijection ψ from the set of equivalence classes of permutations to weighted Motzkin paths, where two permutations are equivalent if they map to the same *unweighted* Motzkin paths.

Definition 4. For an equivalence class $\mathbf{E}_p = \{\pi \in \mathcal{S} | \Psi(\pi) = p\}$ of permutations let $\psi(\mathbf{E}_p) = p$, and let the weight of step k be the sum of weights of node k over permutations in $\mathbf{E_p}$.

Theorem 3. *The mapping $\psi(\mathbf{E})$ is a bijection from the set of equivalence classes of permutations (with the above definition of equivalent) to the set of weighted Motzkin paths, with weights*

$$N_h = E_h = [h+1]_q^x \text{ and } S_h = F_h = [h]_q, \tag{2}$$

such that the sum of weights of permutations in $\mathbf{E}$ is the weight of $\psi(\mathbf{E})$.

Proof. Assume Ψ maps node k to an E step at height h. Then the pair of left and right shapes is either $(\emptyset, \emptyset)$ or $(\rightarrow, \rightarrow)$. Further, to the left of node k there must be $h+m$ shape pairs in the set $\{(\emptyset, \rightarrow), (\emptyset, \rightleftharpoons)\}$ (corresponding to N steps) and m shape pars in the set $\{(\leftarrow, \emptyset), (\rightleftharpoons, \emptyset)\}$ (corresponding to S steps). The nodes corresponding to E and F steps to the left of node k may be disregarded in this discussion.

Now, if the shape pair of node k is $(\emptyset, \emptyset)$, there are h arcs going over node k, and since all these arcs starts at a node to the left of node k, they are drawn after node k is visited. Therefore node k gets the weight xq^h.

If the shape pair is $(\rightarrow, \rightarrow)$ there is h possibilities for the incoming arc (call this arc A). These give weights $1, \ldots, q^{h-1}$ depending on the number of arcs with start node between the start node of arc A and node k.

Thus, in the image of the equivalence class, a step E at height h is given a total weight of $[h+1]_q^x$ as required.

The cases of F, N and S steps are similar, and the details omitted.

That ψ is a bijection follows at once from the fact that Ψ is onto the set of Motzkin paths, and ψ is defined from the set of equivalence classes that maps to the same Motzkin paths. □

The step weights produced by ψ are of the right form, but not exactly what we want. Let $\mathcal{M}_n^*$ denote the set of weighted Motzkin paths with weights given by (2). A bijection ϕ from $\mathcal{M}_{n+1}^*$ to $\mathcal{M}_n$ will finally give paths with the correct weights.

Definition 5. For p in $\mathcal{M}_{n+1}^*$ and for $k \in [n]$, if steps k and $k+1$ is x and y, let step k in $\phi(p)$ be given by

$x\backslash y$	E	F	N	S
E	E	S	E	S
F	N	F	N	F
N	N	F	N	F
S	E	S	E	S

and have the same weight as step $k+1$ in p.

Theorem 4. *The mapping ϕ is a bijection from $\mathcal{M}_{n+1}^*$ to $\mathcal{M}_n$.*

Proof (sketch). That ϕ give the correct step weights follows effortlessly from the definition. To show that ϕ is a bijection, the inverse mapping is easily derived. See [6] for details. $\square$

4 Closed forms

Let $C(t)$ be the Catalan function, $C(t) = \frac{1-\sqrt{1-4t}}{2t}$. It is well known that $C(\gamma t)^2$ is the generating function for (bi-coloured) Motzkin paths in which each step have weight γ.

Define $\bar{\alpha}_i^j = \{\alpha_i, \dots, \alpha_j\}$ and $\bar{\beta}_i^j = \{\beta_i, \dots, \beta_j\}$. Let $g_k(\bar{\alpha}_1^k, \bar{\beta}_1^k, \gamma; t)$ be the generating function for Motzkin paths in which weights are given by

$$\begin{aligned} &N_h S_{h+1} = \beta_h \text{ for } h \leq k, \\ &E_h + F_h = \alpha_h \text{ for } h \leq k, \\ &N_h = S_{h+1} = E_h = F_h = \gamma \text{ for } h > k. \end{aligned}$$

Decomposing on the first return to the x-axis (where E and F steps counts as returns), we find that

$$g_1(\alpha_1, \beta_1, \gamma; t) = 1 + (\alpha_1 t + \beta_1 t^2 C(\gamma t)^2) g_1(\alpha_1, \beta_1, \gamma; t),$$

and in general

$$g_k(\bar{\alpha}_1^k, \bar{\beta}_1^k, \gamma; t) = 1 + (\alpha_1 t + \beta_1 t^2 g_{k-1}(\bar{\alpha}_2^k, \bar{\beta}_2^k, \gamma; t)) g_k(\bar{\alpha}_1^k, \bar{\beta}_1^k, \gamma; t).$$

Now, to find the number of permutations with k occurrences of 2–13, we can count weighted Motzkin paths with all weights truncated at q^k. This is formalised in the following theorem.

Theorem 5. *For $i \leq k$,*

$$\Phi_{i,j}(n) = [q^i x^j t^n] g_k(\{[1]_q + [1]_q^x, \dots, [k]_q + [k]_q^x\}, \{[1]_q [2]_q^x, \dots, [k]_q [k+1]_q^x\}, [k]_q; t).$$

Let $G_k(x,t) = \sum_{m,n} x^m t^n \Phi_{k,m}(n)$. By iteratively calculating g_k and differentiating with respect to q, we find that

$$G_0(x,t) = \frac{C(t)}{1-txC(t)},$$

$$G_1(x,t) = \frac{C(t)(-1+C(t)+x(2-C(t)))(1-C(t))^2}{(2-C(t))(1-xtC(t))^2}$$

and

$$\begin{aligned}G_2(x,t) =& \frac{2(1-C(t))^3}{(2-C(t))^3(1-xtC(t))^3}\Big(\\ &-x^3(2-C(t))^3(1-C(t))\\ &+x^2(2-C(t))^2(3-8C(t)+4C(t)^2)\\ &-x(3-20C(t)+37C(t)^2-24C(t)^3+5C(t)^4)\\ &-(1-C(t))(1-5C(t)+2C(t))\Big).\end{aligned}$$

4.1 Extracting coefficients

The generating functions can be written in the form $P(C,x)(2-C)^{-a}(1-xtC)^{-b}$ for integers a and b, and where $P(C,x)$ is a polynomial in C and x. This allows for a routine, but lengthy, method for extracting coefficients. Consider as an example

$$G_1(x,t) = \frac{A+xB}{(1-xtC(t))^2}$$

where $A = \frac{C(t)(1-C(t))^3}{C(t)-2}$ and $B = C(t)(1-C(t))^2$. Expanding $G_1(x,t)$ in powers of x, we find that

$$[x^k]G_1(x,t) = (k+1)At^kC(t)^k + kBt^{k-1}C(t)^{k-1}.$$

Noting that the above may be written as a sum of powers of $\sqrt{1-4t}$, coefficients may be extracted by applying the binomial theorem. See [6] for details.

Theorem 6.

$$\begin{aligned}\Phi_{0,m}(n) =& \binom{2n-m}{n-m}\frac{m+1}{n+1},\\ \Phi_{1,m}(n) =& \binom{2n-m}{n-m-1}\frac{(m+1)n^2+3(m-1)n+m^2+9m+2}{(n+2)(n+3)},\\ \Phi_{2,m}(n) =& \binom{2n-m}{n-m-1}\frac{1}{2(n+2)(n+3)(n+4)}\\ &\Big((m+1)n^4-(6-5m+m^2)n^3-(29-32m+3m^2)n^2\\ &-(66-72m+12m^2+2m^3)n-20+54m-28m^2-6m^3\Big).\end{aligned}$$

5 Other patterns

There are 12 patterns of type (1,2) or (2,1). As shown by Claesson [3], these fall into three equivalence classes with respect to distribution of non-cyclic occurrences in permutations, namely

$$\{1\text{–}23,\ 12\text{–}3,\ 3\text{–}21,\ 32\text{–}1\},\ \{1\text{–}32,\ 21\text{–}3,\ 23\text{–}1,\ 3\text{–}12\} \text{ and } \{13\text{–}2,\ 2\text{–}13,\ 2\text{–}31,\ 31\text{–}2\}.$$

It is only natural to ask about equivalence classes with respect to cyclic occurrences of patterns of type (1,2) and (2,1). Unfortunately, there are a lot of them. We conjecture that the 144 possible distributions fall into 106 equivalence classes. In any case 106 is lower bound. The conjectured classes of size 2 or more are given in Table 1.

Conjecture 7. *The distributional relations in Table 1 holds, and the table includes all such relations.*

Let $\Pi(p_b, p_w)$ denote the distribution of occurrences of (p_b, p_w) in permutations, where (p_b, p_c) means that we count occurrences of p_b between cycles and of p_w within cycles. Let $\Pi(p_b, p_w; C)$ denote the bivariate distribution of cycles and occurrences of (p_b, p_w).

Write $(p_b, p_w) \sim (q_b, q_w)$ if $\Pi(p_b, p_w) = \Pi(q_b, q_w)$, and $(p_b, p_w) \overset{c}{\sim} (q_b, q_w)$ if $\Pi(p_b, p_w; C) = \Pi(q_b, q_w; C)$.

First note that there are 8 *diagonal* classes.

Theorem 8. *The following distributional equivalences holds.*

$$\begin{aligned}
&(13\text{–}2,\ 13\text{–}2) \sim (2\text{–}13,\ 2\text{–}13),\\
&(1\text{–}23,\ 1\text{–}23) \sim (12\text{–}3,\ 12\text{–}3) \text{ and}\\
&(1\text{–}32,\ 1\text{–}32) \sim (3\text{–}12,\ 3\text{–}12) \overset{c}{\sim} (23\text{–}1,\ 23\text{–}1).
\end{aligned}$$

Proof. The "$\sim$" cases follow from Theorem 9 and the non-cyclic equivalence classes.

It remains to show that $(3\text{–}12, 3\text{–}12) \overset{c}{\sim} (23\text{–}1, 23\text{–}1)$. Given a permutation π in cycle form

$$C(\pi) = (c_1^1 c_2^1 \cdots c_{i_1}^1)(c_1^2 c_2^2 \cdots c_{i_2}^2) \cdots (c_1^k c_2^k \cdots c_{i_k}^k),$$

let

$$\hat{C}(\pi) = (d_{i_k}^k \cdots d_1^k) \cdots (d_{i_1}^1 \cdots d_1^1),$$

where $d_i^j = n+1-c_i^j$. Write $\hat{C}(\pi)$ in standard cycle form. The result is a permutation, say $D(\pi)$, such that each occurrence, between or within cycles, of 3–12 in $C(\pi)$ corresponds exactly to an occurrence of 23–1 in $D(\pi)$. Furthermore, the cycle structure is obviously preserved. □

The seven patterns involved the above theorem share the property that they are equidistributed with the non-cyclic occurrences. Let $\Pi(p)$ denote the distribution of non-cyclic occurrences of the pattern p.

$(31\text{–}2,\ 31\text{–}2) \overset{c}{\sim} (31\text{–}2,\ 2\text{–}31)$
$(13\text{–}2,\ 31\text{–}2) \overset{c}{\sim} (13\text{–}2,\ 2\text{–}13)$
$(13\text{–}2,\ 13\text{–}2) \sim (2\text{–}13,\ 2\text{–}13) \overset{c}{\sim} (2\text{–}13,\ 31\text{–}2)$
$(2\text{–}31,\ 31\text{–}2) \overset{c}{\sim} (2\text{–}31,\ 2\text{–}31)$
$(31\text{–}2,\ 3\text{–}21) \overset{c}{\sim} (31\text{–}2,\ 32\text{–}1)$
$(2\text{–}31,\ 3\text{–}21) \overset{c}{\sim} (2\text{–}31,\ 32\text{–}1)$
$(31\text{–}2,\ 3\text{–}12) \overset{c}{\sim} (31\text{–}2,\ 23\text{–}1)$
$(13\text{–}2,\ 3\text{–}12) \overset{c}{\sim} (13\text{–}2,\ 21\text{–}3)$
$(2\text{–}13,\ 3\text{–}12) \overset{c}{\sim} (2\text{–}13,\ 21\text{–}3)$
$(2\text{–}31,\ 3\text{–}12) \overset{c}{\sim} (2\text{–}31,\ 23\text{–}1)$
$(1\text{–}23,\ 31\text{–}2) \overset{c}{\sim} (1\text{–}23,\ 2\text{–}13) \overset{c}{\sim} (3\text{–}21,\ 31\text{–}2) \overset{c}{\sim} (3\text{–}21,\ 2\text{–}31)$
$(3\text{–}21,\ 2\text{–}13) \overset{c}{\sim} (1\text{–}23,\ 2\text{–}31)$
$(12\text{–}3,\ 31\text{–}2) \overset{c}{\sim} (12\text{–}3,\ 2\text{–}13) \overset{c}{\sim} (12\text{–}3,\ 2\text{–}31)$
$(32\text{–}1,\ 31\text{–}2) \overset{c}{\sim} (32\text{–}1,\ 2\text{–}13) \overset{c}{\sim} (32\text{–}1,\ 2\text{–}31)$
$(3\text{–}21,\ 13\text{–}2) \overset{c}{\sim} (1\text{–}32,\ 13\text{–}2)$
$(1\text{–}32,\ 31\text{–}2) \overset{c}{\sim} (1\text{–}32,\ 2\text{–}13) \overset{c}{\sim} (1\text{–}32,\ 2\text{–}31)$
$(3\text{–}12,\ 31\text{–}2) \overset{c}{\sim} (3\text{–}12,\ 2\text{–}13) \overset{c}{\sim} (3\text{–}12,\ 2\text{–}31)$
$(21\text{–}3,\ 31\text{–}2) \overset{c}{\sim} (21\text{–}3,\ 2\text{–}31)$
$(23\text{–}1,\ 31\text{–}2) \overset{c}{\sim} (23\text{–}1,\ 2\text{–}13)$
$(1\text{–}23,\ 1\text{–}23) \sim (12\text{–}3,\ 12\text{–}3)$
$(3\text{–}21,\ 3\text{–}21) \overset{c}{\sim} (1\text{–}32,\ 32\text{–}1)$
$(1\text{–}32,\ 3\text{–}21) \overset{c}{\sim} (3\text{–}21,\ 32\text{–}1)$
$(1\text{–}23,\ 3\text{–}12) \overset{c}{\sim} (1\text{–}32,\ 21\text{–}3)$
$(3\text{–}21,\ 3\text{–}12) \overset{c}{\sim} (1\text{–}32,\ 23\text{–}1)$
$(3\text{–}21,\ 21\text{–}3) \overset{c}{\sim} (1\text{–}23,\ 23\text{–}1)$
$(1\text{–}32,\ 3\text{–}12) \overset{c}{\sim} (1\text{–}23,\ 21\text{–}3) \overset{c}{\sim} (3\text{–}21,\ 23\text{–}1)$
$(1\text{–}32,\ 1\text{–}32) \sim (3\text{–}12,\ 3\text{–}12) \overset{c}{\sim} (23\text{–}1,\ 23\text{–}1)$

Table 1: Equivalences among cyclic occurrences of patterns of type (1,2) and (2,1).

Theorem 9. *We have*

$$\begin{aligned}
\Pi(2\text{–}13,\ 2\text{–}13) &= \Pi(2\text{–}13),\\
\Pi(13\text{–}2,\ 13\text{–}2) &= \Pi(13\text{–}2),\\
\Pi(1\text{–}23,\ 1\text{–}23) &= \Pi(1\text{–}23),\\
\Pi(12\text{–}3,\ 12\text{–}3) &= \Pi(12\text{–}3),\\
\Pi(1\text{–}32,\ 1\text{–}32) &= \Pi(1\text{–}32),\\
\Pi(23\text{–}1,\ 23\text{–}1) &= \Pi(23\text{–}1) \quad \text{and}\\
\Pi(3\text{–}12,\ 3\text{–}12) &= \Pi(3\text{–}12).
\end{aligned}$$

Proof. We use a standard bijection between permutations written in standard cycle form and permutations. Given a permutation π in cycle form,

$$C(\pi) = (c_1^1 c_2^1 \cdots c_{i_1}^1)(c_1^2 c_2^2 \cdots c_{i_2}^2) \cdots (c_1^k c_2^k \cdots c_{i_k}^k),$$

map it to the permutation

$$\tilde{\pi} = c_1^1 c_2^1 \cdots c_{i_1}^1 c_1^2 c_2^2 \cdots c_{i_2}^2 \cdots c_1^k c_2^k \cdots c_{i_k}^k.$$

Note that the bijection preserves the occurrences of each of the 7 patterns. This is true as we have the restrictions

$$c_m^j > c_1^{j+1}, m = 1, \ldots, i_j, j = 1, \ldots k-1.$$

$\square$

5.1 Increasing cycle order

Using the standard cycle form and listing cycles in *decreasing* order with respect to the cycles minimal elements is equivalent to listing cycles in *increasing* order.

Theorem 10. *Let $\Pi^d(p_b, p_w; C)$ and $\Pi^i(p_b, p_w; C)$ denote the distribution of cyclic occurrence of some pattern pair (p_b, p_w) when the cycles are listed in decreasing respectively increasing order. Let abc be a permutation of* $[3]$*. Then*

$$\begin{aligned}
\Pi^d(a\text{–}bc, p_w; C) &= \Pi^i(bc\text{–}a, p_w; C), \ \text{and}\\
\Pi^d(ab\text{–}c, p_w; C) &= \Pi^i(c\text{-}ab, p_w; C).
\end{aligned}$$

Proof. If a is in a cycle to the left of a cycle containing bc when the cycles are listed in decreasing order, it is to the right when the cycles are listed in increasing order. $\square$

Writing cycles with the maximal element first also gives trivial equivalences.

Theorem 11. *Let $\hat{\Pi}^x$ denote the distributions when cycles are started with their maximal elements, and cycles are ordered in increasing ($x = i$) or decreasing ($x = d$) order. For a pattern p let $r(p)$ denote the reverse pattern. Then*

$$\begin{aligned}
\hat{\Pi}^d(p_b, p_w) &= \Pi^i(r(p_b), r(p_w)), \ \text{and}\\
\hat{\Pi}^i(p_b, p_w) &= \Pi^d(r(p_b), r(p_w)).
\end{aligned}$$

	0	1	2	3
0		(123) (132)	$(23)(1)$ $(3)(12)$ $(2)(13)$	$(3)(2)(1)$
1	$\underline{(123)}$ $\underline{(132)}$	$\underline{(23)}(1), (23)\underline{(1)}$ $\underline{(2)}(13), (2)\underline{(13)}$ $\underline{(3)}(12), (3)\underline{(12)}$	$\underline{(1)}(2)(3)$ $(1)\underline{(2)}(3)$ $(1)(2)\underline{(3)}$	
2	$\underline{(1)(23)}$ $\underline{(2)(13)}$ $\underline{(3)(12)}$	$\underline{(3)(2)}(1)$ $(3)\underline{(2)(1)}$ $\underline{(3)}(2)\underline{(1)}$		
3	$\underline{(3)(2)(1)}$			

Table 2: The set of marked permutations of length 3. The marked cycles are underlined.

6 What about y?

Expanding $F(1, x, y, t)$ we are quickly led to conjecture that $F(1, x, y, t)$ is the generating function for a product of Stirling numbers and binomial coefficients. Using the same bijection as in the proof of Theorem 1, we can prove this.

Theorem 12.

$$[x^i y^j t^n]F(1, x, y, t) = \binom{i+j}{j} |s(n, i+j)|.$$

In other words, $[x^i y^j z^n]F(1, x, y, t)$ is the number of permutations of $[n]$ with $i + j$ cycles of which i are marked. We will call these *marked permutations*, and denote the set of marked permutations of length n with $\underline{\mathcal{S}}_n$. In Table 2 the elements of $\underline{\mathcal{S}}_3$ are listed. As the proof is much the same as that of Theorem 1, we only sketch it here.

Proof (sketch). We again use the arc representation. Give node k weight x if it is the first element in an unmarked cycle, and weight y if it is the first in a marked cycle.

Reasoning as in the proof of Theorem 1 shows that Ψ defines a bijection from equivalence classes of permutations with the above weighting to weighted Motzkin paths with weights

$$N_h = E_h = h + x + y, S_h = F_h = h.$$

The result follows after application of [5, Theorem 1]. $\square$

6.1 What about q and y?

In light of the above, $F(q,x,y,t)$ should count the number of permutations with respect to length, cycles, marked cycles and occurrences of 2–13. Unfortunately, life is not that easy. For instance, $[t^3]F(q,1,1,t) = 14+8q+q^2$, but in the set of 24 marked permutations of length 3 there are only two single occurrences of 2–13.

Perhaps marked permutations are not the natural object for studying $F(q,x,y,t)$. As the number of marked permutations of length n is $(n+1)!$, we should look for a nice (weight preserving) bijection between $\underline{\mathcal{S}}_n$ and $\mathcal{S}_{n+1}$. So far, we have not found such a bijection.

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