Rainbow H-factors

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Abstract

An *H-factor* of a graph G is a spanning subgraph of G whose connected components are isomorphic to H. Given a properly edge-colored graph G, a *rainbow* H-subgraph of G is an H-subgraph of G whose edges have distinct colors. A *rainbow H-factor* is an H-factor whose components are rainbow H-subgraphs. The following result is proved. If H is any fixed graph with h vertices then every properly edge-colored graph with hn vertices and minimum degree $(1-1/\chi(H))hn + o(n)$ has a rainbow H-factor.

1 Introduction

All the graphs considered here are finite, undirected and simple. For a graph G we let $v(G)$ and $e(G)$ denote the cardinality of the vertex set and edge set of G, respectively. Given two graphs G and H where $v(H)$ divides $v(G)$, we say that G has an *H-factor* if G contains $v(G)/v(H)$ vertex-disjoint subgraphs isomorphic to H. Thus, a K_2-factor is simply a perfect matching. The study of H-factors is a major topic of research in extremal graph theory. A seminal result of Hajnal and Szemerédi [6] gave a sufficient condition for the existence of a K_k-factor. They proved that a graph with nk vertices and minimum degree at least $nk(1-1/k)$ has a K_k-factor, and this is best possible. Later, Alon and Yuster proved [3], using the Regularity Lemma [12], a general result guaranteeing the existence of H-factors. They showed that for every fixed graph H with chromatic number $\chi(H)$, any graph with $v(H)n$ vertices and minimum degree at least $v(H)n(1-1/\chi(H)) + o(n)$ has an H-factor, and this is asymptotically tight in terms of the chromatic number. Later, it was proved in [10] that the $o(n)$ term can be replaced with a constant $K = K(H)$.

An edge coloring of a graph is called *proper* if two edges sharing an endpoint receive distinct colors. Vizing's theorem asserts that there exists a proper edge coloring of a graph G which uses at most $\Delta(G)+1$ colors. A *rainbow* subgraph of an edge-colored graph is a subgraph all of whose edges receive distinct colors. Many graph theoretic parameters have corresponding rainbow variants. Erdős and Rado [5] were among the first to consider problems of this type. Jamison, Jiang and Ling [7], and Chen, Schelp and Wei [4] considered Ramsey type variants where an arbitrary number of colors can be used. Alon et. al. [1] studied the function $f(H)$ which is the minimum integer n such that any proper edge coloring of K_n has a rainbow copy of H. Keevash et. al. [8] considered the rainbow Turán number $ex^*(n, H)$ which is the largest integer m such that there exists a properly edge-colored graph with n vertices and m edges and which has no rainbow copy of H.

A *rainbow H-factor* of a properly edge-colored graph is an H-factor whose elements are rainbow copies of H. Our main result provides sufficient conditions for the existence of a rainbow H-factor. It turns out that the same asymptotic conditions that guarantee an H-factor also guarantee a rainbow H-factor.

Theorem 1.1 *Let H be a graph. There exists $K = K(H)$ such that every proper edge coloring of a graph with n vertices, where $v(H)$ divides n, and with minimum degree at least $(1 - 1/\chi(H))n + K$ has a rainbow H-factor.*

The result might seem a bit surprising as a rainbow version of the theorem of Hajnal and Szemerédi ceases to hold for small values of n. An example is provided in the final section. The proof of Theorem 1.1 is a consequence of a lemma that shows that if H is a complete r-partite graph then any proper edge coloring of some fixed (though much larger) complete r-partite graph has a rainbow H-factor. This lemma and the proof of Theorem 1.1 appear in Section 2. In Section 3 we consider the problem of finding an *almost rainbow H-factor*. Given $\epsilon > 0$, an (ϵ, H)-factor of a graph G is a set of vertex-disjoint copies of H that cover at least $(1 - \epsilon)v(G)$ vertices. Komlós [9] showed that the chromatic number in the main result of [2] can be replaced with another parameter, called the *critical chromatic number* (which, in many cases, is strictly smaller than the chromatic number) if one settles for an (ϵ, H)-factor. We prove a simple rainbow version of a strengthened version of his result due to Shokoufandeh and Zhao [11] where $\epsilon v(G)$ can be replaced by a constant depending only on H. The final section contains some concluding remarks.

2 Rainbow H-factors

Let $T_r(k)$ denote the complete r-partite graph with k vertices in each vertex class. Let H be a fixed graph with $v(H) = h$ and $\chi(H) = r$. Clearly, $T_r(h)$ has an H-factor. As $T_r(h)$ and H have the same chromatic number, this essentially means that it suffices to prove Theorem 1.1 for complete partite graphs. Now, if we can also show that for k sufficiently large, any proper edge coloring of $T_r(k)$ has a rainbow $T_r(h)$-factor, we can use the results on (usual) H-factors in order to deduce a similar result for the rainbow analogue. We therefore need to prove the following lemma.

Lemma 2.1 *Let h and r be positive integers. There exists $k = k(h, r)$ such that any proper edge coloring of $T_r(k)$ has a rainbow $T_r(h)$-factor.*

Proof: We shall prove a slightly stronger statement. For $0 \le p \le h$, Let $T_r(h,p)$ be the complete r-partite graph with h vertices in each vertex class, except the last vertex class which has only p vertices. Notice that $T_r(h,0) = T_{r-1}(h,h)$. We prove that there exists $k = k(h,r,p)$ such that any proper edge coloring of $T_r(kh,kp)$ has a rainbow $T_r(h,p)$-factor.

We fix h, and prove the result by induction on r, and for each r, by induction on $p \ge 1$. The base case $r = 2$ and $p = 1$ is trivial since every star subgraph of a proper edge-colored graph is rainbow. Given $r \ge 2$, assuming the result holds for r and $p-1$, we prove it for r and p (if $p = 1$ then $p-1 = 0$ so we use the induction on $T_{r-1}(h,h)$). Let $k = k(h,r,p-1)$ and let t be sufficiently large (t will be chosen later). Consider a proper edge-coloring of $T = T_r(kth, ktp)$. We let $c(x,y)$ denote the color of the edge (x,y). Denote the first $r-1$ vertex classes of T by $V_1, \ldots, V_{r-1}$ and denote the last vertex class by U_r. Let $V_r \subset U_r$ be an arbitrary subset of size $k(p-1)t$ and let $W = U_r \setminus V_r$ be the remaining set with $|W| = kt$. For $i = 1, \ldots, r$, we *randomly* partition V_i into t subsets $V_i(1), \ldots, V_i(t)$, each of the same size. Each of the r random partitions is performed independently, and each partition is equally likely.

Let $S(j)$ be the subgraph of T induced by $V_1(j) \cup V_2(j) \cup \cdots \cup V_r(j)$, for $j = 1, \ldots, t$. Notice that $S(j)$ is a properly edge-colored $T_r(kh, k(p-1))$ and hence, by the induction hypothesis $S(j)$ has a rainbow $T_r(h, p-1)$-factor.

Let $B = (X \cup W, F)$ be a bipartite graph where $X = \{S(j) \; : \; j = 1, \ldots, t\}$ and there exists an edge $(S(j), v) \in F$ if for all $i = 1, \ldots, r-1$ and for all $x \in V_i(j)$, the color $c(x,v)$ does not appear at all in $S(j)$. If we can show that, with positive probability, B has a 1-to-k assignment in which each $S(j) \in X$ is assigned to precisely k elements of W and each $v \in W$ is assigned to a unique $S(j)$ then we can show that T has a rainbow $T_r(h,p)$-factor. Indeed, consider $S(j)$ and the unique set X_j of k elements of W that are matched to $S(j)$. Since $S(j)$ has a rainbow $T_r(h, p-1)$-factor, we can arbitrarily assign a unique element of X_j to each element of this factor and obtain a $T_r(h,p)$ which is also rainbow because all the edges of this $T_r(h,p)$ incident with the assigned vertex from X_j have colors that did not appear at all in other edges of this $T_r(h,p)$.

In order to prove that B has the required 1-to-k assignment we shall use the 1-to-k extension of Hall's Theorem. Namely, we will show that, with positive probability, $|N(Y)| \ge k|Y|$ for each $Y \subset X$. (Hall's Theorem is simply the case $k = 1$. The 1-to-k generalization reduces to the 1-to-1 version by taking k vertex-disjoint copies of X.) To guarantee this condition, it suffices to prove that, with positive probability, each vertex of X has degree greater than $(k - 1/2)t$ in B and each vertex of W has degree greater than $t/2$ in B.

The second part is easy to guarantee, and randomness plays no role. Consider $S(j) \in X$. Let $C(j)$ be the set of all colors appearing in $S(j)$. As $S(j)$ is a $T_r(kh, k(p-1))$ we have that $|C(j)| < k^2h^2\binom{r}{2}$. For each vertex x of $S(j)$, let $W_x \subset W$ be the set of vertices $v \in W$ such that $c(v,x) \in C(j)$. Clearly, $|W_x| < |C(j)|$ since no color appears more than once in edges incident with x. Let $W(j)$ be the union of all W_x taken over all vertices of $S(j)$. Hence, $|W(j)| < (khr)(k^2h^2\binom{r}{2})$. Each $v \in W \setminus W(j)$ is a neighbor of $S(j)$ in B. Thus, if we take $t > k^3h^3r^3$, we have that each $S(j)$ has more than $(k-1/2)t$ neighbors in B.

For the first part, fix some $v \in W$ and let $d_B(v)$ denote the degree of v in B. As $d_B(v)$ is a random variable, and since $|W| = kt$, it suffices to prove that

$$\Pr[d_B(v) \leq t/2] < 1/kt$$

which implies that

$$\Pr[\exists v \ : \ d_B(v) \leq t/2] < 1.$$

To simplify notation we let s_i be the size of the i'th vertex class of each $S(j)$. Thus $s_i = kh$ for $i = 1, \ldots, r-1$ and $s_r = k(p-1)$. Recall that the i'th vertex class of $S(j)$ is formed by taking the j'th block of a random partition of V_i into t blocks of equal size s_i. Alternatively, one can view the i'th vertex class of $S(j)$ as the elements $s_i(j-1)+1, \ldots, s_i j$ of a random permutation of V_i for $i = 1, \ldots, r$. Let, therefore, π_i be a random permutation of V_i. Thus, for $i = 1, \ldots, r$, $\pi_i(\ell) \in V_i$ for $\ell = 1, \ldots, s_i t$. We define the ℓ'th vertex of vertex class i of $S(j)$ to be $\pi_i(s_i(j-1)+\ell)$ for $i = 1, \ldots, r$ and $\ell = 1, \ldots, s_i$.

We define the following events. For three vertex classes $V_\alpha, V_\beta, V_\gamma$ with $1 \leq \alpha < \beta \leq r$, and $1 \leq \gamma \leq r-1$, for a block j where $1 \leq j \leq t$ and for three positive indices $\ell_1 \leq s_\alpha$, $\ell_2 \leq s_\beta$, $\ell_3 \leq s_\gamma$, let x be the ℓ_1'th vertex of vertex class α in $S(j)$, let y be the ℓ_2'th vertex of vertex class β in $S(j)$, and let z be the ℓ_3'th vertex of vertex class γ in $S(j)$. Let $A(\alpha, \beta, \gamma, j, \ell_1, \ell_2, \ell_3)$ be the event that $c(x, y) = c(v, z)$. (Notice that if $\gamma = \alpha$ and $\ell_1 = \ell_3$ or $\gamma = \beta$ and $\ell_2 = \ell_3$ then the corresponding event never holds as our coloring is proper.) We now prove the following claim.

Claim 2.2 *If $d_B(v) \leq t/2$ then there exist $\alpha, \beta, \gamma, \ell_1, \ell_2, \ell_3$ and there exists $J \subset \{1, \ldots, t\}$ with $|J| > t/(khr)^3$ such that for each $j \in J$ the event $A(\alpha, \beta, \gamma, j, \ell_1, \ell_2, \ell_3)$ holds.*

Proof: If $d_B(v) \leq t/2$ then there exists $J' \subset \{1, \ldots, t\}$ with $|J'| \geq t/2$ such that for each $j \in J$ some event $A(.,.,.,j,.,.,.)$ holds. There are $\binom{r}{2}$ choices for α and β. There are $r-1$ choices for γ. There are at most kh choices for each of ℓ_1, ℓ_2 and ℓ_3. Hence for some $J \subset J'$ with

$$|J| \geq \frac{|J'|}{k^3 h^3 \binom{r}{2}(r-1)} > \frac{t}{(khr)^3}$$

the 6-tuple $(\alpha, \beta, \gamma, \ell_1, \ell_2, \ell_3)$ is the same for all $j \in J$. ■

For each $\alpha, \beta, \gamma, \ell_1, \ell_2, \ell_3$ where $\ell_1 \leq s_\alpha$, $\ell_2 \leq s_\beta$ and $\ell_3 \leq s_\gamma$ and for each subset $J \subset \{1, \ldots, t\}$ of cardinality $|J| = \lceil \frac{t}{(khr)^3} \rceil$, let

$$A(J, \alpha, \beta, \gamma, \ell_1, \ell_2, \ell_3) = \cap_{j \in J} A(\alpha, \beta, \gamma, j, \ell_1, \ell_2, \ell_3).$$

Claim 2.3 *If the probability of each of the events $A(J, \alpha, \beta, \gamma, \ell_1, \ell_2, \ell_3)$ is smaller than $k^{-4}h^{-3}r^{-3}t^{-1}2^{-t}$ then $\Pr[d_B(v) \leq t/2] < 1/kt$.*

Proof: The proof of the claim follows immediately from Claim 2.2 and from the fact that there are less than 2^t possible choices for J and less than $k^3h^3r^3$ possible choices for $\alpha, \beta, \gamma, \ell_1, \ell_2, \ell_3$ where $\ell_1 \leq s_\alpha$, $\ell_2 \leq s_\beta$ and $\ell_3 \leq s_\gamma$. ■

By Claim 2.3, in order to complete the proof of Lemma 2.1 it suffices to prove the following claim.

Claim 2.4 *Let $1 \leq \alpha < \beta \leq r$, let $1 \leq \gamma \leq r-1$, let $\ell_1 \leq s_\alpha$, $\ell_2 \leq s_\beta$, $\ell_3 \leq s_\gamma$ and let $J \subset \{1\ldots,t\}$ with $|J| = \lceil \frac{t}{(khr)^3} \rceil$. Then,*

$$\Pr[A(J,\alpha,\beta,\gamma,\ell_1,\ell_2,\ell_3)] < \frac{1}{k^4h^3r^3t2^t}.$$

Proof: For convenience, let $A = A(J,\alpha,\beta,\gamma,\ell_1,\ell_2,\ell_3)$ and let $\Delta = \lceil \frac{t}{(khr)^3} \rceil$. We may assume, without loss of generality, that $J = \{1,\ldots,\Delta\}$. For $j \in J$, let x_j be the ℓ_1'th vertex of vertex class α in $S(j)$, let y_j be the ℓ_2'th vertex of vertex class β in $S(j)$, and let z_j be the ℓ_3'th vertex of vertex class γ in $S(j)$. Suppose that we are *given* the identity of the $3j-2$ vertices $x_1,y_1,z_1,\ldots,x_{j-1},y_{j-1},z_{j-1}$ and z_j (we assume here that all vertices are distinct since if $z_{j'}$ equals either $x_{j'}$ or $y_{j'}$ then $pr[A]=0$ in this case, as our coloring is proper). If we can show that given this information, the probability that $c(x_j,y_j) = c(v,z_j)$ is less than q where q only depends on t,r,h then, by the product formula of conditional probabilities we have $Pr[A] < q^\Delta$. Thus, assume that we are *given* the identity of the $3j-2$ vertices $x_1,y_1,z_1,\ldots,x_{j-1},y_{j-1},z_{j-1}$ and z_j. In particular, we know the color $c = c(v,z_j)$. What is the probability that $c(x_j,y_j) = c$? If $\alpha \neq \gamma$, let $X = V_\alpha \setminus \{x_1,\ldots,x_{j-1}\}$ and if $\alpha = \gamma$ let $X = V_\alpha \setminus \{x_1,\ldots,x_{j-1},z_1,\ldots,z_j\}$. If $\beta \neq \gamma$, let $Y = V_\beta \setminus \{y_1,\ldots,y_{j-1}\}$ and if $\beta = \gamma$ let $Y = V_\beta \setminus \{y_1,\ldots,y_{j-1},z_1,\ldots,z_j\}$. Each vertex of X has an equal chance of being x_j and each vertex of Y has an equal chance of being y_j. Thus, each edge of $X \times Y$ has an equal chance of being the edge (x_j,y_j). Clearly $|X| \geq tkh - 2\Delta$ and $|Y| \geq tk(p-1) - 2\Delta$ (if $\beta \neq r$ then, in fact, $|Y| \geq tkh - 2\Delta$ and if $p=1$ then, trivially, $\beta \neq r$). Since our coloring is proper, the color c appears at most tkh times in $V_\alpha \times V_\beta$. Hence,

$$\Pr[c(x_j,y_j)=c] \leq \frac{tkh}{|X||Y|} \leq \frac{tkh}{(tk-2\Delta)^2} < \frac{tkh}{(tk-tk/2)^2} = \frac{4h}{tk}.$$

It follows that for t sufficiently large as a function of k,h,r we have

$$\Pr[A] < \left(\frac{4h}{tk}\right)^\Delta \leq \left(\frac{4h}{tk}\right)^{t/(khr)^3} < \frac{1}{k^4h^3r^3t2^t}.$$

■

This completes the induction step and the proof of Lemma 2.1. ■

Proof of Theorem 1.1: Let H be a graph with $\chi(H) = r$ and $v(H) = h$. By Lemma 2.1 there exists $k = k(h,r)$ such that every proper edge coloring of $T_r(k)$ has a rainbow $K(h,r)$-factor, and hence also a rainbow H-factor. By [10], the exists $K_0 = K_0(k,r)$ such that every graph with n vertices, where kr divides n, and with minimum degree at least $n(1-1/r)+K_0$ has a $T_r(k)$-factor. Let $K = K_0 + kr$ and let G be a properly edge-colored graph with n vertices where h divides n, and with minimum degree at least

$n(1-1/r)+K$. Let $n^* \le n$ be the largest integer which is a multiple of kr. Any graph obtained from G by deleting $n-n^*$ vertices has n^* vertices and minimum degree at least $n(1-1/r)+K_0 \ge n^*(1-1/r)+K_0$ and hence has a $T_r(k)$-factor. In particular, we can greedily delete from G a set of $(n-n^*)/h$ vertex-disjoint rainbow copies of H, and the remaining graph has a $T_r(k)$-factor. As each $T_r(k)$ in this factor is properly colored, each has a rainbow H-factor. Thus, G has a rainbow H-factor. ■

3 Rainbow "almost" H-factors

For an r-chromatic graph H on h vertices, let $u = u(H)$ be the smallest possible color-class size in any r-coloring of H. The *critical chromatic number* of H is $\chi_{cr}(H) = (r-1)h/(h-u)$. It is easy to see that $\chi(H) - 1 < \chi_{cr}(H) \le \chi(H)$ and $\chi_{cr}(H) = \chi(H)$ if and only if every r-coloring of H has equal color-class sizes. In [9], Komlós proved the following result.

Theorem 3.1 *[Komlós [9]] Let $\epsilon > 0$ and let H be a graph. There exists $n_0 = n_0(H, \epsilon)$ such that every graph with $n > n_0$ vertices and minimum degree at least $(1 - 1/\chi_{cr}(H))n$ has a set of vertex-disjoint copies of H that cover all but at most ϵn vertices.*

Solving a conjecture of Komlós, Shokoufandeh and Zhao proved the following strengthened version in [11].

Theorem 3.2 *[Shokoufandeh and Zhao [11]] For every graph H there exists $K_0 = K_0(H)$ such that every graph with n vertices and minimum degree at least $(1 - 1/\chi_{cr}(H))n$ has a set of vertex-disjoint copies of H that cover all but at most K_0 vertices.*

Let T be a complete r-partite graph with vertex class sizes $u_1 \le u_2 \le \ldots \le u_r$. For a positive integer k, let kT denote the complete r-partite graph with vertex class sizes $ku_1 \le ku_2 \le \ldots \le ku_r$. Clearly,

$$\chi_{cr}(kT) = \chi_{cr}(T) = \frac{(r-1)\sum_{i=1}^r u_i}{\sum_{i=2}^r u_i}.$$

The following is a slight generalization of Lemma 2.1 whose proof is almost identical.

Lemma 3.3 *Let T be a complete r-partite graph with vertex class sizes $u_1 \le u_2 \le \ldots \le u_r$. There exists $k = k(T)$ such that any proper edge coloring of kT has a rainbow T-factor.* ■

Let H be a graph, and consider a coloring of H in which the smallest vertex class has size $u(H)$. Adding edges between any two vertices in distinct vertex classes we obtain a complete r-partite graph T with $\chi_{cr}(T) = \chi_{cr}(H)$. Thus, exactly as in the proof of Theorem 1.1 we can use Lemma 3.3 and Theorem 3.2 to obtain the following.

Proposition 3.4 *For every graph H there exists $K = K(H)$ such that every properly edge-colored graph with n vertices and minimum degree at least $(1 - 1/\chi_{cr}(H))n$ has a set of vertex-disjoint rainbow copies of H that cover all but at most K vertices.* ■

4 Concluding remarks

- The proof of Lemma 2.1 yields a *huge* constant $k = k(h, r)$. It is an interesting combinatorial problem to *determine* the minimum integer $k = k(h, r)$ which guarantees that a properly edge-colored $T_r(k)$ has a rainbow $T_r(h)$-factor. Even for the case $h = 1$ (the case of complete graphs) we do not know the precise answer. Trivially $k(1, 2) = 1$ and $k(1, 3) = 1$. However, $k(1, 4) > 1$ since a proper edge coloring of K_4 need no be rainbow. The following example shows that $k(1, 4) > 2$. Assume the four vertex classes of $T_4(2)$ are $V_i = \{x_i, y_i\}$ for $i = 1, 2, 3, 4$. Color with 1 the edges $x_1x_2, y_1y_2, x_3x_4, y_3y_4$. Color with 2 the edges $x_1y_2, x_2y_1, x_3y_4, x_4y_3$. Color with 3 the edges $x_2x_3, y_2y_3, x_1y_4, y_1x_4$. Color with 4 the edges $x_2y_3, y_2x_3, x_1x_4, y_1y_4$. Color the remaining 8 edges in any way as to obtain a proper edge coloring. It is easily verified that any K_4 of this $T_4(2)$ is not rainbow. In particular, this example shows that the rainbow version of the theorem of Hajnal and Szemerédi ceases to hold for small values of n.

- An edge coloring of a graph is called m-good if each color appears at most m times at each vertex. A slightly more complicated version of Lemma 2.1 also holds in this setting. Namely, Let h, r and m be positive integers. There exists $k = k(h, r, m)$ such that any m-good edge coloring of $T_r(k)$ has a rainbow $T_r(h)$-factor. We omit the details. Given this extended version of Lemma 2.1 it is straightforward to show that Theorem 1.1 also holds for m-good colored graphs.

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