Lyndon words and transition matrices between elementary, homogeneous and monomial symmetric functions

Andrius Kulikauskas*
Minciu Sodas Laboratory
Vilnius, Lithuania
La Jolla, CA 92093-0112. USA
ms@ms.lt

Jeffrey Remmel†
Department of Mathematics
University of California, San Diego
La Jolla, CA 92093-0112. USA
jremmel@ucsd.edu

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Abstract

Let h_λ, e_λ, and m_λ denote the homogeneous symmetric function, the elementary symmetric function and the monomial symmetric function associated with the partition λ respectively. We give combinatorial interpretations for the coefficients that arise in expanding m_λ in terms of homogeneous symmetric functions and the elementary symmetric functions. Such coefficients are interpreted in terms of certain classes of bi-brick permutations. The theory of Lyndon words is shown to play an important role in our interpretations.

1 Introduction

Let Λ_n denote the space of homogeneous symmetric functions of degree n in infinitely many variables $x_1, x_2, \ldots$. There are six standard bases of Λ_n: $\{m_\lambda\}_{\lambda\vdash n}$ (the monomial symmetric functions), $\{h_\lambda\}_{\lambda\vdash n}$ (the complete homogeneous symmetric functions), $\{e_\lambda\}_{\lambda\vdash n}$ (the elementary symmetric functions), $\{p_\lambda\}_{\lambda\vdash n}$ (the power symmetric functions), $\{s_\lambda\}_{\lambda\vdash n}$ (the Schur functions) and $\{f_\lambda\}_{\lambda\vdash n}$ (the forgotten symmetric functions) where $\lambda \vdash n$ denotes that λ is a partition of n. We let $\ell(\lambda)$ denote the length of λ, i.e. $\ell(\lambda)$ equals the number of parts of λ. The entries of the transition matrices between these bases of symmetric functions all have combinatorial significance. For example, Doubilet [2] showed that all such entries could be interpreted via the lattice of set partitions π_n and

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its Möbius function. More recently, Beck, Remmel, and Whitehead [1] gave a complete list of combinatorial interpretations of such entries.

The main purpose of this paper is to provide proofs for two of the combinatorial interpretations described in [1] that have not previously been published, namely, the entries of the transition matrices which allow one to express the monomial symmetric function m_μ in terms of the homogeneous symmetric functions h_λ and the elementary symmetric functions e_λ.

More formally, given two bases of Λ_n, $\{a_\lambda\}_{\lambda\vdash n}$ and $\{b_\lambda\}_{\lambda\vdash n}$, we fix some standard ordering of the set of partitions of n, such as the lexicographic order, and then we think of the bases as row vectors, $\langle a_\lambda\rangle_{\lambda\vdash n}$ and $\langle b_\lambda\rangle_{\lambda\vdash n}$. We define the transition matrix $M(a,b)$ by the equation

$$\langle b_\lambda\rangle_{\lambda\vdash n} = \langle a_\lambda\rangle_{\lambda\vdash n} M(a,b). \tag{1}$$

Thus $M(a,b)$ is the matrix that transforms the basis $\langle a_\lambda\rangle_{\lambda\vdash n}$ into the basis $\langle b_\lambda\rangle_{\lambda\vdash n}$ and the (λ,μ)-th entry of $M(a,b)$ is defined by the equation

$$b_\mu = \sum_{\lambda\vdash n} a_\lambda M(a,b)_{\lambda,\mu}. \tag{2}$$

We note that our convention for the transition matrix $M(a,b)$ differs from that of Macdonald [6] since Macdonald interprets $\langle a_\lambda\rangle_{\lambda\vdash n}$ as a column vector.

The goal of this paper is to give combinatorial interpretations for $M(h,m)_{\lambda,\mu}$ and $M(e,m)_{\lambda,\mu}$. To describe our interpretations of $M(h,m)_{\lambda,\mu}$ and $M(e,m)_{\lambda,\mu}$, we must first introduce the concept of a primitive bi-brick permutation. Given partitions $\lambda = (\lambda_1,\ldots,\lambda_\ell)$ and $\mu = (\mu_1,\ldots,\mu_k)$ of n, define a (λ,μ)-*bi-brick permutation* as follows. We shall consider cycles C which are nothing more than circles which are partitioned in s equal arcs or cells for some $s \geq 1$. The length, $|C|$, of any such cycle C is defined to be the number of cells of C. Let $C_1, C_2, \ldots, C_t$ be a multiset of cycles whose lengths sum to n. Assume we have a set of bricks of sizes $\lambda_1,\ldots,\lambda_\ell$ called λ-bricks and a set of bricks of size $\mu_1,\ldots,\mu_k$ called μ-bricks. On each cycle, place an outer tier of λ-bricks and an inner tier of μ-bricks whose lengths sum to the length of the cycle. The resulting set of bi-brick cycles will be called a (λ,μ)-bi-brick permutation. If the bricks are placed in such a way that no cycle has rotational symmetry, then the bi-brick permutation is called *primitive*. For example, suppose $\lambda = (2^5)$, $\mu = (1^2, 2^4)$, and $C_1 = 4$, $C_2 = 4$, and $C_3 = 2$. Figure 1(a) shows a (λ,μ)-bi-brick permutation which is not primitive since the first and second cycles have rotational symmetry. Figure 1(b) shows a (λ,μ)-bi-brick permutation which is primitive since no cycle has rotational symmetry.

An alternative way to understand the notion of a primitive bi-brick cycle C is to use the theory of Lyndon words. Given an ordered alphabet $X = \{x_1 < \ldots < x_r\}$, let X^* denote the set of all words over the alphabet X. We then can use the lexicographic order to give a total ordering to X^* by declaring that for two words $w = w_1\cdots w_n$ and $v = v_1\cdots v_n$, $v \leq_\ell w$ if and only if either (a) there is an $i \leq min\{m,n\}$ such that $v_i < w_i$ and $v_j = w_j$ for $j < i$ or (b) $m < n$ an $v_j = w_j$ for all $j \leq m$. We let ϵ denote the empty word which has length 0 by definition. If $w = w_1\cdots w_s$, then we say w has length s and write $|w| = s$. We let $X^+ = X^* - \{\epsilon\}$. If $w = w_1\cdots w_s$ and $v = v_1\cdots v_t$, then

$wv = w_1 \cdots w_s v_1 \cdots v_t$. For any word w with $|w| \geq 1$, we define w^r for $r \geq 1$ by induction as $w^1 = w$, and for $r > 1$, $w^r = w^{r-1}w$. We say that a nonempty word $w = w_1 \cdots w_s$ is *Lyndon* if either $s = 1$ or $s > 1$ and w is the lexicographically least element in its cyclic rearrangement class. For example, if $w = x_1x_2x_1x_3$, then the cyclic rearrangement class of w is

$$\{x_1x_2x_1x_3,\ x_2x_1x_3x_1,\ x_1x_3x_1x_2,\ x_3x_1x_2x_1\}$$

so that w is Lyndon since it is the lexicographically least element in its set of cyclic rearrangement class. In fact, one can show that if w has length greater than or equal to 2 and w is not Lyndon, then $w = u^r$ for some word $u \in X^+$ and $r \geq 2$, see [5].

We shall associate to each bi-brick cycle a word in the ordered alphabet $A = \{B < L < N < M\}$ as follows. First, read the cycle clockwise and, for each cell of the cycle, record a B if both a λ-brick and a μ-brick start in the cell, record an L if a λ-brick starts at the cell and a μ-brick does not, record an M if a μ-brick starts at the cell and a λ-brick does not, and record an N if neither a λ-brick nor a μ-brick starts at the cell. We then define the word of the cycle, $W(C)$, to be the lexicographically least circular rearrangement of the cycle of letters associated with C. For example, consider the first cycle C_1 of Figure 1(a). Starting at the top and reading clockwise, the cycle of letters associated with C_1 is $NBNB = w$. There are just two cyclic rearrangements of ω, namely $NBNB$ and $BNBN$. Since $BNBN$ is the lexicographically least of these two words, $W(C_1) = BNBN$. Below each of the cycles in Figure 1(a) and 1(b), we have listed the word of the cycle. Now if a bi-brick cycle C has rotational symmetry, then $W(C)$ will be a power of a smaller word, i.e. $W(C) = u^r$ where $r > 1$ and $|u| \geq 1$. Thus a bi-brick cycle C is *primitive* if $W(C)$ is a Lyndon word. Note that each bi-brick cycle C in a (λ, μ)-bi-brick permutation has at least one λ-brick and at least one μ-brick. Thus $W(C)$ must contain a B if a λ-brick and μ-brick start at the same cell or, if $W(C)$ contains no B, then it must contain both an L and an M. Vice versa, it is easy to see that any word w over A such that either (a) w contains a B or (b) w contains no B but w does contain both an L and an M is of the form $W(C)$ for some bi-brick cycle C.

We say that a bi-brick permutation is *primitive* is it consists of entirely of primitive bi-brick cycles. Thus we can think of a primitive bi-brick permutation with k cycles as a multiset $\{w_1 \leq_\ell \cdots \leq_\ell w_k\}$ of Lyndon words over A where each w_i either contains a B or contains both an L and M if $w_i \in \{L, M, N\}^*$. Here $\leq_\ell$ denotes the lexicographic order on A^* relative to ordering of letters $B < L < N < M$. We say a primitive (λ, μ)-bi-brick permutation is *simple* if its bi-brick cycles are pairwise distinct. Thus we can think of a simple primitive bi-brick permutation with k cycles as a set $\{w_1 <_\ell \cdots <_\ell w_k\}$ of Lyndon words over A where each w_i either contains a B or contains both an L and M if $w_i \in \{L, M, N\}^*$. We let $PB(\lambda, \mu)$ be the set of primitive (λ, μ)-bi-brick permutations and $SPB(\lambda, \mu)$ be the set of simple primitive bi-brick permutations. Define the sign of a bi-brick permutation θ, $sgn(\theta)$, to be $(-1)^{n-c}$ where $\lambda, \mu \vdash n$ and c is the number of cycles of θ. This given, the main result of this paper is to prove the following.

Theorem 1 *Let λ and μ be partitions of n. Then*

$$\text{(i)} \quad M(h, m)_{\lambda,\mu} = (-1)^{\ell(\lambda)+\ell(\mu)} |PB(\lambda, \mu)| \tag{3}$$

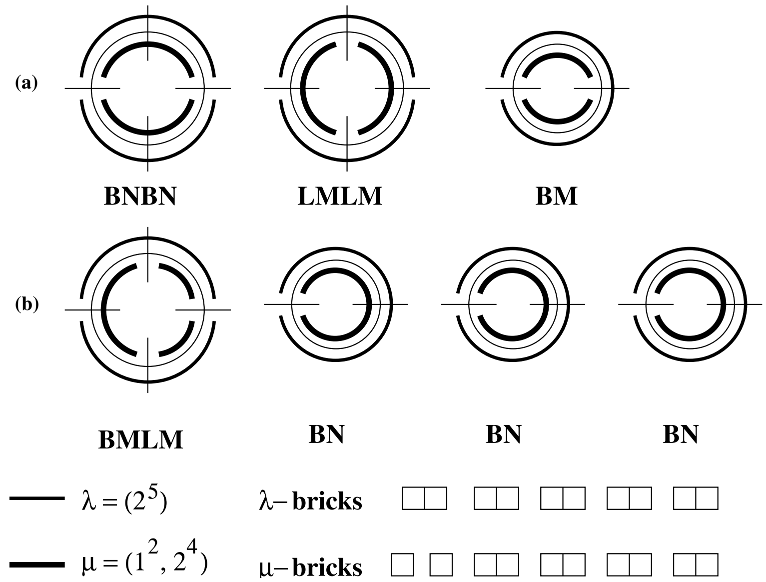

Figure 1: Bi-brick permutations.

and

$$(ii) \quad M(e,m)_{\lambda,\mu} = (-1)^{\ell(\lambda)+\ell(\mu)} \sum_{\theta \in SPB(\lambda,\mu)} sgn(\theta). \tag{4}$$

For example, Figures 2-6 picture all the (λ,μ)-brick permutations such that $\lambda = \mu = (1^2,2)$ where we have partitioned the (λ,μ)-bi-brick permutations according to type of the underlying cycles. In Figure 2, we picture the (λ,μ)-bi-brick permutations whose cycles induce the partition $(1,1,2)$. We see there are 2 (λ,μ)-bi-brick permutations according to which $(2,2)$-cycles we pick. Neither of the resulting (λ,μ)-bi-brick permutations is simple so that the (λ,μ)-bi-brick permutations in Figure 2 contribute 2 to $M(h,m)_{\lambda,\mu}$ and 0 to $M(e,m)_{\lambda,\mu}$. In Figure 3, we picture the unique (λ,μ)-bi-brick permutation whose cycles induce the partition $(2,2)$ and where one cycle is a $((1^2),(2))$ cycle and the other cycle is a $((2),(1^2))$ cycle. It is primitive and simple and has a positive sign so that the bi-brick permutation pictured in Figure 3 contributes 1 to $M(h,m)_{\lambda,\mu}$ and 1 to $M(e,m)_{\lambda,\mu}$. In Figure 4, we picture the other possibilities for a (λ,μ)-bi-brick permutation whose cycles induce the partition $(2,2)$. One can see that the $((1,1),(1,1))$-cycle is not primitive so there is no contribution to either $M(h,m)_{\lambda,\mu}$ or $M(e,m)_{\lambda,\mu}$ in this case. Figure 5 pictures all the possibilities of (λ,μ)-bi-brick permutations whose cycles induce the partition $(1,3)$. We see that there are 3 such (λ,μ)-bi-brick permutations according to which cycle of type $((1,2)(1,2))$ we pick. All three resulting bi-brick permutations are primitive and simple and have positive sign so that the (λ,μ)-bi-brick permutations in Figure 5 contribute 3 to both $M(h,m)_{\lambda,\mu}$ and $M(e,m)_{\lambda,\mu}$. Finally there are 4 (λ,μ)-bi-brick permutations consisting of single cycles which we picture in Figure 6. We see that these (λ,μ)-bi-brick

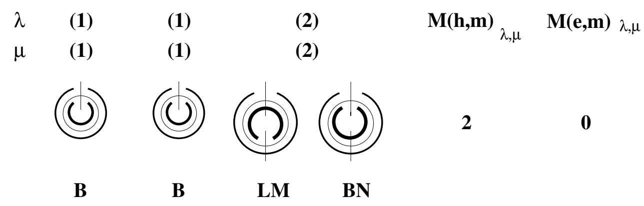

Figure 2: Bi-brick permutations of type $(1, 1, 2)$.

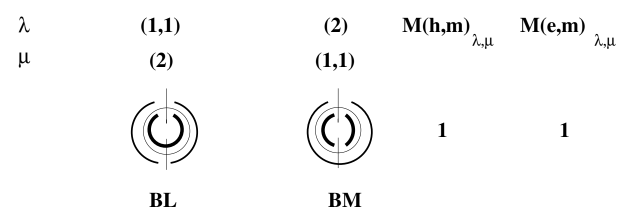

Figure 3: Bi-brick permutations of type $(2, 2)$.

permutations all have sign -1 and, hence, they contribute 4 to $M(h,m)_{\lambda,\mu}$ and -4 to $M(e,m)_{\lambda,\mu}$. Thus $M(h,m)_{(1^2,2),(1^2,2)} = 10$ and $M(e,m)_{(1^2,2),(1^2,2)} = 0$.

As one can see from figures 2-6, there is considerable cancellation in our expression for $M(e,m)_{\lambda,\mu}$. Thus in section 3, we shall define some sign reversing involutions which will simplify our expression for $M(e,m)_{\lambda,\mu}$. For example, we shall define a sign reversing involution which shows that to compute $M(e,m)_{\lambda,\mu}$, we can restrict ourselves to summing the signs of those simple primitive (λ, μ)-bi-brick permutations θ such that there are at most one cell c where both a λ-brick and a μ-brick start at c or, equivalently, the number of B's occuring in the corresponding set of Lyndon words for θ is ≤ 1.

We should note that equivalent interpretations for $M(h,m)_{\lambda,\mu}$ and $M(e,m)_{\lambda,\mu}$ first appeared in the first author's thesis [4] although the methods used to find such an interpretation were completely different than the ones presented in this paper.

We note that there are a number of restrictions on the values of $M(h,m)_{\lambda,\mu}$ and

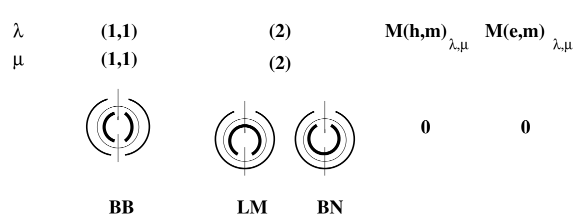

Figure 4: More bi-brick permutations of type $(2, 2)$.

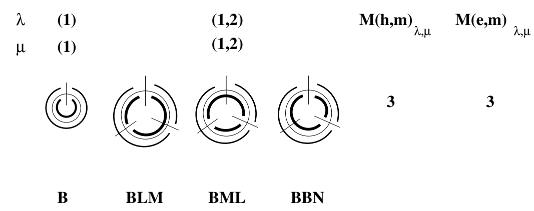

Figure 5: Bi-brick permutations of type $(1,3)$.

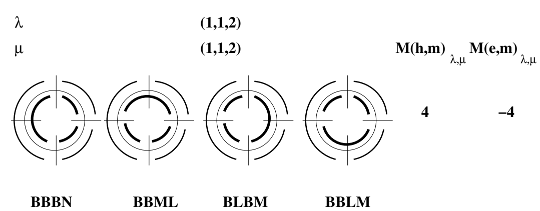

Figure 6: Bi-brick permutations of type (4).

$M(h,m)_{\lambda,\mu}$ that follows from the combinatorial interpretations of well known combinatorial interpretations of the entries of the matrices $M(m,h)$ and $M(m,e)$. That is, suppose $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ and $\mu = (\mu_1 \geq \cdots \geq \mu_\ell)$ are partitions of n. Then we define the dominance order $\leq_D$ on the partitions of n by defining $\lambda \geq_D \mu$ if and only if for all $j \leq max(\{k,\ell\})$, $\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i$. For $k \times \ell$ matrix M with entries from $\mathbb{N} = \{0,1,\ldots\}$, let $r(M) = (r_1(M),\ldots,r_k(M))$ where for each i, $r_i(M) = \sum_{j=1}^{\ell} M_{i,j}$ is the i-th row sum of M. Similarly, let $c(M) = (c_1(M),\ldots,c_\ell(M))$ where for each i, $c_i(M) = \sum_{j=1}^{k} M_{j,i}$ is the i-th column sum of M. Let $\mathbb{N}M_{\lambda,\mu}$ denote the number non-negative integer valued $k \times \ell$ matrices M such that $r(M) = \lambda$ and $c(M) = \mu$ and let $Z_2M_{\lambda,\mu}$ denote the number $\{0,1\}$-valued $k \times \ell$ matrices M such that $r(M) = \lambda$ and $c(M) = \mu$. Then

$$M(m,h)_{\lambda,\mu} = \mathbb{N}M_{\lambda,\mu} \text{ and} \tag{5}$$

$$M(m,e)_{\lambda,\mu} = Z_2M_{\lambda,\mu}, \tag{6}$$

see [6]. It then easily follows that

$$M(m,h)_{\lambda,\mu} = M(m,h)_{\mu,\lambda}, \tag{7}$$

$$M(m,e)_{\lambda,\mu} = M(m,e)_{\mu,\lambda}, \tag{8}$$

$$M(m,e)_{\lambda,\mu} \neq 0 \text{ implies } \mu \leq_D \lambda', \text{ and} \tag{9}$$

$$M(m,e)_{\lambda,\lambda'} = 1, \tag{10}$$

where λ' denotes the conjugate of λ, see [6]. Thus $M(m,h)^T = M(m,h)$ and $M(m,e)^T = M(m,e)$ where for any matrix M, M^T denotes the transpose of M. It follows that

$M(h,m)^T = M(h,m)$ and $M(e,m)^T = M(e,m)$ so that

$$\begin{aligned} M(h,m)_{\lambda,\mu} &= M(h,m)_{\mu,\lambda} && (11) \\ M(e,m)_{\lambda,\mu} &= M(e,m)_{\mu,\lambda}. && (12) \end{aligned}$$

Note that (11) and (12) also follow from our combinatorial interpretations of $M(h,m)_{\lambda,\mu}$ and $M(e,m)_{\lambda,\mu}$ given in Theorem 1. Finally, let $\prec$ be any total order on partitions which refines the dominance partial order and suppose that $\lambda^{(1)} \prec \cdots \prec \lambda^{(p(n))}$ is the $\prec$-increasing list of all partitions of n. Since for all partitions λ and μ of n, $\lambda \leq_D \mu$ if and only if $\mu' \leq_D \lambda'$, it follows from (9) and (10) that the $p(n) \times p(n)$ matrix $E = ||E_{i,j}||$ where $E_{i,j} = M(m,e)_{\lambda^{(i)},(\lambda^{(j)})'}$ is an upper triangular matrix with 1's on the diagonal. Thus $E^{-1} = ||E^{-1}_{i,j}||$ where $E_{i,j} = M(e,m)_{(\lambda^{(i)})',\lambda^{(j)}}$ is also an upper triangular matrix with 1's on the diagonal and hence

$$M(e,m)_{\lambda',\mu} = 0 \text{ if } \mu <_D \lambda \tag{13}$$

and

$$M(e,m)_{\lambda',\lambda} = 1. \tag{14}$$

We also should note that similar results hold for two other transition matrices. Namely, let $\omega : \bigoplus_{n\geq 0} \Lambda_n \rightarrow \bigoplus_{n\geq 0} \Lambda_n$ be the algebra isomorphism defined by declaring $\omega(h_n) = e_n$ for all n where $h_0 = e_0 = 1$ and $h_n = h_{(n)} = \sum_{1\leq i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n}$ and $e_n = e_{(n)} = \sum_{1\leq i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}$. In [6], it is shown that ω is an involution and for all partitions λ, $\omega(h_\lambda) = e_\lambda$, $\omega(m_\lambda) = f_\lambda$, $\omega(s_\lambda) = s_{\lambda'}$ and $\omega(p_\lambda) = (-1)^{n-\ell(\lambda)} p_\lambda$. It is easy to see that for any bases $\{a_\lambda\}_{\lambda\vdash n}$ and $\{b_\lambda\}_{\lambda\vdash n}$ of Λ_n, the transition matrix from $\{\omega(a_\lambda)\}_{\lambda\vdash n}$ to $\{\omega(b_\lambda)\}_{\lambda\vdash n}$ is given by

$$M(\omega(a), \omega(b)) = M(a,b). \tag{15}$$

Thus combining Theorem 1 and (15), we have

$$M(e,f)_{\lambda,\mu} = (-1)^{\ell(\lambda)+\ell(\mu)} |PB(\lambda,\mu)| \tag{16}$$

and

$$M(h,f)_{\lambda,\mu} = (-1)^{\ell(\lambda)+\ell(\mu)} \sum_{\theta \in SPB(\lambda,\mu)} sgn(\theta). \tag{17}$$

The outline of this paper is as follows. In section 2, we shall prove Theorem 1. In section 3, we shall define a series of involutions which will allow us to give a more refined interpretation of $M(e,m)_{\lambda,\mu}$. That is, we shall show that $M(e,m)_{\lambda,\mu} = (-1)^{\ell(\lambda)+\ell(\mu)} \sum_{\theta\in SPB^*(\lambda,\mu)} sgn(\theta)$ for certain subsets of $SPB(\lambda,\mu)$. For example, we will show that $SPB^*(\lambda,\mu)$ cannot contain any bi-brick permutations θ such that there are two distinct cells in θ where both a λ and μ brick start at those cells. These involutions will be defined in terms of our alternative interpretation of primitive bi-brick permutations as sequences of certain Lyndon words and we will heavily use the basic properties of Lyndon words

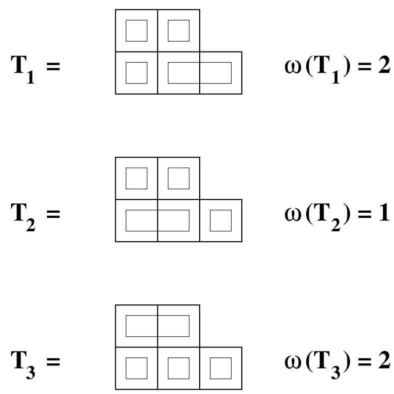

Figure 7: Brick tabloids.

to show that our involutions are well defined. Finally, in section 4, we shall use our interpretations to give the formulas for $M(h,m)_{\lambda,\mu}$ and $M(e,m)_{\lambda,\mu}$ in a number of special cases, In particular, we shall give explicit formulas for $M(h,m)_{\lambda,\mu}$ and $M(e,m)_{\lambda,\mu}$ when $\lambda = \mu = (k^n)$ for some k and n, when both λ and μ are two row shapes or when both λ and μ are hook shapes. Finally we shall also give formulas for $M(e,m)_{\lambda,\mu}$ when both λ and μ are two column shapes.

2 Proof of Theorem 1

Our proof of Theorem 1 depends on the combinatorial interpretation of the entries of $M(h,p)$ and $M(p,m)$ due to Eğecioğlu and Remmel [3]. If $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of n which has α_i parts of size i for $i = 1, \ldots, n$, then we write $\lambda = (1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n})$. This given, we set $z_\lambda = 1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n} \alpha_1! \cdots \alpha_n!$. It is well known that $\frac{n!}{z_\lambda} = |\mathcal{C}_\lambda|$ where $\mathcal{C}_\lambda$ is the set of permutations σ of the symmetric group $\mathcal{S}_n$ whose cycle lengths induce the partition λ. A λ-brick tabloid T of shape μ is a filling of the Ferrers diagram of μ, F_μ, with λ-bricks such that (i) each brick lies in a single row of F_μ and (ii) no two bricks overlap. For example, if $\lambda = (1^3, 2)$ and $\mu = (2,3)$, there are three λ-brick tabloids of shape μ and these are pictured in Figure 2.

We define the weight of a λ-brick tabloid T, $\omega(T)$, to be the product of the lengths of the bricks that are at the ends of the rows of T. Let $\mathcal{B}_{\lambda,\mu}$ denote the set of λ-brick tabloids of shape μ and let

$$\omega(B_{\lambda,\mu}) = \sum_{T \in \mathcal{B}_{\lambda,\mu}} \omega(T). \tag{18}$$

Then Eğecioğlu and Remmel [3] proved the following.

$$M(h,p)_{\lambda,\mu} = (-1)^{\ell(\lambda)+\ell(\mu)} \omega(B_{\lambda,\mu}), \tag{19}$$

$$M(e,p)_{\lambda,\mu} = (-1)^{n-\ell(\lambda)} \omega(B_{\lambda,\mu}), \tag{20}$$

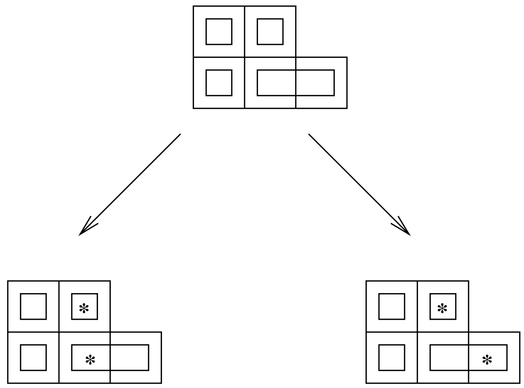

Figure 8: Elements of $B^*_{(1^3,2),(2,3)}$.

and

$$M(p,m)_{\lambda,\mu} = (-1)^{\ell(\lambda)+\ell(\mu)}\frac{\omega(B_{\mu,\lambda})}{z_\lambda}. \tag{21}$$

For the proof of part (i) of Theorem 1, note that

$$M(h,m) = M(h,p)M(p,m)$$

and hence

$$\begin{aligned} M(h,m)_{\lambda,\mu} &= \sum_{\nu\vdash n} M(h,p)_{\lambda,\nu}M(p,m)_{\nu,\mu} \\ &= \sum_{\nu\vdash n}(-1)^{\ell(\lambda)+\ell(\nu)}\omega(B_{\lambda,\nu})(-1)^{\ell(\nu)+\ell(\mu)}\frac{\omega(B_{\mu,\nu})}{z_\nu} \\ &= \frac{(-1)^{\ell(\lambda)+\ell(\mu)}}{n!}\sum_{\nu\vdash n}\frac{n!}{z_\nu}\omega(B_{\lambda,\nu})\omega(B_{\mu,\nu}). \end{aligned} \tag{22}$$

Next we want to give a combinatorial interpretation to $\sum_{\nu\vdash n}\frac{n!}{z_\nu}\omega(B_{\lambda,\nu})\omega(B_{\mu,\nu})$. We let $\mathcal{B}^*_{\lambda,\mu}$ denote the set of λ brick tabloids of shape μ where we mark one cell in the last brick of each row with an $*$. It is easy to see that $\omega(B_{\lambda,\mu}) = |\mathcal{B}^*_{\lambda,\mu}|$ since each $T \in \mathcal{B}_{\lambda,\mu}$ gives rise to $\omega(T)$ elements of $\mathcal{B}^*_{\lambda,\mu}$. For example, the λ-brick tabloid T_1 pictured in Figure 2 with $\omega(T_1) = 2$ gives rise to the two tabloids in $\mathcal{B}^*_{\lambda,\mu}$ pictured in Figure 3.

Thus,

$$\sum_{\nu\vdash n}\frac{n!}{z_\nu}\omega(B_{\lambda,\nu})\omega(B_{\mu,\nu}) = \sum_{\nu\vdash n}|\mathcal{C}_\nu \times \mathcal{B}^*_{\lambda,\nu} \times \mathcal{B}^*_{\mu,\nu}|. \tag{23}$$

Next we shall describe how we can associate to each triple $(\sigma, B_1, B_2) \in \mathcal{C}_\nu \times \mathcal{B}^*_{\lambda,\nu} \times \mathcal{B}^*_{\mu,\nu}$, a labeled sequence of primitive bi-brick cycles $\psi(\sigma, B_1, B_2)$. The construction of

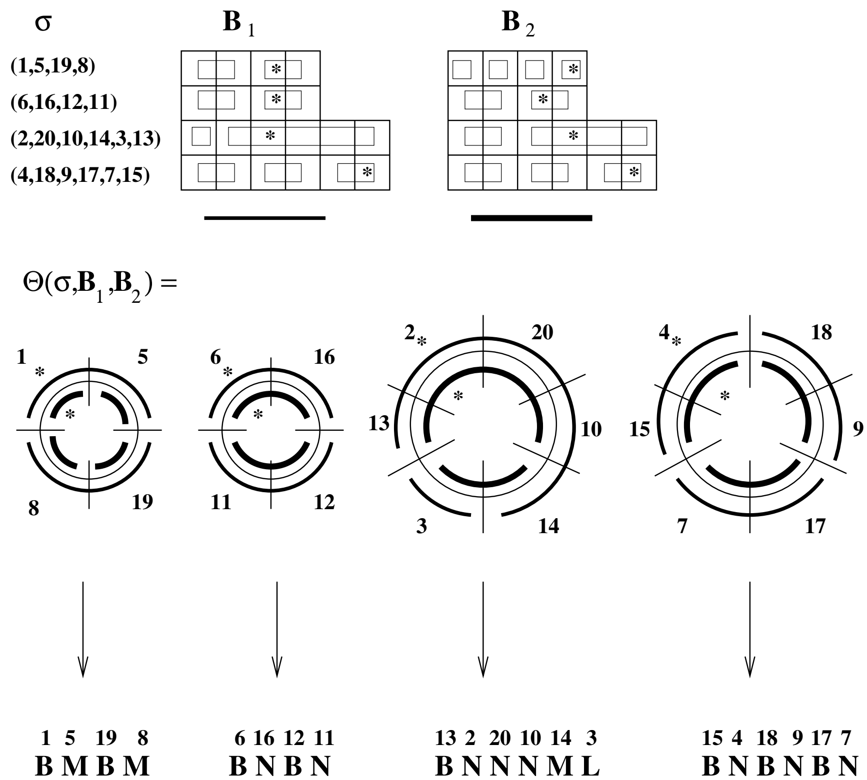

Figure 9: $\Theta(\sigma, B_1, B_2)$.

$\psi(\sigma, B_1, B_2)$ is best described by referring to an example. Let $\lambda = (1, 2^7, 5)$, $\mu = (1^4, 2^6, 4)$, and $\nu = (4^2, 6^2)$.

We start with a triple $(\sigma, B_1, B_2) \in \mathcal{C}_\nu \times \mathcal{B}^*_{\lambda,\nu} \times \mathcal{B}^*_{\mu,\nu}$ as pictured at the top of Figure 4. Each cycle c of σ is associated to a row of B_1 and B_2 of the same size as c. If there is more than one cycle of size i in σ, then we list the cycles of σ of size i in increasing order according to their smallest elements, say $c^i_1, c^i_2, \ldots, c^i_{k_i}$. Then $c^i_1, \ldots, c^i_{k_i}$ are associated with the rows of size i in B_1 and B_2 reading from top to bottom.

We then construct a bi-brick cycle out of each pair of corresponding rows of B_1 and B_2 by having the cells with $*$'s correspond to the same cell in the bi-brick cycle. Next we label the bi-brick cycles with the elements of the corresponding cycle in σ by having the smallest element of σ correspond to the cell with the $*$'s in the λ and μ bricks in the bi-brick cycle. This process yields a labeled bi-brick permutation $\Theta(\sigma, B_1, B_2)$ as pictured in Figure 4. Note that since the smallest label corresponds to the cells with the $*$'s, there is no loss in erasing the $*$'s. Clearly we can use $\Theta(\sigma, B_1, B_2)$ to reconstruct, σ, B_1 and B_2 since we can (1) reconstruct the $*$ by picking the cell with the smallest label, (2) for each cycle, construct a pair of corresponding rows of B_1 and B_2 by placing the brick with the $*$ at the end of the row, and (3) order the rows of B_1 and B_2 of the same size by ensuring that the smallest elements in the corresponding cycles of σ increase when we

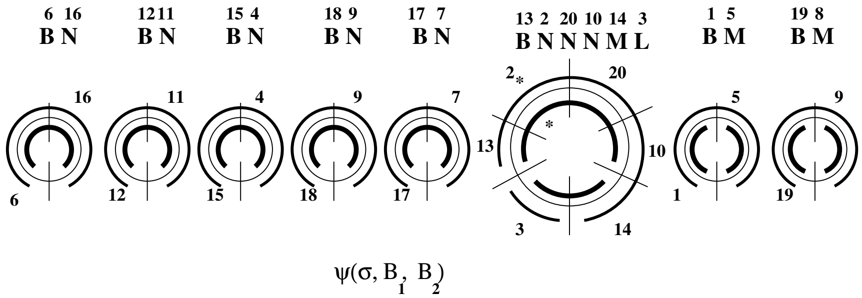

Figure 10: $\psi(\sigma, B_1, B_2)$.

read the cycles from top to bottom. Thus, Θ is a one-to-one correspondence between $\bigcup_\nu \mathcal{C}_\nu \times \mathcal{B}^*_{\lambda,\nu} \times \mathcal{B}^*_{\mu,\nu}$ and all labeled (λ, μ)-bi-brick permutations.

Next we can replace each cycle by its word $W(C)$ and label $W(C)$ in the obvious manner to get a set of labeled words $\bar{W}(C_1), \ldots, \bar{W}(C_k)$ as pictured at the bottom of Figure 4. Now if the underlying word $W(C_i) \in \{B, L, M, N\}^*$ of $\bar{W}(C_i)$ factors into $\omega_i^{r_i}$ where ω_i is a Lyndon word, then we can factor $\bar{W}(C_i)$ into labeled Lyndon words $\bar{\omega}_{i,1} \cdots \bar{\omega}_{i,r_i}$. The rotational symmetry of C_i automatically ensures that ω_i corresponds to a primitive bi-brick cycle. We let m_i denote the minimal label in C_i and we cyclically arrange the labeled factors so that m_i is a label in $\bar{\omega}_{i,1}$. Now in this process, there may be more than one cycle that factors into a power of a given Lyndon word u. For example, in Figure 4, the second and fourth cycles factor into labeled Lyndon words whose underlying Lyndon word is BN. For any such Lyndon word u, let $C_{i_1}, \ldots, C_{i_k}$ be the set of cycles such that $\bar{W}(C_{i_s}) = \bar{u}_{1,i_s} \cdots \bar{u}_{t_s,i_s}$ where $m_{i_1} > \cdots > m_{i_k}$. This gives us a block of labeled words $\bar{u}_{1,i_1} \cdots \bar{u}_{t_1,i_1} \bar{u}_{1,i_2} \cdots \bar{u}_{t_2,i_2} \cdots \bar{u}_{1,i_k} \cdots \bar{u}_{t_k,i_k} = \vec{u}$ of labeled Lyndon words which all correspond to the same underlying Lyndon word u. Note that we easily reconstruct each $\bar{u}_{1,i_j} \cdots \bar{u}_{t_j,i_j}$ from $\vec{u}$ as follows. First by construction $\bar{u}_{1,i_k}$ is the labeled word with the smallest label in $\vec{u}$ so that $\bar{u}_{1,i_k} \cdots \bar{u}_{t_k,i_k}$ consists of the word with the smallest label in $\vec{u}$ together with all words of $\vec{u}$ to its right. Once we remove $\bar{u}_{1,i_k} \cdots \bar{u}_{t_k,i_k}$ from $\vec{u}$ to get $\vec{u}'$, then $\bar{u}_{1,i_{k-1}}$ is the word with the smallest label in $\vec{u}'$ so that $\bar{u}_{1,i_{k-1}} \cdots \bar{u}_{t_{k-1},i_{k-1}}$ consists of the word of $\vec{u}'$ with the smallest label in $\vec{u}'$ together with all words to its right. Continuing on in this manner we can reconstruct $\bar{W}(C_{i_1}), \ldots, \bar{W}(C_{i_k})$. Thus we have shown that each labeled (λ, μ)-bi-brick permutation corresponds to a sequence of labeled Lyndon words where we order the blocks of labeled Lyndon words by the lexicographic order of their underlying Lyndon words as pictured in Figure 5. This sequence of labeled Lyndon words corresponds to the sequence of labeled primitive bi-brick cycles as pictured in Figure 4.

We call this sequence of labeled primitive bi-brick cycles $\psi(\sigma, B_1, B_2)$. The key point to observe is that the labels on the primitive cycles or, equivalently, on the sequence of Lyndon words is completely arbitrary since the reconstruction procedure described above will always produce a labeled (λ, μ)-bi-brick permutation. It follows that each primitive (λ, μ)-bi-brick permutation gives rise to $n!$ labeled primitive (λ, μ)-bi-brick permutations and hence to $n!$ elements of $\bigcup_{\nu} \mathcal{C}_\nu \times \mathcal{B}^*_{\lambda,\nu} \times \mathcal{B}^*_{\mu,\nu}$. It thus follows that

$$\sum_{\nu \vdash n} |\mathcal{C}_\nu \times \mathcal{B}^*_{\lambda,\nu} \times \mathcal{B}^*_{\mu,\nu}| = n!|PB(\lambda,\mu)|. \tag{24}$$

Combining (22), (23), and (24), we get that

$$M(h,m)_{\lambda,\mu} = (-1)^{\ell(\lambda)+\ell(\mu)}|PB(\lambda,\mu)|$$

as desired.

For part (ii) of Theorem 1, note that $M(e,m) = M(e,p)M(p,m)$ and hence

$$\begin{aligned} M(e,m)_{\lambda,\mu} &= \sum_{\nu \vdash n} M(e,p)_{\lambda,\nu} M(p,m)_{\nu,\mu} \\ &= \sum_{\nu \vdash n} (-1)^{n-\ell(\lambda)} \omega(B_{\lambda,\nu})(-1)^{\ell(\nu)+\ell(\mu)} \frac{\omega(B_{\mu,\nu})}{z_\nu} \\ &= \frac{(-1)^{\ell(\lambda)+\ell(\mu)}}{n!} \sum_{\nu \vdash n} (-1)^{n-\ell(\nu)} \frac{n!}{z_\nu} \omega(B_{\lambda,\nu})\omega(B_{\mu,\nu}) \\ &= \frac{(-1)^{\ell(\lambda)+\ell(\mu)}}{n!} \sum_{\nu \vdash n} (-1)^{n-\ell(\nu)} |\mathcal{C}_\nu \times \mathcal{B}^*_{\lambda,\nu} \times \mathcal{B}^*_{\mu,\nu}|. \end{aligned} \tag{25}$$

We can then proceed exactly as in the proof of part (i) of Theorem 1 and associate to each triple (σ, B_1, B_2) in $\bigcup_{\nu \vdash n} \mathcal{C}_\nu \times \mathcal{B}^*_{\lambda,\nu} \times \mathcal{B}^*_{\mu,\nu}$ a sequence of labeled primitive bi-brick cycles $\psi(\sigma, B_1, B_2)$ or, equivalently, a sequence of labeled Lyndon words $W(\psi(\sigma, B_1, B_2))$. The only difference in this case is that $\psi(\sigma, B_1, B_2)$ carries a sign which is $(-1)^{n-c}$ where c is the number of cycles of the labeled bi-brick permutation $\Theta(\sigma, B_1, B_2)$. We can define a simple sign reversing involution f on the set of all such labeled sequences of Lyndon words $W(\psi(\sigma, B_1, B_2))$ with $(\sigma, B_1, B_2) \in \bigcup_{\nu \vdash n} \mathcal{C}_\nu \times \mathcal{B}^*_{\lambda,\nu} \times \mathcal{B}^*_{\mu,\nu}$. That is, if the underlying bi-brick permutation of $\psi(\sigma, B_1, B_2)$ is simple, we let $f(W(\psi(\sigma, B_1, B_2))) = W(\psi(\sigma, B_1, B_2))$. Otherwise, let u be the lexicographically least word v such that there are at least two occurrences labeled Lyndon words in $W(\psi(\sigma, B_1, B_2))$ whose underlying Lyndon words is v. Let $\vec{u}$ be the block of all labeled Lyndon words in $W(\psi(\sigma, B_1, B_2))$ whose underlying Lyndon words is u. We then define $f(W(\psi(\sigma, B_1, B_2))$ to be the labeled sequence of Lyndon words which results from interchanging the two labeled words in $\vec{u}$ with the two smallest minimal labels. For example, suppose that

$$\vec{u} = \bar{u}_{1,i_1} \cdots \bar{u}_{t_1,i_1} \cdots \bar{u}_{1,i_{k-1}} \cdots \bar{u}_{t_{k-1},i_{k-1}} \bar{u}_{1,i_k} \cdots \bar{u}_{t_k,i_k}$$

is as described in our proof of part (i). Then $\bar{u}_{1,i_k}$ is the word with the smallest label. There are two possibilities for the word $\bar{u}$ whose minimal label is the next smallest. Namely either (a) $\bar{u} = \bar{u}_{1,i_{k-1}}$ if $\bar{u}$ occurs to the left of $\bar{u}_{1,i_k}$ or (b) $\bar{u} = \bar{u}_{j,i_k}$ with $j > 1$ if $\bar{u}$ occurs to the right of $\bar{u}_{1,i_k}$. In case (a), $\vec{u}$ is replaced by

$$\bar{u}_{1,i_1} \cdots \bar{u}_{t_1,i_1} \cdots \bar{u}_{1,i_{k-2}} \cdots \bar{u}_{t_{k-2},,i_{k-2}} \bar{u}_{1,i_k} \bar{u}_{2,i_{k-1}} \cdots \bar{u}_{t_{k-1},i_{k-1}} \bar{u}_{1,i_{k-1}} \bar{u}_{2,i_k} \cdots \bar{u}_{t_k,i_k}$$

in $f(W(\psi(\sigma,B_1,B_2)))$. Now suppose that (σ',B_1',B_2') is the triple such that $W(\psi(\sigma',B_1',B_2')) = f(W(\psi(\sigma,B_1,B_2)))$. Then it easy to see that the sequence

$$\bar{u}_{1,i_k}\bar{u}_{2,i_{k-1}}\cdots\bar{u}_{t_{k-1},i_{k-1}}\bar{u}_{1,i_{k-1}}\bar{u}_{2,i_k}\cdots\bar{u}_{t_k,i_k}$$

will be associated with a single cycle C in $\Theta(\sigma',B_1',B_2')$ whereas the sequence

$$\bar{u}_{1,i_{k-1}}\bar{u}_{2,i_{k-1}}\cdots\bar{u}_{t_{k-1},i_{k-1}}\bar{u}_{1,i_k}\bar{u}_{2,i_k}\cdots\bar{u}_{t_k,i_k}$$

produces two cycles in $\Theta(\sigma,B_1,B_2)$. In case (b), $\vec{u}$ is replaced by

$$\bar{u}_{1,i_1}\cdots\bar{u}_{t_1,i_1}\cdots\bar{u}_{1,i_{k-1}}\cdots\bar{u}_{t_{k-1},i_{k-1}}\bar{u}_{j,i_k}\bar{u}_{2,i_k}\cdots\bar{u}_{j-1,i_k},\bar{u}_{1,i_k}\bar{u}_{j+1,i_k}\cdots\bar{u}_{t_k,i_k}$$

in $f(W(\psi(\sigma,B_1,B_2)))$. Again if (σ',B_1',B_2') is the triple such that

$$W(\psi((\sigma',B_1',B_2'))) = f(W(\psi(\sigma,B_1,B_2)))$$

, then the sequence

$$\bar{u}_{j,i_k}\bar{u}_{2,i_k}\cdots\bar{u}_{j-1,i_k}\bar{u}_{1,i_k}\bar{u}_{j+1,i_k}\cdots\bar{u}_{t_k,i_k}$$

will be associated with two cycles in $\Theta(\sigma',B_1',B_2')$ whereas the sequence

$$\bar{u}_{1,i_k}\bar{u}_{2,i_k}\cdots\bar{u}_{j-1,i_k}\bar{u}_{j,i_k}\bar{u}_{j+1,i_k}\cdots\bar{u}_{t_k,i_k}$$

is associated to one cycle in $\Theta(\sigma,B_1,B_2)$. It follows that

$$sgn(\Theta((\sigma,B_1,B_2)) = -sgn(\Theta(\sigma',B_1',B_2')$$

in both cases (a) and (b). For example, if we start with $(\sigma,B_1,B_2))$ of Figure 5, then (σ',B_1',B_2'), $f(W(\psi(\sigma,B_1,B_2)))$, and $\Theta(\sigma',B_1',B_2')$ are pictured in Figure 6.

Our involution f shows that

$$\begin{aligned}&\frac{(-1)^{\ell(\lambda)+\ell(\mu)}}{n!}\sum_{\nu\vdash n}(-1)^{n-\ell(\nu)}|\mathcal{C}_\nu\times\mathcal{B}^*_{\lambda,\nu}\times\mathcal{B}^*_{\mu,\nu}| = \\ &\frac{(-1)^{\ell(\lambda)+\ell(\mu)}}{n!}\sum sgn(\Theta(\sigma,B_1,B_2))\end{aligned} \tag{26}$$

where the second sum runs over all (σ,B_1,B_2) such that $W(\psi(\sigma,B_1,B_2)$ has no repeated words or, equivalently, over all (σ,B_1,B_2) such that underlying bi-brick permutation of $\Theta(\sigma,B_1,B_2) = \psi(\sigma,B_1,B_2)$ is simple. Once again, the labels on such labeled simple (λ,μ)-bi-brick permutations are completely arbitrary so that each simple (λ,μ)-bi-brick permutation gives rise to $n!$ labeled simple (λ,μ)-bi-brick permutations. Moreover, the signs of all these $n!$ labeled simple bi-brick permutations are the same. Thus (25) and (26) imply that

$$M(e,m)_{\lambda,\mu} = (-1)^{\ell(\lambda)+\ell(\mu)}\sum_{\theta\in SPB(\lambda,\mu)} sgn(\theta). \tag{27}$$

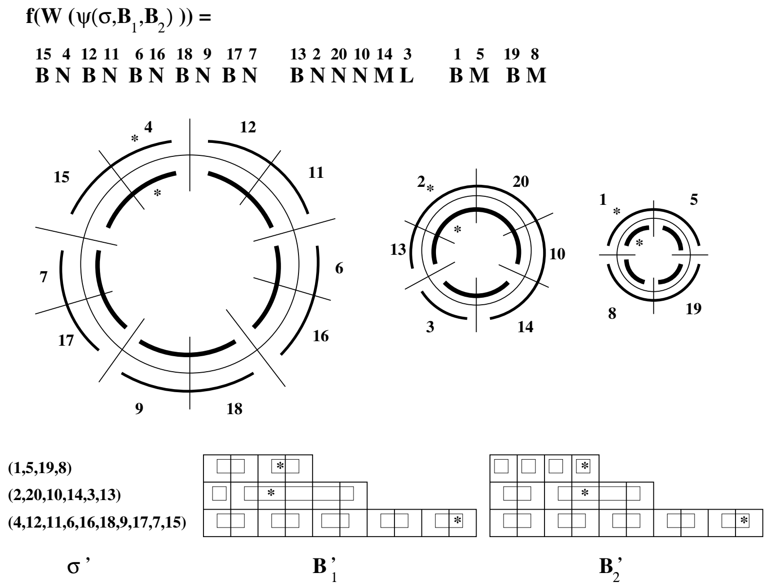

Figure 11: $f(W(\psi(\sigma, B_1, B_2)))$.

3 Further Involutions for the $M(e, m)_{\lambda,\mu}$

In Section 2, we proved that

$$M(e,m)_{\lambda,\mu} = (-1)^{\ell(\lambda)+\ell(\mu)} \sum_{\theta \in SPB(\lambda,\mu)} sgn(\theta). \tag{28}$$

As we can see from our example in Figures 2-6, there is a considerable amount of cancellation that can occur in (28). In this section, we shall show that we can define further involutions on the set $SPB(\lambda, \mu)$ to explain some of this cancellation.

Recall that we can code each primitive bi-brick cycle by a Lyndon word over the alphabet $A = \{B, L, M, N\}$. Note that each bi-brick cycle C has at least one λ-brick and at least one μ-brick. Thus either (a) $W(C)$ must contain a B if a λ-brick and μ-brick start at the same cell or (b) $W(C)$ contains no B but it does contain both an L and M. Vice versa, it is easy to see that any word w over A which either (a) contains a B or (b) contains no B but does contain both an L and a M is of the form $W(C)$ for some bi-brick cycle C. Thus any simple primitive bi-brick permutation θ can be identified with a sequence of Lyndon words $W(\theta) = (w_1, \ldots, w_p)$ where $w_1 <_\ell w_2 <_\ell \cdots <_\ell w_p$ and $<_\ell$ denotes the lexicographic order relative to our ordering of the alphabet $B < L < N < M$. Moreover it must be the case that for all $1 \leq i \leq p$, either (a) w_i contains a B or (b) w_i contains both an L and an M if $w_i \in \{L, N, M\}^*$. We let $\mathcal{SL}$ denote the set of all such sequences of Lyndon words over the alphabet A. Given a sequence $(w_1, \ldots, w_p) \in \mathcal{SL}$, we define the sign of $(w_1, \ldots, w_p)$, $sgn(w_1, \ldots, w_p)$, to be $(-1)^{\sum_{i=1}^{p}(|w_i|-1)}$. Thus if $(w_1, \ldots, w_p) = W(\theta)$ for some bi-brick permutation θ, then $sgn(\theta) = sgn(w_1, \ldots, w_p)$. We shall define a series of sign reversing involutions on $\mathcal{SL}$ which have the property that the collection of λ and μ bricks in the corresponding simple primitive bi-brick permutations is preserved. These involutions will show that we can replace the sum on the right hand side of (28) by a more restricted sum. For example, let $\mathcal{SL}_{B\leq 1}$ denote the set of all sequences of Lyndon words $(w_1,\ldots,w_p) \in \mathcal{SL}$ such that $(w_1,\ldots w_p)$ contains at most one B. The sequences $(w_1,\ldots w_p) \in \mathcal{SL}_{B\leq 1}$ correspond to simple primitive bi-brick permutations θ such that as we traverse the cycles in a clockwise manner, there is at most one cell in θ which is the start of both a λ and a μ brick. Our first result of this section will be to construct a sign reversing involution on $\mathcal{SL}$ which proves the following.

Theorem 2

$$M(e,m)_{\lambda,\mu} = (-1)^{\ell(\lambda)+\ell(\mu)} \sum_{\substack{\theta \in SPB(\lambda,\mu) \\ W(\theta) \in \mathcal{SL}_{B\leq 1}}} sgn(\theta).$$

Proof. Before we can define our desired involution on $\mathcal{SL}$, we need to establish some notation. Let $X = \{x_1,\ldots,x_n\}$ be an ordered alphabet where $x_1 < x_2 < \cdots < x_n$. Let X^* denote the set of all words over X and $Lyn(X)$ denote the set of all Lyndon words in X^*. Given $x \in X$, we let x-Lyn denote the set of all words in $Lyn(X)$ which start with x. If $w = uv$ where $u,v \in X^*$, then we say u is an initial segment of w and write $u \sqsubseteq w$. If in addition, $|v| \geq 1$ and $|u| \geq 1$, then we say u is a *head* of w and v is a *tail* of w. Recall that $<_\ell$ denotes the lexicographic order on X^*. We shall write $w <<_\ell u$ if $w <_\ell u$ and $w \not\sqsubseteq u$.

This given, we recall two characterizations of Lyndon words over X which we shall use in our proofs which can be found in [5].

Lemma 1 *(Proposition 5.1.2 in [5], page 65.)*
Let $w \in X^*$. Then $w \in Lyn(X)$ if and only if $w <<_\ell v$ for any tail v of w.

Lemma 2 *(Proposition 5.1.3 in [5], page 66.)*
Let $w \in X^*$. Then $w \in Lyn(X)$ if and only if either (i) $w \in X$ or (ii) $w = u_1u_2$ where $u_1 <_\ell u_2$ and $u_1,u_2 \in Lyn(X)$. In fact, if $w \in Lyn(X), |w| \geq 2$, and $w = uv$ where v is the longest tail of w which is in $Lyn(X)$, then $u \in Lyn(X)$ and $u <_\ell w <_\ell v$.

This given, we define the following involution $I_B : \mathcal{SL} \to \mathcal{SL}$. Suppose $(w_1,\ldots,w_t) \in \mathcal{SL}$ where $w_1 <_\ell w_2 <_\ell \cdots <_\ell w_t$. Let m be the smallest $s \geq 0$ such that $w_{s+1} \notin B$-$Lyn(A)$ if there is such an s and $m = t$ if $w_t \in B$-$Lyn(A)$. Note that all words in B-$Lyn(A)$ are lexicographically less than the words in $Lyn(A) \setminus B$-$Lyn(A)$. Hence it must be the case that $w_{m+1},\ldots,w_t \in Lyn(A) \setminus B$-$Lyn(A)$. The definition of I_B proceeds according to the following five cases.

Case 1 $m = 0$ so that no B's occur in $(w_1,\ldots,w_t)$. Then $I_B(w_1,\ldots,w_t) = (w_1,\ldots,w_t)$.

Case 2 $m = 1$ and w_1 contains exactly one B. Then $I_B(w_1,\ldots,w_t) = (w_1,\ldots,w_t)$.

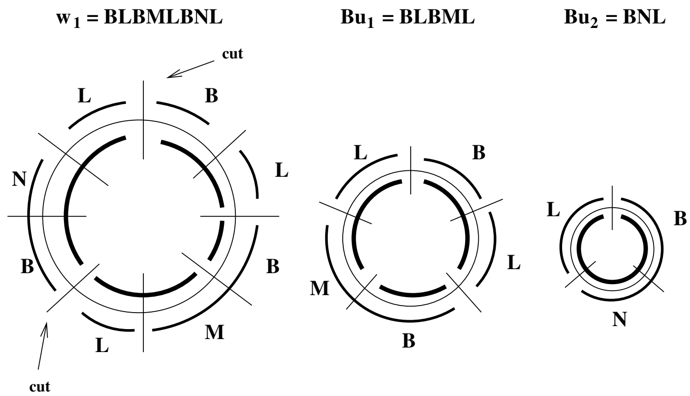

Figure 12: Cutting a bi-brick cycle at B's.

Case 3 $m = 1$ and w_1 contains two or more B's. Let $I_B(w_1, \ldots, w_t) = (u_1, v_1, w_2, \ldots, w_t)$ where v_1 is the shortest tail of w_1 such that $w_1 = u_1v_1$ where $u_1, v_1 \in B\text{-}Lyn(A)$ and $u_1 <_\ell v_1$.

Case 4 $m > 1$ and there is a tail v of w_m such that $w_m = uv$ where $u, v \in B\text{-}Lyn(A)$ and $w_{m-1} <_\ell u <_\ell v$. Let $w_m = u_m v_m$ where v_m is the shortest such tail v of w_m such that $w_m = uv, w_{m-1} <_\ell u <_\ell v$ and $u, v \in B\text{-}Lyn(A)$. We then set $I_B(w_1, \ldots, w_t) =$
$(w_1, \ldots w_{m-1}, u_m, v_m, w_{m+1}, \ldots, w_t)$.

Case 5 $m > 1$ and not case 4. Then set $I_B(w_1, \ldots, w_t) =$
$(w_1, \ldots w_{n-2}, w_{m-1}w_m, w_{m+1}, \ldots, w_t)$.

Before we proceed to show that I_B is indeed a well defined involution, we pause to make a few remarks about the properties of I_B. First observe that if w is the word of a bi-brick cycle C and $w = Bu_1Bu_2$ where $u_1, u_2 \in A^*$, then the bi-brick cycles C_1 and C_2 corresponding to Bu_1 and Bu_2 respectively can be constructed from C by cutting C at two cells which are the start of both λ and μ-bricks so that C_1 and C_2 contain the same λ and μ-bricks as C. See Figure 12 for an example. Thus if $\theta_1 \in SPB(\lambda, \mu)$ is such that $W(\theta_1) = (w_1, \ldots, w_t)$, then there is a $\theta_2 \in SPB(\lambda, \mu)$ such that $W(\theta_2) = I_B(w_1, \ldots, w_t)$.

Second we observe that if θ_2 arises from θ_1 by either splitting one cycle of θ_1 into two cycles or combining two cycles of θ_1 into one cycle, then $sgn(\theta_1) = -sgn(\theta_2)$. Thus once we have proved that I_B is a well defined involution, it will follow that for any $n > 0$ and partitions λ and μ of n,

$$\sum_{\theta\in SPB(\lambda,\mu)} sgn(\theta) = \sum_{\substack{\theta \in SPB(\lambda,\mu)\\ I_B(W(\theta))=W(\theta)}} sgn(\theta)$$

$$= \sum_{\substack{\theta \in SPB(\lambda,\mu)\\ W(\theta)\in \mathcal{SL}_{B\leq 1}}} sgn(\theta).$$

Thus Theorem 2 immediately follows once we have proved that I_B is a well defined involution.

To see that I_B is well defined, first consider Case 3. Thus w_1 is the only word in $(w_1, w_2, \ldots, w_t)$ which contains a B and w_1 contains at least two B's. Thus we can write $w_1 = Bu_1Bu_2$ where $u_1 \in A^*$ and $u_2 \in \{L, M, N\}^*$. It is easy to see that Bu_2 has the property that every tail v of Bu_2 satisfies $Bu_2 <_\ell v$. Thus Bu_2 is Lyndon by Lemma 1. Now let v' be the longest tail v of w_1 such that $v \in Lyn(A)$. By Lemma 2, $w_1 = u'v'$ where $u' <_\ell v'$ and $u' \in B\text{-}Lyn(A)$. Note that since Bu_2 is a tail of w_1 in $Lyn(A)$, $|v'| \geq |Bu_2|$ so that v' must contain a B. But if v' contains a B, it must start with a B since $v' \in Lyn(A)$. Thus $u', v' \in B\text{-}Lyn(A)$ and $u' <_\ell v'$ so that v' is a candidate to be the v_1 of Case 3. Hence u_1 and v_1 exist in Case 3. It easily follows that I_B is well defined in all cases. Thus we need only show that I_B is an involution.

To see that I_B is an involution, first consider Case 3. Thus v_1 is the shortest tail v of w_1 such that $w_1 = uv$ where $u, v \in B\text{-}Lyn(A)$ and $u <_\ell v$. We claim that it cannot be the case that $v_1 = \alpha\beta$ where $|\alpha|, |\beta| \geq 1$, $\alpha, \beta \in B\text{-}Lyn(A)$ and $u_1 <_\ell \alpha <_\ell \beta$. That is, if such α and β exist, then $u_1\alpha \in B\text{-}Lyn(A)$ by Lemma 2. But then $u_1\alpha <_\ell u_1\alpha\beta = w_1$ and $w_1 <_\ell \beta$ by Lemma 1. Thus $u_1\alpha <_\ell \beta$ and $u_1\alpha, \beta \in B\text{-}Lyn(A)$ which would mean that v_1 is not the shortest tail v of w_1 such that $w_1 = uv$ where $u, v \in B\text{-}Lyn(A)$ and $u <_\ell v$. It follows that there can be no such α and β so that $(u_1, v_1, w_2, \ldots, w_t)$ is in Case 5 and hence $I_B((u_1, v_1, w_2, \ldots, w_t)) = (w_1, \ldots, w_t)$. Similarly suppose that in Case 4, $v_m = \alpha\beta$ where $|\alpha|, |\beta| \geq 1, \alpha, \beta \in B\text{-}Lyn(A)$ and $v_{m-1} <_\ell \alpha <_\ell \beta$. Then $w_{m-1} <_\ell u_m <_\ell u_m\alpha <_\ell w_m <_\ell \beta$ so that $w_{m-1} <_\ell u_m\alpha <_\ell \beta$. Again $u_m\alpha \in B\text{-}Lyn(A)$ by Lemma 2 so that β would violate our choice of v_m as the shortest tail v of w_m such that $w_m = uv$ where $u, v \in B\text{-}Lyn(A)$ and $w_{m-1} <_\ell u <_\ell v$. Thus in Case 4, $(w_1, \ldots, w_{m-1}, u_m, v_m, w_{m+1}, \ldots, w_t)$ is in Case 5 so that $I_B((w_1, \ldots, w_{m-1}, u_m, v_m, w_{m+1}, \ldots, w_t)) = (w_1, \ldots, w_t)$.

Finally consider Case 5. In this case, we must show that w_m is the shortest tail v of $w_{m-1}w_m$ such that $w_{m-1}w_m = uv$ where $u, v \in B\text{-}Lyn(A)$ and $w_{m-2} <_\ell u <_\ell v$. If not, there exists $\alpha, \beta \in A^*$ such that $w_m = \alpha\beta, |\alpha|, |\beta| \geq 1, \beta \in B\text{-}Lyn(A), w_{m-1}\alpha \in B\text{-}Lyn(A)$ and $w_{m-2} <_\ell w_{m-1}\alpha <_\ell \beta$. Assume β is the longest possible tail of w_m with this property. By Lemma 1, $w_{m-1}\alpha <_\ell \alpha$ and $w_m <_\ell \beta$. Thus $w_{m-1} <_\ell w_{m-1}\alpha <_\ell \alpha <_\ell w_m <_\ell \beta$ so that $w_{m-1} <_\ell \alpha <_\ell \beta$. Since $\beta \in B\text{-}Lyn(A)$ and we are not in Case 4, we must conclude that $\alpha \notin Lyn(A)$. Hence by Lemma 1, there is a tail v of α such that $v \leq_\ell \alpha$. Let δ be the shortest tail v of α such that $v \leq_\ell \alpha$. Thus $\alpha = \gamma\delta$ where $\gamma, \delta \in A^*$ and $|\gamma|, |\delta| \geq 1$. It cannot be the case that $\delta <<_\ell \alpha$ since otherwise $\delta\beta$ is a tail of w_m such that $\delta\beta <<_\ell \alpha\beta = w_m$ which would violate the fact that $w_m \in Lyn(A)$. Thus $\delta \sqsubseteq \alpha$. We claim that $\delta \in B\text{-}Lyn(A)$ and $|\delta| \leq |\gamma|$. That is, if $\delta \notin Lyn(A)$, then there is a tail θ of δ such that $\theta \leq_\ell \delta$. But then $\theta \leq_\ell \delta \leq_\ell \alpha$ so that θ would be a shorter tail v of α such that $v \leq_\ell \alpha$ which violates our choice of δ. Moreover, since α starts with a B, then δ must start with a B and hence $\delta \in B\text{-}Lyn(A)$. If $|\gamma| < |\delta|$, then $\delta = \gamma\theta$ where $\theta \in A^*$. On the other hand, since $\delta \sqsubseteq \alpha$, $\alpha = \gamma\theta\psi$ for some $\psi \in A^*$. But then since $\alpha = \gamma\delta$, it must be the case that $\delta = \theta\psi$. But that would imply that θ is a tail of $\gamma\delta$ and $\theta \sqsubseteq \delta \sqsubseteq \gamma\delta = \alpha$ which would again violate our choice of δ. Thus $|\delta| \leq |\gamma|$ as claimed. It follows that we can write α in the form $\alpha = \delta^k\xi\delta^\ell$ for some $k, \ell \geq 1$ where $\xi \in A^*$ is such that δ is neither a head nor tail of ξ. We note that it is possible that $\xi = \emptyset$ (the empty word), in which case, we assume $k = 1$. Next observe that $\delta <_\ell \alpha <_\ell \alpha\beta = w_m$ and $w_m <_\ell \beta$ since β is a tail of the Lyndon word w_m. Thus $\delta <_\ell \beta$. But then by Lemma 2, $\delta\beta \in Lyn(A)$. Hence $\delta <_\ell \delta\beta$ so that $\delta^2\beta \in Lyn(A)$. Continuing on in this way, $\delta^\ell\beta \in Lyn(A)$ and since δ starts with a $B, \delta^\ell\beta \in B\text{-}Lyn(A)$. But then $\delta^\ell\beta$ is a tail of $w_{m-1}w_m$ and $w_{m-1}w_m \in Lyn(A)$ so that $w_{m-2} <_\ell w_{m-1} <_\ell w_{m-1}\delta^k\xi <_\ell w_{m-1}w_m <_\ell \delta^\ell\beta$. Hence $w_{m-2} <_\ell w_{m-1}\delta^k\xi <_\ell \delta^\ell\beta$. Our choice of β forces us to conclude that $w_{m-1}\delta^k\xi \notin Lyn(A)$. Thus there is a tail Θ of $w_{m-1}\delta^k\xi$ such that $\Theta \leq_\ell w_{m-1}\delta^k\xi$. We shall show that the existence of such a Θ leads to a contradiction so that there can be no such α and β. First observe that it cannot be that $\Theta <<_\ell w_{m-1}\delta^k\xi$ since otherwise $\Theta\delta^\ell\beta <<_\ell w_{m-1}\delta^k\xi\delta^\ell\beta = w_{m-1}w_m$ so that $\Theta\delta^\ell\beta$ would be a tail of $w_{m-1}w_m$ which is $\leq_\ell w_{m-1}w_m$. But $w_{m-1}w_m \in Lyn(A)$ so there can be no such tail. Thus $\Theta \sqsubseteq w_{m-1}\delta^k\xi$. We now have three cases.

Case (i) $|\Theta| > |\delta^k\xi|$.

In this case $\Theta = \psi\delta^k\xi$ for some $\psi \in A^*$ with $|\psi| \geq 1$. Thus ψ is a tail of w_{m-1}. On the other hand, $\Theta \sqsubseteq w_{m-1}\delta^k\xi$ so that ψ is a head of w_{m-1}. Thus $\psi \leq_\ell w_{m-1}$ which violates the assumption that $w_{m-1} \in Lyn(A)$.

Case (ii) $|\delta^k\xi| \geq |\Theta| > |\xi|$.

It follows that either (a) $\Theta = \delta^j\xi$ for some $1 \leq j \leq k$ or (b) $\Theta = \psi\delta^j\xi$ where $0 \leq j < k$ and ψ is some tail of δ. In case (a), δ would be both a head and a tail of $w_{m-1}\delta^k\xi\delta^\ell = w_{m-1}\alpha$ which would violate the fact that $w_{m-1}\alpha \in Lyn(A)$. Similarly in case (b), ψ would be both a tail and head of $w_{m-1}\alpha$ which again would violate our assumption that $w_{m-1}\alpha \in Lyn(A)$. Thus Case (ii) cannot hold.

Case (iii) $|\xi| \geq |\Theta|$.

In this case $\Theta \sqsubseteq w_{m-1}\delta^k\xi <_\ell w_{m-1}\alpha <_\ell \delta$. Thus $\Theta <_\ell \delta$. It cannot be that $\Theta <<_\ell \delta$ since otherwise $\Theta\delta^\ell\beta$ is a tail of w_m such that $\Theta\delta^\ell\beta <<_\ell \delta \sqsubseteq \alpha \sqsubseteq w_m$ which would violate the fact that $w_m \in Lyn(A)$. Thus it must be the case that $\Theta \sqsubseteq \delta$. It cannot be that $\Theta = \delta$ since then δ would be both a tail and a head of $w_{m-1}\delta^k\xi\delta^\ell = w_{m-1}\alpha$. Thus Θ is a head of δ. Suppose that $\delta = \Theta\psi$ where $\psi \in A^*$ and $|\psi| = h \geq 1$. Next let $\delta = \eta\phi$ where $\eta, \phi \in A^*$ and $|\eta| = h$. Since $\delta \in Lyn(A)$, $\delta <_\ell \psi$ and, in fact, $\eta <<_\ell \psi$. Thus $\Theta\eta <<_\ell \Theta\psi$. But $\Theta\eta \sqsubseteq \Theta\delta^\ell\beta$ and $\Theta\delta^\ell\beta$ is a tail of w_m while $\delta = \Theta\psi$ is a head of w_m. Thus $\Theta\delta^\ell\beta <<_\ell w_m$ which violates the fact that $w_m \in Lyn(A)$. Thus case (iii) cannot hold.

Thus in Case 5, the assumption that there is a tail β of w_m such that $w_m = \alpha\beta$ and $w_{m-2} <_\ell w_{m-1}\alpha <_\ell \beta$ where $w_{m-1}\alpha, \beta \in B\text{-}Lyn(A)$ leads to a contradiction. Thus w_m is the shortest tail v of $w_{m-1}w_m$ such that $w_{m-1}w_m = uv$ where $w_{m-2} <_\ell u <_\ell v$ and $u, v \in B\text{-}Lyn(A)$. Hence in Case 5, we can conclude that $I_B((w_1, \ldots, w_{m-2}, w_{m-1}w_m, w_{m+1}, \ldots, w_t)) = (w_1, \ldots, w_t)$. Thus I_B is a well defined involution as claimed. ■

It is not difficult to show that our next result is a consequence of the fact that $M(e,m)_{\lambda,\mu} = 0$ if $\mu <_D \lambda'$. However, we can use Theorem 2 to give a combinatorial proof of this result.

Theorem 3 *If λ and μ are partitions of u and $\ell(\lambda) + \ell(\mu) \geq n+2$, then $M(e,m)_{\lambda,\mu} = 0$.*

Proof. Suppose that $\theta = (\theta_1, \ldots, \theta_k)$ is a simple primitive bi-brick permutation such that $W(\theta) = (W(\theta_1), \ldots, W(\theta_k)) \in \mathcal{SL}_{B\leq 1}$. For each i, suppose θ_i is a primitive bi-brick permutation of size n_i of type $(\lambda^{(i)}, \mu^{(i)})$. Thus $\lambda = \bigcup_{i=1}^{k} \lambda^{(i)}$, $\mu = \bigcup_{i=1}^{k} \mu^{(i)}$, $\ell(\lambda) = \sum_{i=1}^{k} \ell(\lambda^{(i)})$, $\ell(\mu) = \sum_{i=1}^{k} \ell(\mu^{(i)})$ and $n = \sum_{i=1}^{k} n_i$. Now $\ell(\lambda^{(i)})$ is the number of cells of θ_i where a λ-brick starts and $\ell(\mu^{(i)})$ is the number of cells of θ_i where a μ-brick starts. It is easy to see that if $\ell(\lambda^{(i)}) + \ell(\mu^{(i)}) \geq n_i + 2$, then there must be at least two cells of θ_i where both a λ-brick and a μ-brick start and hence $W(\theta_i)$ would contain two B's. But by assumption, there is at most one B in $W(\theta)$ and hence we can conclude that $\ell(\lambda^{(i)}) + \ell(\mu^{(i)}) \leq n_i + 1$ for all i. If $\ell(\lambda^{(i)}) + \ell(\mu^{(i)}) = n_i + 1$, then there must be at least one cell of θ_i where both a λ-brick and a μ-brick start so that $W(\theta_i)$ will have at least one B. Thus if $W(\theta_1) <_\ell \cdots <_\ell W(\theta_k)$ and $W(\theta)$ has one B, then that B must occur in $W(\theta_1)$. But then $\ell(\lambda^{(1)}) + \ell(\mu^{(1)}) \leq n_1 + 1$ and for all $j > 1$, $\ell(\lambda^{(j)}) + \ell(\mu^{(j)}) \leq n_j$ since $W(\theta_j)$ has no B's. Thus $\ell(\lambda) + \ell(\mu) = \sum_{i=1}^{k} \ell(\lambda^{(i)}) + \ell(\mu^{(i)}) \leq n_1 + 1 + \sum_{j=2}^{k} n_i = n + 1$. By a similar argument we can show that if $W(\theta)$ has no B's, then $\ell(\lambda) + \ell(\mu) \leq n$. Thus if $\ell(\lambda) + \ell(\mu) \geq n+2$, there can be no simple primitive bi-brick permutations θ such that $W(\theta) \in \mathcal{SL}_{B\leq 1}$ and hence $M(e,m)_{\lambda,\mu} = 0$ by Theorem 2. ■

Next we shall consider involutions on primitive bi-brick permutations θ such that the word of θ, $W(\theta) = (w_1, w_2, \ldots, w_m)$, does not contain a B. In this case every word w_i must contain both an L and an M. Recall that in our alphabet $L < N < M$. Thus each w_i must have an initial segment of the form ϕ where $\phi \in L\{L,N\}^*M$. In fact, it is easy to see that ϕ must be a Lyndon word. That is, suppose for a contradiction that δ is a tail of ϕ such that $\delta \leq_\ell \phi$. Then $\delta = \alpha M$ where $\alpha <<_\ell \phi$. That is, if $\alpha \sqsubseteq \phi$, then the $(|\alpha|+1)$st letter of δ, namely M, is greater than the $(|\alpha|+1)$st letter of ϕ which is in $\{L,N\}$ so that $\phi <<_\ell \delta$. Since $\alpha <<_\ell \phi$, $w_i = \phi\beta$ where $\beta \in A^*$ and hence w_i has a tail $\alpha M\beta$ such that $\alpha M\beta << \phi\beta = w_i$. Thus ϕ must be a Lyndon word.

Given any Lyndon word $\phi \in L\{L,N\}^*M$, we say that a word $w \in A^*$ is ϕ-Lyndon if $w \in Lyn(A)$ and ϕ is an initial segment of w. We let $\phi\text{-}Lyn(A)$ denote the set of all ϕ-Lyndon words in A^*. Now suppose that ψ and ϕ are Lyndon words in $L\{L,N\}^*M$ such

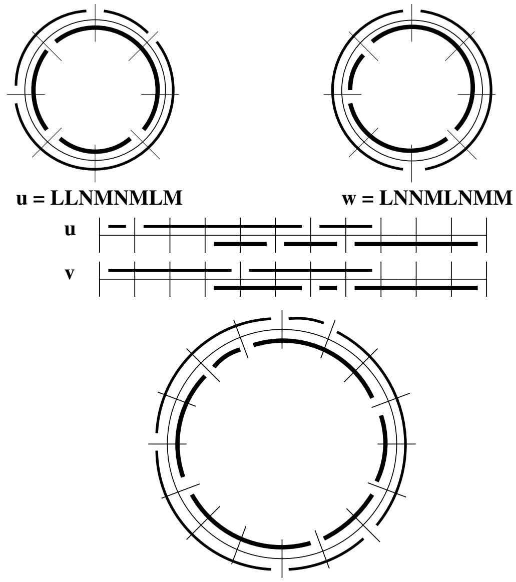

Figure 13: Combining two cycles arising from ϕ-Lyndon words.

that $|\psi| = |\phi|$, u is a ψ-Lyndon word, w is a ϕ-Lyndon word, and $u <_\ell w$. Moreover assume that u is the word of some bi-brick cycle of type (α, β) and w is the word of some bi-brick cycle of type (γ, δ). Then we claim uw is also the word of a bi-brick cycle of type $(\alpha \cup \gamma, \beta \cup \delta)$. This is best explained by an example. Consider Figure 13. Here $\psi = LLNM$, $\phi = LNNM$, $u = LLNMNMLM$, and $w = LNNMLNMM$. Thus $\alpha = (1, 2, 5)$, $\beta = (2, 2, 4)$, $\gamma = (4, 4)$, and $\delta = (1, 3, 4)$. It is then easy to see that we can break apart the cycle corresponding to u and draw the two sets of bricks in a line so that the α-bricks are on top starting with the bricks corresponding to the initial segment of ψ of length $|\psi|-1$, i.e. LLN, and the β-bricks are on the bottom starting with the brick that corresponds with the M of ψ. In this case since $|\psi| = 4$, the α-bricks will start $|\psi|-1 = 3$ squares ahead of the first β brick and the last β brick will extend 3 squares beyond the last α-brick. Similarly, we can break apart the cycle corresponding to w and draw the two sets of bricks so that the α-bricks are on top starting with the bricks corresponding to the initial segment of ϕ of length $|\phi| - 1$, i.e. LNN, and the δ-bricks are on the bottom starting with the brick that corresponds with the M of ϕ. Again the γ-bricks will start 3 squares ahead of the first δ-brick and the last δ-brick will extend 3 squares beyond the last γ-brick. It is then easy to see that we can hook these two sequences together by having the γ-brick start immediately after the α-bricks since the initial segment of 3 squares of γ-bricks fits over the 3 squares that the last β-brick extends beyond the last α-brick. Then the combined sequence can be wrapped around a bi-brick cycle of length $|uw|$ so that the word of the bi-brick cycle is uw. Note that uw must be the word of a primitive bi-brick cycle since uw is in $Lyn(A)$ by Lemma 2.

This given, for any Lyndon word $\phi \in L\{L, N\}^*M$, we can define an involution I_ϕ :

$\mathcal{SL} \to \mathcal{SL}$ in much the same way as we defined the involution I_B. That is, suppose that $(z_1, \ldots, z_t) \in \mathcal{SL}$ where $z_1 <_\ell \cdots <_\ell z_t$ and let $(w_1, \ldots, w_m)$ be the subsequence of $(z_1, \ldots, z_t)$ consisting of all words which are ϕ-Lyndon. Then the definition of I_ϕ proceeds according to the following five cases.

Case 1. $m = 0$ so that there are no ϕ-Lyndon words in $(z_1, \ldots, z_t)$. Then $I_\phi(z_1, \ldots, z_t) = (z_1, \ldots, z_t)$.

Case 2. $m = 1$ and w_1 contains exactly one occurrence of ϕ. (Here we say ϕ *occurs* in w if $w = \alpha\phi\beta$ for some $\alpha, \beta \in A^*$.) Then $I_\phi(z_1, \ldots, z_t) = (z_1, \ldots, z_t)$.

Case 3. $m = 1$ and w_1 contains two or more occurrences of ϕ. Then let v_1 be the shortest tail of w_1 such that $w_1 = uv$ where $u, v \in \phi\text{-}Lyn(A)$ and $u <_\ell v$. Assume $w_1 = u_1 v_1$ where $u_1 \in \phi\text{-}Lyn(A)$ and $u_1 <_\ell v_1$. Then $I_\phi(z_1, \ldots, z_t) = (z'_1, \ldots, z'_{t+1})$ where $z'_1 <_\ell \cdots <_\ell z'_{t+1}$ is obtained from $(z_1, \ldots, z_t)$ by replacing the single word w_1 by two words u_1 and v_1.

Case 4. $m > 1$ and there is a tail $v \in w_m$ such that $w_m = uv$ where $u, v \in \phi\text{-}Lyn(A)$ and $w_{m-1} <_\ell u <_\ell v$. Then let $w_m = u_m v_m$ where v_m is the shortest such tail of w_m such that $w_{m-1} <_\ell u_m <_\ell v_m$ and $u_m, v_m \in \phi\text{-}Lyn(A)$ and define $I_\phi(z_1, \ldots, z_t) = (z'_1, \ldots, z'_{t+1})$ where $z'_1 <_\ell \cdots <_\ell z'_{t+1}$ is obtained from $(z_1, \ldots, z_t)$ by replacing the single word w_m by two words u_m and v_m.

Case 5. $m > 1$ and not case 4. Then set $I_\phi(z_1, \ldots, z_t) = (z'_1, \ldots, z'_{t-1})$ where $z'_1 < \cdots < z'_{t-1}$ is obtained from $(z_1, \ldots, z_t)$ by replacing the two words w_{m-1} and w_m by the single word $w_{m-1}w_m$.

The proof that I_ϕ is a well defined involution is almost word for word the same as the proof that I_B is a well defined involution with two exceptions. That is, first we must show that in case 3, u_1 and v_1 are defined and, second, we must show that in case 5 where the two words w_{m-1} and w_m get replaced by the single word $w_{m-1}w_m$, w_m is the shortest ϕ-Lyndon tail v of $w_{m-1}w_m$ such that $w_{m-1}w_m = uv$, $u, v \in \phi\text{-}Lyn(A)$ and $w_{m-2} <_\ell u <_\ell v$. That is, these are the only two places in the proof that I_B is a well defined involution that we used any special properties of B-Lyndon words. Thus we shall only verify these two facts. First suppose that we are in case 3 and that w_1 has two occurrences of ϕ. It is easy to see that since $\phi \in L\{L, N\}^*M$ that no two occurrences of ϕ in w_1 can overlap. Now consider the longest Lyndon tail v of w_1. By Lemma 2, $w_1 = uv$ where $u <_\ell v$ and $u, v \in Lyn(A)$. We claim that ϕ occurs in v. That is, since there are two occurrences of ϕ in w_1, there is a tail β of w_1 such that $\beta = \phi\gamma$ where there are no occurrences of ϕ in γ. We claim that β is Lyndon. If not, $\beta = \alpha_1\alpha_2$ where $\alpha_1, \alpha_2 \in A^*$ and $\phi \neq \alpha_2 \leq_\ell \beta$. It cannot be that $\alpha_2 <<_\ell \phi$ since otherwise α_2 would be a tail of w_1 such that $\alpha_2 <<_\ell \phi \sqsubseteq w_1$ violating the fact $w_1 \in Lyn(A)$. Similarly it cannot be that α_2 is a head of ϕ since then α_2 is a head of w_1 which again violates the fact that $w_1 \in Lyn(A)$. Thus it must be that $\phi \sqsubseteq \alpha_2$. But this is impossible because then β has only one occurrence of ϕ. Thus β is Lyndon. But then since v is the longest Lyndon tail of w_1, β must be a final segment of v. We claim that this forces ϕ to be an initial segment of v. That is, if $v = \beta$, then certainly ϕ is an initial segment of v. If $v \neq \beta$, then β is a tail of v and hence $v <_\ell \beta$ since $v \in Lyn(A)$. Since β is a tail of v, it must be the case that $v <<_\ell \beta$. However it cannot be that $v <<_\ell \phi$ since otherwise v is a tail of w_1 such that $v <<_\ell \phi \sqsubseteq w_1$. Hence ϕ must be an initial segment of v. Thus v is in $\phi\text{-}Lyn(A)$. Since no two copies of ϕ can overlap in w_1, it must be the case that v is a final segment of δ where $w_1 = \phi\delta$. Thus if $w_1 = uv$, then ϕ must be an initial segment of u. Hence by Lemma 2, $u <_\ell v$ and $u, v \in Lyn(A)$ so that $u, v \in \phi\text{-}Lyn(A)$. Hence there is at least one tail v' of w_1 such that $w_1 = u'v'$, $u', v' \in \phi\text{-}Lyn(A)$ and $u' <_\ell v'$. Thus I_ϕ is defined in case 3.

Next suppose that we are in case 5. Thus we cannot write $w_m = uv$ where u and v are ϕ-Lyndon words such that $w_{m-1} <_\ell u <_\ell v$. Now suppose there exist α, β such that $w_m = \alpha\beta$, $|\alpha|, |\beta| \geq 1$, β and $w_{m-1}\alpha$ are ϕ-Lyndon words, and $w_{m-2} <_\ell w_{m-1}\alpha <_\ell \beta$. Assume that β is the longest possible tail of w_m with this property. First observe that since $w_m = \alpha\beta$ and β are ϕ-Lyndon words and no two copies of ϕ in w_m can overlap, ϕ must be an initial segment of α. By Lemma 1, $w_{m-1}\alpha <_\ell \alpha$ and $w_m <_\ell \beta$. Thus $w_{m-1} <_\ell w_{m-1}\alpha <_\ell \alpha <_\ell w_m <_\ell \beta$. Thus $w_{m-1} <_\ell \alpha <_\ell \beta$. Since $\beta \in \phi\text{-}Lyn(A)$ and we are not in case 4, we must conclude that $\alpha \notin Lyn(A)$. By Lemma 1, there is a tail v of α such that $v \leq_\ell \alpha$. Pick δ to be the shortest tail v of α such that $v \leq_\ell \alpha$ and write $\alpha = \gamma\delta$ where $\gamma, \delta \in A^*$ and $|\gamma|, |\delta| \geq 1$. It cannot be that $\delta <<_\ell \alpha$ since otherwise $\delta\beta$ is a tail of w_m such that $\delta\beta <<_\ell \alpha\beta = w_m$ which would violate the fact that $w_m \in Lyn(A)$. Thus $\delta \sqsubseteq \alpha$. We claim that $\delta \in \phi\text{-}Lyn(A)$ and that $|\delta| \leq |\gamma|$. It cannot be that δ is an initial segment of ϕ since δ would then be both a tail and a head of $w_{m-1}\alpha$ which would violate the fact that $w_{m-1}\alpha \in Lyn(A)$. Thus $\phi \sqsubseteq \delta$. Next suppose $\delta \notin Lyn(A)$. Then there is a tail θ of δ such that $\theta \leq_\ell \delta$. But then θ is a tail of α such that $\theta \leq_\ell \delta \leq_\ell \alpha$ which would violate our choice of δ. Thus $\delta \in Lyn(A)$ and since $\phi \sqsubseteq \delta$, $\delta \in \phi\text{-}Lyn(A)$. Next assume that $|\gamma| < |\delta|$. Thus $\delta = \gamma\theta$ where $\theta \in A^*$ and $|\theta| \geq 1$. But then $\alpha = \gamma\delta = \delta\psi$ for some $\psi \in A^*$ so that $\alpha = \gamma\theta\psi$. However, this would mean that $\delta = \gamma\theta = \theta\psi$ and hence θ would be both a head and a tail of δ which would violate the fact that $\delta \in Lyn(A)$. Thus δ is a ϕ-Lyndon word which is an initial segment of γ as claimed. It follows that we can write α in the form $\alpha = \delta^k\xi\delta^\ell$ for some $k, \ell \geq 1$ where $\xi \in A^*$ and either $\xi = \emptyset$ or δ is neither a head nor a tail of ξ. We can now argue exactly as in the proof that I_B is a well defined involution in case 5 that such a factorization leads to a contradiction. It follows that there can be no such β and hence w_m is the shortest ϕ-Lyndon tail v of $w_{m-1}w_m$ such that $w_{m-1}w_m = uv$ where $u, v \in \phi\text{-}Lyn(A)$ and $w_{m-2} <_\ell u <_\ell v$.

We can thus conclude that I_ϕ is a well defined involution. Moreover if $I_\phi(z_1, \ldots, z_t) \neq (z_1, \ldots, z_t)$, then there are simple primitive bi-brick permutations θ_1 and θ_2 of type (λ, μ) for some partitions λ and μ such that $W(\theta_1) = (z_1, \ldots, z_t)$, $W(\theta_2) = I_\phi(z_1, \ldots, z_t)$ and $sgn(\theta_1) = -sgn(\theta_2)$. Since I_ϕ affects only the ϕ-Lyndon words in $(z_1, \ldots, z_2)$, we can apply the involutions I_B and I_ϕ for all Lyndon words $\phi \in L\{L, N\}^*M$ sequentially to conclude the following.

Theorem 4 *Let $\mathcal{SLS}$ consist of all sequences of Lyndon words $(w_1, \ldots, w_m)$ such that $w_1 <_\ell \cdots <_\ell w_m$, $(w_1, \ldots, w_m)$ contains at most one B and for any Lyndon word $\phi \in L\{L, N\}^*M$, $(w_1, \ldots, w_m)$ contains at most one ϕ-Lyndon word and if there is an i such that w_i is a ϕ-Lyndon word, then there is exactly one occurrence of ϕ in w_i. Then for all λ and μ which are partitions of n,*

$$M(e, m)_{\lambda,\mu} = (-1)^{\ell(\lambda)+\ell(\mu)} \sum_{\substack{\theta \in SPB(\lambda,\mu) \\ W(\theta) \in \mathcal{SLS}}} sgn(\theta).$$

There are still more involutions that can be applied to the set of all $\theta \in SPB(\lambda,\mu)$ with $W(\theta) \in \mathcal{SLS}$. We define a word w to be L-s-M-Lyndon if w does not contain a B, w has a head of the form $L\psi M$ where ψ is a word of $\{L,N\}^*$ of length s and w has only one occurrence of $L\psi M$. Our observations above show that if α and β are L-s-M-Lyndon words which come from primitive bi-brick cycles θ_1 and θ_2 and $\alpha <_\ell \beta$, then there is a primitive bi-brick cycle θ such that $\alpha\beta$ is the word of θ. This suggests that we could define further involutions by combining L-s-M-Lyndon words. That is, suppose $\theta \in SPB(\lambda,\mu)$ with $W(\theta) \in \mathcal{SLS}$ and $W(\theta) = (w_1,\ldots,w_m)$ and $w_{i_1} <_\ell \cdots <_\ell w_{i_k}$ is the subsequence of $W(\theta)$ consisting of all w_i which are L-s-M-Lyndon. Now suppose $k \geq 2$ and, for $1 \leq j \leq k$, w_{i_j} is ϕ_j-Lyndon where ϕ_j is a Lyndon word in $L\{L,N\}^*M$ of length $s+2$. If ϕ_{k-1} does not occur in w_{i_k}, then we know that we can combine the cycles corresponding to $w_{i_{k-1}}$ and w_{i_k} into a single cycle C such that $w(C) = w_{i_{k-1}}w_{i_k}$. Thus there will be a cycle $\theta' \in SPB(\lambda,\mu)$ such that $W(\theta') \in \mathcal{SLS}$, $W(\theta')$ arises from θ by replacing the two words $w_{i_{k-1}}$ and w_{i_k} by the single word $w_{i_{k-1}}w_{i_k}$ and $sgn(\theta) = -sgn(\theta')$. One could use this observation to try to construct an involution I_s much like the involutions I_B and I_ϕ described above. The problem is to find conditions which will allow us to recover $w_{i_{k-1}}$ and w_{i_k} from $w_{i_{k-1}}w_{i_k}$. The following example will show that we cannot proceed exactly as before. That is, suppose that $s = 5$ and our subsequence

$$(w_{i_1},\ldots,w_{i_k}) = (w_1,w_2) = (LLLLLLM, LLNLLNMLLNLNNNNNM).$$

There are three occurrences of seven letter Lyndon words in $L\{L,N\}^*M$ in (w_1,w_2), namely $\phi_1 = LLLLLLM$, $\phi_2 = LLNLLNM$ and $\phi_3 = LNNNNNM$. Note that we cannot break off ϕ_3 from w_2 since $LLNLLNMLLN$ is not Lyndon. However if we combine w_1w_2, then we can break off ϕ_3 from w_1w_2 since $LLLLLLMLLNLLNMLLN$ is Lyndon.

Despite this example, one can define an involution I_s for each $s \geq 0$ on the set of all $\theta \in SPB(\lambda,\mu)$ with $W(\theta) \in \mathcal{SLS}$ as follows. Let $\theta \in SPB(\lambda,\mu)$ and suppose $W(\theta) = (w_1,\ldots,w_m) \in \mathcal{SLS}$ and $w_{i_1} <_\ell \cdots <_\ell w_{i_k}$ is the subsequence of $W(\theta)$ consisting of all w_i which are L-s-M-Lyndon. Let w_{i_j} be ϕ_j-Lyndon where ϕ_j is a Lyndon word in $L\{L,N\}^*M$ of length $s+2$ for $j = 1,\ldots,k$. Let ψ^* be the lexicographically largest L-s-M-Lyndon word which occurs as a subword in some w_{i_j}. We say that w_{i_k} has a *good ψ^*-tail* if either (i) $\psi^* = \phi_k$ or (ii) $\phi_k \neq \psi^*$ and $w_{i_k} = \alpha\psi^*\beta$ where α is ϕ_k-Lyndon and $\psi^*\beta$ is ψ^*-Lyndon and there is no occurrence of a L-s-M-Lyndon word in β. Observe that if $\phi_k \neq \psi^*$, then the good ψ^*-tail of w_{i_k} is uniquely defined. The involution I_s is defined as follows.

Case 1 If $\phi_k = \psi^*$ and $k \geq 2$, then $I_s(\theta) = \theta^*$ where $W(\theta^*)$ comes from $W(\theta)$ by replacing the two words $w_{i_{k-1}}$ and w_{i_k} by a single word $w_{i_{k-1}}w_{i_k}$.
Note that in this case, w_{i_k} does not contain any L-s-M-Lyndon subword other than the initial ψ^*. That is, by definition, w_{i_k} has only one occurrence of ψ^*. Now if δ is another L-s-M-Lyndon occurring in w_{i_k}, then since δ and ψ^* cannot overlap, it must be the case that $w_{i_k} = \psi^*\alpha\delta\beta$ for some $\alpha,\beta \in A^*$. But our choice of ψ^* ensures that $\delta <<_\ell \psi^*$ so that $\delta\beta <<_\ell \psi^*\alpha \sqsubseteq w_{i_k}$ which would violate the fact that w_{i_k} is Lyndon. It follows that $w_{i_{k-1}}w_{i_k}$ has only one occurrence of ϕ_{k-1} so that $W(\theta^*) \in \mathcal{SLS}$. Note also that in this

case, w_{i_k} is the good ψ^*-tail of $w_{i_{k-1}}w_{i_k}$.

Case 2 If $\psi^* \neq \phi_k$ and w_{i_k} has a good ψ^*-tail β, then let $w_{i_k} = \alpha\beta$ and define $I_s(\theta) = \theta^*$ where $W(\theta^*)$ comes from $W(\theta)$ by replacing the w_{i_k} by the two words α and β.
Note that if $k \geq 1$, then we have $\phi_{k-1} <<_\ell \phi_k <<_\ell \psi^*$ and hence $w_{i_{k-1}} <_\ell \alpha <_\ell \beta$.

It is easy to see that I_s is an involution for each $s \geq 1$ and that we can apply these involutions independently. Note that there can be no such involution for $s = 0$ because there is only one L-0-M-Lyndon word, namely, LM. Thus the fixed point set of all the involutions defined so far is the set $FSPB(\lambda, \mu)$ consisting of $\theta \in SPB(\lambda, \mu)$ such that $W(\theta) = (w_1, \ldots, w_m)$ satisfies the following properties:

1. $(w_1, \ldots, w_m)$ contains at most one B,

2. for any Lyndon word $\phi \in L\{L, N\}^*M$, $(w_1, \ldots, w_m)$ contains at most one ϕ-Lyndon word and if w_i is a ϕ-Lyndon word, then there is only one occurrence of ϕ in w_i, and

3. For each $s \geq 1$, if $w_{i_1} <_\ell \cdots <_\ell w_{i_k}$ is the subsequence of $W(\theta)$ consisting of all w_i such that w_i is L-s-M-Lyndon and, for all $j = 1, \ldots, k$, w_{i_j} is ϕ_j-Lyndon where ϕ_j is a Lyndon word in $L\{L, N\}^*M$ of length $s + 2$, and ψ^* is the lexicographically largest L-s-M-Lyndon word which occurs in some w_{i_j}, then either (i) $\phi_k = \psi^*$ and $k = 1$ or (ii) $\phi_k \neq \psi^*$ and w_{i_k} does not have a good ψ^*-tail.

Thus we have the following.

Theorem 5 *For all $n \geq 1$ and for all partitions, λ and μ of n,*

$$M(e, m)_{\lambda,\mu} = (-1)^{\ell(\lambda)+\ell(\mu)} \sum_{\theta \in FSPB(\lambda,\mu)} sgn(\theta).$$

4 Some special cases of $M(h, m)_{\lambda,\mu}$ and $M(e, m)_{\lambda,\mu}$

In this section, we shall apply Theorems 1–5 to prove a few simple results about the values of $M(h, m)_{\lambda,\mu}$ and $M(e, m)_{\lambda,\mu}$ for certain classes of λ and μ. In particular, we shall give explicit formulas for $M(h, m)_{\lambda,\mu}$ and $M(e, m)_{\lambda,\mu}$ when $\lambda = \mu = (k^n)$ for some k and n, when both λ and μ are two-row shapes or when both λ and μ are hook shapes. Finally we shall also find formulas $M(e, m)_{\lambda,\mu}$ when both λ and μ are two-column shapes.

Theorem 6 *For all $n \geq 1$ and $k \geq 1$,*

$$M(h, m)_{(k^n),(k^n)} = \binom{n+k-1}{n} \text{ and } \tag{29}$$

$$M(e, m)_{(k^n),(k^n)} = (-1)^{n(k-1)} \binom{k}{n} \tag{30}$$

where we set $\binom{k}{n} = 0$ if $n > k$.

PROOF. It is easy to see that any bi-brick cycle of type $((k^p),(k^p))$ with $p \geq 2$ will have rotational symmetry. Thus the only primitive bi-brick cycles which contain only bricks of size k must be of type $((k),(k))$. It is easy to see that there are exactly k primitive bi-brick cycles of type $((k),(k))$. Thus any primitive bi-brick permutation of type $((k^n),(k^n))$ consists of n cycles of type $((k),(k))$. Hence the number of primitive bi-brick permutations of type $((k^n),(k^n))$ equals the number of non-negative integer valued solutions to $x_1 + \cdots + x_k = n$ which is equal to $\binom{n+k-1}{n}$. Thus $M(h,m)_{(k^n),(k^n)} = \binom{n+k-1}{n}$.

A simple primitive bi-brick permutation of type $((k^n),(k^n))$ must consist of n pairwise distinct primitive bi-brick cycles of type $((k),(k))$. Since there are k primitive bi-brick cycles of type $((k),(k))$, there are $\binom{k}{n}$ simple primitive bi-brick permutations of type $((k^n),(k^n))$. Clearly the sign of any such simple primitive bi-brick permutation of type $((k^n),(k^n))$ is $(-1)^{n(k-1)}$ so that $M(e,m)_{(k^n),(k^n)} = (-1)^{n(k-1)}\binom{k}{n}$. ■

Next we shall give formulas for $M(h,m)_{\lambda,\mu}$ and $M(e,m)_{\lambda,\mu}$ when both λ and μ are two-row shapes, i.e., when $\lambda = (a,b)$ and $\mu = (c,d)$ where $a+b = c+d = n$. Note that since both $M(h,m)_{\lambda,\mu} = M(h,m)_{\mu,\lambda}$ and $M(e,m)_{\lambda,\mu} = M(e,m)_{\mu,\lambda}$, there is no loss in generality in assuming that $a \leq c$.

Theorem 7 *Suppose $\lambda = (a,b)$ and $\mu = (c,d)$ are two-part partitions of n where $a \leq c$. Then*

$$M(h,m)_{\lambda,\mu} = \begin{cases} n & \text{if } a < c < d \\ n/2 & \text{if } a < c = d \\ n + ab & \text{if } a = c < d \\ \binom{a+1}{2} & \text{if } a = b = c = d \end{cases}$$

$$M(e,m)_{\lambda,\mu} = \begin{cases} (-1)^{n-1}n & \text{if } a < c < d \\ (-1)^{n-1}n/2 & \text{if } a < c = d \\ (-1)^{n-1}(n - ab) & \text{if } a = c < d \\ \binom{a}{2} & \text{if } a = b = c = d. \end{cases}$$

PROOF. This result easily follows from Theorem 1 once we make the following observations. First it is easy to see that if $a < c$, then the only primitive bi-brick permutation of type $((a,b),(c,d))$ consists of a single n-cycle. If $c < d$, there are clearly n such primitive bi-brick cycles while if $c = d$, then there are $n/2$ such primitive bi-brick cycles.

Next suppose $a = c < b = d$. Then there are n primitive bi-brick cycles of size n of type $((a,b),(c,d))$. The only other $((a,b),(a,b))$-primitive bi-brick permutations consist of two cycles, one of type $((a),(a))$ and the other of type $((b),(b))$. Clearly there are a primitive bi-brick cycles of type $((a),(a))$ and there are b primitive bi-brick cycles of type $((b),(b))$.

Finally if $a = b = c = d$, then our formulas follow from Theorem 6. ■

Next we shall give formulas for $M(h,m)_{\lambda,\mu}$ and $M(e,m)_{\lambda,\mu}$ when both λ and μ are hook shapes, i.e., when $\lambda = (1^a,b)$ and $\mu = (1^c,d)$ where $a+b = c+d = n$. Note that since both $M(h,m)_{\lambda,\mu} = M(h,m)_{\mu,\lambda}$ and $M(e,m)_{\lambda,\mu} = M(e,m)_{\mu,\lambda}$, there is no loss in generality in assuming that $b \leq d$.

Theorem 8 *Let $\lambda = (1^a, b)$ and $\mu = (1^c, d)$ where $a + b = c + d = n$ and $b \leq d$.*

1. *If $d = n$ so that $\mu = (n)$, then*

$$M(h,m)_{(1^a,b),(n)} = \begin{cases} (-1)^a n & \text{if } b \geq 2 \\ (-1)^{n+1} & \text{if } b = 1 \end{cases} \tag{31}$$

and

$$M(e,m)_{(1^a,b),(n)} = \begin{cases} (-1)^{a+n-1} n & \text{if } b \geq 2 \\ 1 & \text{if } b = 1. \end{cases} \tag{32}$$

2. *If $b = 1$ and $d \leq n - 1$, then*

$$M(h,m)_{(1^n),(1^c,d)} = \begin{cases} (-1)^{n+c-1}(c+1) & \text{if } d \geq 2 \\ 1 & \text{if } d = 1 \end{cases} \tag{33}$$

and

$$M(e,m)_{(1^n),(1^c,d)} = 0. \tag{34}$$

3. *If $2 \leq b \leq d \leq n - 1$, then*

$$M(h,m)_{\lambda,\mu} = (-1)^{a+c} \left((c+1)d + \binom{c+1}{2} + \binom{n-b-d+2}{2} \right) \tag{35}$$

and

$$M(e,m)_{\lambda,\mu} = \begin{cases} (-1)^{a+c+n+1} & \text{if } b + d \geq n + 1 \\ 0 & \text{if } b + d \leq n. \end{cases} \tag{36}$$

Proof. For (1), note that there are n bi-brick cycles of type $((1^a, b), (n))$ if $b \geq 2$ depending on where the outside brick of size b starts relative to the start of the inside brick of size n. Clearly all such bi-brick cycles are primitive. Thus $M(h,m)_{(1^a,b),(n)} = (-1)^{\ell((1^a,b))+\ell((n))} n = (-1)^a n$ if $b \geq 2$. Since all such bi-brick cycles have sign $(-1)^{n-1}$, it follows that $M(e,m)_{(1^a,b),(n)} = (-1)^{a+n-1} n$ if $b \geq 2$.

In the case when $b = 1$, there is a unique bi-brick cycle of type $((1^n), (n))$ which is also primitive. It then easily follows that

$$\begin{aligned} M(h,m)_{(1^n),(n)} &= (-1)^{\ell((1^n))+\ell((n))} 1 = (-1)^{n+1} \text{ and} \\ M(e,m)_{(1^n),(n)} &= (-1)^{\ell((1^n))+\ell((n))} (-1)^{n-1} = 1. \end{aligned}$$

For (2), note that $M(e,m)_{(1^n),(1^c,d)} = 0$ since $(1^c, d) <_D (n)$. One can also use Theorem 3 to conclude that $M(e,m)_{(1^n),(1^c,d)} = 0$ since $\ell((1^n)) + \ell((1^c, d)) \geq n + 2$.

To compute $M(h,m)_{(1^n),(1^c,d)}$, first observe that if $d \geq 2$, then we can classify the primitive bi-brick permutations σ of type $((1^n),(1^c,d))$ by the size of the bi-brick cycle θ which contains the brick of size d. That is, if θ is of type $((1^{j+d}),(1^j,d))$, then the rest of the bi-brick cycles of σ must be of type $((1),(1))$ since the only primitive bi-brick cycle made up entirely of bricks of size 1 is of type $((1),(1))$. Moreover, θ is uniquely determined by j. Thus there is one primitive $((1^n),(1^c,d))$-bi-brick permutation for each $0 \leq j \leq c$. It easily follows that $M(h,m)_{(1^n),(1^c,d)} = (-1)^{n+c+1}(c+1)$ if $d \geq 2$. If $d = 1$, there is a unique primitive bi-brick permutation of type $((1^n),(1^n))$ consisting of n bi-brick cycles of type $((1),(1))$. Thus $M(h,m)_{(1^n),(1^n)} = (-1)^{n+n}1 = 1$.

For (3), we observe that when $2 \leq b \leq d \leq n-1$, the primitive $((1^a,b),(1^c,d))$ bi-brick permutations σ fall into one of two categories. First, the bricks of size b and d can be in a single bi-brick cycle θ of type $((1^{j+d-b},b),(1^j,d))$ for some $0 \leq j \leq c$ and the rest of σ must consist of $c-j$ bi-brick cycles of type $((1),(1))$. Any bi-brick cycle θ of type $((1^{j+d-b},b),(1^j,d))$ is automatically primitive since neither the inside bricks nor the outside bricks have rotational symmetry. Thus there are $d+j$ choices for θ depending on the relative placement of the brick of size b with respect to the start of the brick of size d. Thus there are a total of $\sum_{j=0}^{c} d+j = (c+1)d + \binom{c+1}{2}$ primitive $((1^a,b),(1^c,d))$-bi-brick permutations where the brick of size b and the brick of size d lie in the same bi-brick cycle. The only other possibility is that the brick of size b and the brick of size d lie in different bi-brick cycles. Of course, this is only possible if $b+d \leq n$. In that case, a primitive bi-brick permutation σ must consist of a bi-brick cycle θ_1 of type $((1^{x+d}),(1^x,d))$, a bi-brick cycle θ_2 of type $((1^y,b),(1^{b+y}))$ and z bi-brick cycles of type $((1),(1))$ where $x+b+y+z = c = n-d$. Note that for fixed x and y, the bi-brick cycles θ_1 and θ_2 are unique. It follows that the number of primitive $((1^a,b),(1^c,d))$-bi-brick permutations where the brick of size b and the brick of size d lie in different bi-brick cycles is the number of solutions of $x+y+z = n-b-d$ where $x,y,z \geq 0$. But clearly for each fixed x, there are $1+n-b-d-x$ choices for y and z. Thus there there are a total of $\sum_{x=0}^{n-b-d} 1+n-b-d-x = \binom{n-b-d+2}{2}$ primitive $((1^a,b),(1^c,d))$-bi-brick permutations where the brick of size b and the brick of size d lie in different bi-brick cycles. Note that when $b+d > n$, then $\binom{n-b-d+2}{2} = 0$ as it should be. Thus when $2 \leq b \leq d \leq n-1$,there are a total of $(c+1)d + \binom{c+1}{2} + \binom{n-b-d+2}{2}$ primitive $((1^a,b),(1^c,d))$-bi-brick permutations. Hence $M(h,m)_{(1^a,b),(1^c,d)} = (-1)^{a+c+2}\left((c+1)d + \binom{c+1}{2} + \binom{n-b-d+2}{2}\right)$ when $2 \leq b \leq d \leq n-1$. This proves (35).

Finally, we consider $M(e,m)_{(1^a,b),(1^c,d)}$ when $2 \leq b \leq d \leq n-1$. By Theorem 3, $M(e,m)_{(1^a,b),(1^c,d)} = 0$ if $a+c+2 \geq n+2$. But note that

$$a+c+2 \geq n+2 \iff n-b+n-d+2 \geq n+2 \iff n \geq b+d.$$

This means $M(e,m)_{(1^a,b),(1^c,d)} = 0$ if $b+d \leq n$. Now suppose $b+d \geq n+1$. Clearly, in this case, a primitive $((1^a,b),(1^c,d))$-bi-brick permutation must have the brick of size b and the brick of size d in the same bi-brick cycle since we do not have enough room to have the brick of size b and the brick of size d in two different bi-brick cycles. By our analysis above, the only possible $((1^a,b),(1^c,d))$-bi-brick permutations consist of a primitive bi-brick cycle θ of type $((1^{d-b+j},b),(1^j,d))$ plus $c-j$ bi-brick cycles of type $((1),(1))$. However only simple primitive bi-brick permutations contribute to $M(e,m)_{\lambda,\mu}$ so that we only have to analyze two types of $((1^a,b),(1^c,d))$-bi-brick permutations, namely, (i) the $((1^a,b),(1^c,d))$-bi-brick permutations σ where there is one bi-brick cycle of type $((1),(1))$ and one bi-brick cycle of type $((1^{a-1},b),(1^{c-1},d))$ and (ii) the $((1^a,b),(1^c,d))$-bi-brick permutations τ where there is a single bi-brick cycle of type $((1^a,b),(1^c,d))$. Moreover, by Theorem 2, we can assume that all such σ and τ have the property that $W(\sigma),W(\tau)\in\mathcal{SL}_{B\leq 1}$.

First consider the $((1^a,b),(1^c,d))$-bi-brick permutations σ where there is one bi-brick cycle of type $((1),(1))$ and one bi-brick cycle of type $((1^{a-1},b),(1^{c-1},d))$ and $W(\sigma)\in\mathcal{SL}_{B\leq 1}$. Clearly the word of the bi-brick cycle of type $((1),(1))$ in σ is B. Thus the bi-brick cycle θ of type $((1^{a-1},b),(1^{c-1},d))$ in σ cannot have a B in $W(\theta)$. This means that there cannot be two bricks in θ that start at the same cell. Note that since $c+d=n$ and we are assuming that $b+d\geq n+1$, it must be the case that $c<b$. Also $a\geq 1$ since $2\leq b\leq d\leq n-1$. Let us draw the bi-brick cycle θ as pictured in Figure 14 where the outside brick of size b starts in cell 1 at the top. We shall consider two cases, depending on whether $a>1$ or $a=1$. First suppose that $a>1$. Labeling the cells clockwise with $1,\ldots,n-1$ starting with cell 1, we see that the outside of the cells $b+1,\ldots n-1$ following the end of the outside brick of size b must be covered by bricks of size 1. Thus an inside brick cannot start at any of the cells $b+1,\ldots,n-1,1$. It follows that the $c-1$ inside bricks must be cover a consecutive sequence of cells between cell 2 and cell $b-1$. If the $c-1$ inside bricks of size 1 end precisely at cell $b-1$, then they must start at cell $b-1-(c-2)=b-c+1$. Thus the start of the sequence of $c-1$ consecutive inside bricks of size 1 can start anywhere from cell 2 to cell $b-c+1$ and hence there are $b-c$ choices for θ. In the case where $a=1$, then $b=n-1$ and the outside cells of θ are completely covered by the outside brick of size b. However, it still follows that the $c-1$ inside bricks must be cover a consecutive sequence of cells between cell 2 and cell $b-1$ so that there are $b-c$ choices for θ in this case as well. Thus the $((1^a,b),(1^c,d))$-bi-brick permutations σ where there is one bi-brick cycle of type $((1),(1))$ and one bi-brick cycle of type $((1^{a-1},b),(1^{c-1},d))$ contribute a total of $(-1)^{a+c+2}(-1)^{n-2}(b-c)=(-1)^{a+c+n}(b-c)$ to $M(e,m)_{(1^a,b),(1^c,d)}$.

Finally consider the $((1^a,b),(1^c,d))$-bi-brick permutations τ where there is a single bi-brick cycle θ of type $((1^a,b),(1^c,d))$ and $W(\theta)\in\mathcal{SL}_{B\leq 1}$. Hence there cannot be two cells c in θ in which both an outside and an inside brick start at c. Again let us draw the bi-brick cycle θ as pictured in Figure 14 where the outside brick of size b starts in cell 1 and we label the cells clockwise starting a cell 1. Thus the outside of the cells $b+1,\ldots,n$ are covered by outside bricks of size 1. We claim that the inside brick of size d cannot start in cell 1. That is, if the outside brick of size d starts at cell 1, then cell 1 contributes a B to $W(\theta)$. But then the inside of cell $d+1$ must be covered by an inside brick of size 1 since $d\leq n-1$. But since $b\leq d$, the outside of cell $d+1$ must be covered by an outside brick of size 1 and hence cell $d+1$ would also contribute a B to $W(\theta)$ which would violate our assumption that $W(\theta)$ has at most one B. Similarly the inside brick of size d cannot start at any cells $b+2,\ldots,n$. That is, suppose the inside brick of size d starts at cell c_1 where $b+1<c_1\leq n$. Then the cell $b+1$ must be covered by an inside brick of size 1 so that cell $b+1$ would contribute a B to $W(\theta)$. But cell c_1 is covered by a outside brick of size 1 and

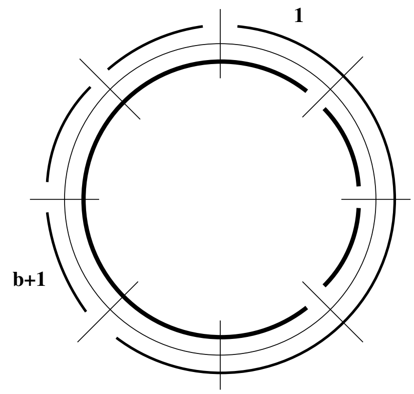

Figure 14: A bi-brick cycle θ of type $((1^{a-1}, b), (1^{c-1}, d))$ such that $W(\theta)$ has no B's.

hence cell c_1 would contribute a second B to $W(\theta)$. Thus the inside brick of size d must start somewhere between cell 2 and cell $b+1$. An alternative way to say this is that if we consider the block of c inside bricks of size 1 read in a clockwise manner, then it must be the case that this block ends in one of the cells $1, \ldots, b$. We claim that the block of c inside bricks of size 1 cannot end in any of the cells $1, \ldots, c-1$ since otherwise the cell n and cell 1 would be covered by inside bricks of size 1 and yet both the cells are the start of outside bricks which would imply that $W(\theta)$ contain two B's. It follows that the block of c inside bricks can end in cells $c, c+1, \ldots, b$ and hence there are $b-c+1$ possibilities for θ. Thus the $(1^a, b), (1^c, d)$-bi-brick permutations σ where there is a single bi-brick cycle of type $((1^a, b), (1^c, d))$ contribute a total of $(-1)^{a+c+2}(-1)^{n-1}(b-c+1) = (-1)^{a+c+n+1}(b-c+1)$ to $M(e, m)_{(1^a,b),(1^c,d)}$. Hence we have shown that

$$\begin{aligned} M(e,m)_{(1^a,b),(1^c,d)} &= (-1)^{a+c+n+1}(b-c+1) + (-1)^{a+c+n}(b-c) \\ &= (-1)^{a+c+n+1}. \end{aligned}$$

Our final result will show that if λ and μ are partitions of n with two or fewer columns, i.e. partitions of the form $(1^s, 2^t)$, then $M(e, m)_{\lambda,\mu} = 0$ if $m \geq 5$.

Theorem 9 *Suppose $\lambda = (1^a, 2^b)$ and $\mu = (1^c, 2^d)$ are partitions of n. Then*

(1) if $n = 2s$ is even, then $M(e, m)_{\lambda,\mu} = 0$ unless $\lambda = \mu = (2^s)$ or $\{\lambda, \mu\} = \{(1^2, 2^{s-1}), (2^s)\}$,

(2) if $n = 2s+1$ is odd, then $M(e, m)_{\lambda,\mu} = 0$ unless $\lambda = \mu = (1, 2^s)$,

(3) $M(e,m)_{(2^s),(2^s)} = \begin{cases} -2 & \text{if } s = 1, \\ 1 & \text{if } s = 2, \\ 0 & \text{if } s > 2. \end{cases}$

(4) $M(e,m)_{(1^2,2^{s-1}),(2^s)} = \begin{cases} 1, & \text{if } s = 1 \\ 0 & \text{if } s \geq 2. \end{cases}$

(5) $M(e,m)_{(1,2^s),(1,2^s)} = \begin{cases} 1 & \text{if } s \leq 1, \\ 0 & \text{if } s > 1. \end{cases}$

Proof. Parts (1) and (2) follow immediately from Theorem 3. That is, if $n = 2s$, the condition $\ell(\lambda) + \ell(\mu) \leq n + 1 = 2s + 1$ is satisfied only when $\lambda = \mu = (2^s)$ or when $\{\lambda, \mu\} = \{(1^2, 2^{s-1}), (2^s)\}$. Similarly when $n = 2s+1$, the condition $\ell(\lambda)+\ell(\mu) \leq n+1 = 2s+2$ is satisfied only when $\lambda = \mu = (1, 2^s)$. Part (3) is just a special case of Theorem 6.

For part (4), it is easy to check that there is one primitive bi-brick permutation of type $((1^2), (2)$ whose sign is -1 so that $M(e,m)_{(1^2),(2)} = 1$. If $s \geq 2$, then $(2^s) <_D (s-1, s+1) = (1^2, 2^{s-1})'$ so that $M(e,m)_{(1^2,2^{s-1}),(2^s)} = 0$. Similarly for part (5), it is easy to check by direct calculation that $M(e,m)_{(1),(1)} = 1$ and $M(e,m)_{(1,2),(1,2)} = 1$. For $s \geq 2$, $(1, 2^s) <_D (s, s+1) = (1, 2^s)'$ so that $M_{(1,2^s),(1,2^s)} = 0$.

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