On subgraphs induced by transversals in vertex-partitions of graphs

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Abstract

For a fixed graph H on k vertices, we investigate the graphs, G, such that for any partition of the vertices of G into k color classes, there is a transversal of that partition inducing H. For every integer $k \geq 1$, we find a family $\mathcal{F}$ of at most six graphs on k vertices such that the following holds. If $H \notin \mathcal{F}$, then for any graph G on at least $4k-1$ vertices, there is a k-coloring of vertices of G avoiding totally multicolored induced subgraphs isomorphic to H. Thus, we provide a vertex-induced anti-Ramsey result, extending the induced-vertex-Ramsey theorems by Deuber, Rödl et al.

1 Introduction

Let $G = (V, E)$ be a graph. Let $c : V(G) \to [k]$ be a vertex-coloring of G. We say that G is *monochromatic* under c if all vertices have the same color and we say that G is *rainbow* or *totally multicolored* if all vertices of G have distinct colors. Investigating the existence of monochromatic or rainbow subgraphs isomorphic to H in vertex-colored graphs, the following questions naturally arise:

Question M: Can one find a small graph G such that in any vertex-coloring of G with fixed number of colors, there is an induced **monochromatic** subgraph isomorphic to H?

Question M-R: Can one find a small graph G so that any vertex coloring of G contains an induced subgraph isomorphic to H which is either **monochromatic** or **rainbow**?

Question R: Can one find a large graph G such that any vertex-coloring of G in a fixed number of colors has a **rainbow** induced subgraph isomorphic to H?

The first two questions are well-studied, e.g., [7], [8], [2]. Together with specific bounds given by Brown and Rödl [3], the following is known:

Theorem 1 (Vertex-Induced Graph Ramsey Theorem). *For any graph H, any integer t, $t \geq 2$, there exists a graph $R_t(H)$ such that if the vertices of $R_t(H)$ are colored with t colors then there is an induced subgraph of $R_t(H)$ isomorphic to H which is monochromatic. Let the smallest order of such a graph be $r_t(H)$. There are constants C_1, C_2 such that*

$$C_1 k^2 \leq \max\{r_t(H)) : |V(H)| = k\} \leq C_2 k^2 \log_2 k.$$

The topic of the second question belongs to the area of "canonization", see, for example, a survey by Deuber [5]. The following result of Eaton and Rödl [6] provides specific bounds for vertex-colorings of graphs.

Theorem 2 (Vertex-Induced-Canonical Graph Ramsey Theorem). *For any graph H, there is a graph $R_{can}(H)$ such that if $R_{can}(H)$ is vertex-colored then there is an induced subgraph of $R_{can}(H)$ isomorphic to H which is either monochromatic or rainbow. Let the smallest order of such a graph be $r_{can}(H)$. There is a constant C such that*

$$Ck^3 \leq \max\{r_{can}(H) : |V(H)| = k\} \leq k^4 \log k.$$

In this paper we initiate the study of Question R when the number of colors in the coloring corresponds to the number of vertices in a graph H. We call a vertex-coloring using exactly k colors *a k-coloring*. In this manuscript we consider only simple graphs with no loops or multiple edges.

Definition 3. For a fixed graph H on k vertices, let $f(H)$ be the maximum order of a graph G such that any coloring of $V(G)$ in k colors has an induced rainbow subgraph isomorphic to H. Note that $f(H) \geq k$.

Since a vertex-coloring of G gives a partition of vertices, finding a rainbow induced copy of a graph H corresponds to finding a copy of H induced by a transversal of this partition. Note that $f(H) = \infty$ if and only if for any $n_0 \in \mathbb{N}$ there is $n > n_0$ and a graph G on n vertices such that any k-coloring of vertices of G produces a rainbow induced copy of H. The results we obtain have a flavor quite different from of those answering Questions M and M-R. In particular, there are few exceptional graphs for which function f is not finite.

Let Λ be a graph on 4 vertices with exactly two adjacent edges and one isolated vertex. Let K_n, E_n, S_n be a complete graph, an empty graph and a star on n vertices, respectively. We define a class of graphs

$$\mathcal{F} = \{K_n, E_n, S_n, \overline{S}_n, \Lambda, \overline{\Lambda} : n \in \mathbb{N}\}.$$

Note that any graph on at most three vertices is in $\mathcal{F}$.

Theorem 4. *Let H be a graph on k vertices. If $H \in \mathcal{F}$ then $f(H) = \infty$, otherwise $f(H) \leq 4k-2$.*

Corollary 1. *Let H be a graph on k vertices, $H \notin \mathcal{F}$. For every graph G on at least $4k-1$ vertices there is a k-vertex coloring of G avoiding rainbow induced subgraphs isomorphic to H.*

2 Proof of Theorem 4

Let H be a graph on k vertices and let $\mathcal{I}n(H)$ be the set of graphs on at most $k-1$ vertices which are isomorphic to **induced** subgraphs of H.
One of our tools is the following theorem of Akiyama, Exoo and Harary, later strengthened by Bosák.

Proposition 1 ([1], [4]). *Let G be a graph on n vertices such that all induced subgraphs of G on t vertices have the same size. If $2 \leq t \leq n-2$ then G is either a complete graph or an empty graph.*

Proposition 2. *Let H be a graph on k vertices. If G is a graph on at least k vertices such that G has an induced subgraph on at most $k-1$ vertices not isomorphic to any graph from $\mathcal{I}n(H)$, then there is a k-coloring of G with no rainbow induced copy of H.*

Proof. Let a set, S, of at most $k-1$ vertices in G induce a graph not in $\mathcal{I}n(H)$. Color the vertices of S with colors $1, 2, \dots, |S|$ and assign all colors from $\{|S|+1, \dots, k\}$ to other vertices arbitrarily. Any rainbow subgraph of G on k vertices must use all of the vertices from S, but these vertices do not induce a subgraph of H. Therefore there is no rainbow induced copy of H in this vertex-coloring of G. □

We call a graph G, H-good if any induced subgraph of G on at most $|V(H)|-1$ vertices is isomorphic to some graph from $\mathcal{I}n(H)$.

Corollary 2. *Let $H \notin \mathcal{F}$ be a regular graph on k vertices. Then $f(H) = k$.*

Proof. Note that each graph in $\mathcal{I}n(H)$ on $k-1$ vertices has the same size. Let G be a graph on $k+1$ vertices. By Proposition 2 we can assume that G is H-good. Thus all $(k-1)$-subgraphs of G have the same size. It follows from Proposition 1 that G is either a complete or an empty graph. Therefore G does not contain H as an induced subgraph and any k-coloring of G does not result in a rainbow induced copy of H. □

We use the following notations for a graph $H = (V, E)$. Let $\alpha(H)$ be the size of the largest independent set of H, let $\omega(H)$ be the order of the largest complete subgraph of H. Let $\delta(H), \Delta(H)$ be the minimum and the maximum degrees of H respectively. For two vertices x, y, such that $\{x, y\} \notin E$, $e = \{x, y\}$ is a non-edge, for a vertex v, $d(v)$ and $cd(v)$ are the degree and the codegree of v, i.e., the number of edges and non-edges incident to v, respectively. A $(k-1)$-subgraph of H is an induced subgraph of H on $k-1$ vertices. For all other definitions and notations we refer the reader to [9].

Next several lemmas provide some preliminary results for the proof of Theorem 4. We consider the graph H according to the following cases:

a) $\alpha(H) = k-1$ or $w(H) = k-1$,

b) $2 \leq \delta(H) \leq \Delta(H) \leq k-3$,

c) $\delta(H) \leq 1$ or $\Delta(H) \geq k-2$.

The cases **a)** and **b)** give us easy upper bounds on $f(H)$, the case **c)** requires some more delicate analysis. The first lemma follows immediately from the definition of function f.

Lemma 1. $f(H) = f(\overline{H})$.

Lemma 2. *Let H be a graph on k vertices such that $2 \leq \delta(H) \leq \Delta(H) \leq k-3$. Then $f(H) \leq 2k-6$.*

Proof. If a graph G has a vertex of degree at least $k-2$ or of codegree at least $k-3$, then G contains a subgraph on $k-1$ vertices not in $\mathcal{I}n(H)$ and by Proposition 2, there is a k-coloring of G avoiding rainbow induced copies of H. Therefore, if any k-coloring of G contains a rainbow induced copy of H then for $v \in V(G)$ we have $|V(G)| \leq d(v) + cd(v) + 1 \leq (k-3) + (k-4) + 1 = 2k-6$. □

Lemma 3. *Let $H \notin \mathcal{F}$ be a graph on k vertices, such that $\alpha(H) = k-1$ or such that $w(H) = k-1$. Then $f(H) = k$, for $k \geq 5$ and $f(H) = k+2$ for $k = 4$.*

Proof. Let H be a graph on k vertices with $\alpha(H) = k-1$, $H \notin \mathcal{F}$. Then H is a disjoint union of a star with k' edges and $k-k'-1$ isolated vertices, $1 \leq k' \leq k-2$.

Assume first that $k \geq 5$. Let G be a graph on n vertices, $n \geq k+1$. If G has two nonadjacent edges e, e', or a triangle, or no edges at all, by Proposition 2 there is a coloring of G avoiding a rainbow induced copy of H. Therefore, G must be a disjoint union of a star S with l edges and $n-l-1$ isolated vertices, $1 \leq l \leq n-1$. Then either $l > k'$ or $n-l-1 > k-k'-1$. If $l > k'$, we can use colors from $\{1, \ldots, k'+1\}$ on the vertices of S and colors from $\{k'+2, \ldots, k\}$ on isolated vertices of G. If $n-l-1 > k-k'-1$ then we can use colors from $\{1, \ldots, k-k'\}$ on isolated vertices of G and other colors on the vertices of S. These colorings do not contain an induced rainbow subgraph isomorphic to H.

Let $k = 4$. Since $H \notin \mathcal{F}$, we have that H is a disjoint union of an edge and two vertices. If a graph G has two adjacent edges e, e', we are done by Proposition 2. Otherwise, G is a vertex disjoint union of isolated edges and vertices. Lets color G so that the adjacent vertices get the same color. This coloring does not contain an induced rainbow copy of H. Moreover, if $|V(G)| \geq 7$ then there is such a coloring using 4 colors. Thus, $f(H) < 7$. On the other hand, any 4-coloring of a graph G consisting of three disjoint edges gives a rainbow induced H, thus $f(H) \geq 6$. We have then that $f(H) = 6$.

If $w(H) = k-1$, Lemma 1 implies the same result. □

Lemma 4. *Let H be a graph on k vertices, $H \notin \mathcal{F}$, $\alpha(H) < k-1$, $\omega(H) < k-1$. If H has at least two nontrivial components then $f(H) \leq 2k-1$.*

Proof. Note that if H has at least two nontrivial components and $\delta(H) \geq 2$, then we are done by Lemma 2. Let m be the largest order of a connected component in H. Let G be a graph on $n \geq 2k$ vertices. We can assume by Proposition 2 that G is H-good. Then there is no component in G of order larger than m. Moreover, since H is contained in G as an induced subgraph, all components of H of order m appear in G as connected components. Let $F_1, F_2, \ldots, F_t$ be components of G of order m, let $x_i, y_i \in V(F_i)$, $i = 1, \ldots, t$. Assign color i to both vertices x_i and y_i, $i = 1, \ldots, t$, and assign all colors from $\{t+1, \ldots, k\}$ to other vertices arbitrarily. Since $k \leq n/2$, $t \leq n/2$, we have that $t + k \leq n$ and such coloring exists. Consider a copy of H in G. It contains at least one of the components of order m, thus it has at least two vertices of the same color. Therefore there is no rainbow induced subgraph of G isomorphic to H in this coloring. $\square$

Lemma 5. *Let $H \notin \mathcal{F}$ be a graph on k vertices such that $\delta(H) \leq 1$, $\alpha(H) < k-1$ and $w(H) < k-1$. Then $f(H) \leq 4k-2$.*

Proof. Let H be a graph on k vertices, $H \notin \mathcal{F}$ such that $\alpha(H) < k-1$ and $\omega(H) < k-1$. Let G be a graph on $n \geq 4k-1$ vertices. We can assume by Proposition 2 that G is H-good.

Claim 0. If all graphs from $\mathcal{I}n(H)$ on $k-1$ vertices with a spanning star are isomorphic or do not exist, then $\Delta(G) \leq k-1$. If all graphs from $\mathcal{I}n(H)$ on $k-1$ vertices with an isolated vertex are isomorphic or do not exist, then $\Delta(\overline{G}) \leq k-1$.

To prove the Claim, assume that all graphs from $\mathcal{I}n(H)$ on $k-1$ vertices with a spanning star are isomorphic. Consider S, a neighborhood of a vertex v of maximum degree in G. Then, all subsets of S of size $k-2$ induce isomorphic graphs. Therefore, if $|S| \geq k$ we have, by Proposition 1, that S induces an empty or a complete graph on at least k vertices, a contradiction. Thus, $|S| = \Delta(v) \leq k-1$. If there is no graph from $\mathcal{I}n(H)$ on $k-1$ vertices with a spanning star and G has a vertex v of degree at least $k-2$, then v and $k-2$ of its neighbors induce a subgraph with a spanning star on $k-1$ vertices, a contradiction. The second statement can be proved in the same manner, concluding the proof of Claim 0.

Case 1. $\delta(H) = 0$.

We can assume by Lemma 4 that H has exactly one nontrivial component. Observe that either there is no $(k-1)$-vertex subgraph of H with a spanning star, or all such subgraphs are isomorphic. Thus, by Claim 0, $\Delta(G) \leq k-1$. Consider two adjacent vertices of G, u and v. There is a set T of vertices, $|T| \geq n-2-2(k-1) = n-2k$, such that neither u nor v is adjacent to any vertex in T. Observe also, that since G has no independent set of size $k-1$, the largest size of an independent set induced by vertices of T is at most $k-2$. Let $T' \subset T$ induce the largest independent set in $G[T]$. Then, for each

$x \in T \setminus T'$, there is $x' \in T'$ such that $xx' \in E(G)$. Since $|T \setminus T'| \geq n-2k-k+2 \geq k$, it is clear that we can build a subgraph of $G[T]$ on $k-3$ vertices with no isolated vertices using some vertices from $T \setminus T'$ and some of their neighbors from T' (provided that $k \geq 5$). Together with uv it forms a subgraph on $(k-1)$ vertices with at least two nontrivial components and no isolated vertices. But each disconnected subgraph of H on $k-1$ vertices has an isolated vertex, a contradiction.

Let $k=4$. Since $\delta(H)=0$ and $\alpha(H)<3$, H must be a disjoint union of an isolated vertex and K_3. But then $H \in \mathcal{F}$, which is impossible.

Case 2. $\delta(H)=1$.

Lets call the vertices of degree 1, leaves. We can assume that H is connected by Lemma 4.

Case 2.1. All leaves in H have a common neighbor, v.

Then all $(k-1)$-subgraphs of H which have an isolated vertex are isomorphic to $H-v$, thus, by Claim 0, we have that $\Delta(\overline{G}) \leq k-1$. Note that all $(k-1)$-subgraphs of H having two adjacent vertices of degree $k-2$ are either isomorphic or do not exist. Consider x, y, two adjacent vertices of G. Since the codegree of each vertex is at most $k-1$ we have that there is a set S of vertices, $|S| \geq n-2-2(k-1) \geq k-1$, such that each vertex of S is adjacent to x and to y. Thus, all $(k-3)$-subsets of S induce isomorphic graphs, and S must induce a complete or an empty graph on at least $k-1$ vertices by Proposition 1, a contradiction.

Case 2.2. There are at least two leaves in H which do not have a common neighbor.

It is easy to see that either H does not have a vertex of degree $k-2$ or all subgraphs of H on $k-1$ vertices with a spanning star are isomorphic. Then, by Claim 0, $\Delta(G) \leq k-1$. Consider a set S of vertices of G inducing H and let $S' \subseteq S$ correspond to the set of leaves in H. Let l be the largest number of leaves in H having a common neighbor, let $x(l)$ be the number of distinct vertices in H each adjacent to l leaves.

If $l \leq 2$ or ($l=3$ and $x(l)=1$) then all $(k-1)$-subgraphs of H with at least three isolated vertices either do not exist or isomorphic. Consider three pairwise nonadjacent vertices w, w', w'' in G. Since $\Delta(G) \leq k-1$, there are at least $n-3-3(k-1) \geq k-1$ vertices of G non-adjacent to either of w, w', w''. This is either impossible, or these vertices must induce an independent set or a clique, a contradiction.

Thus, we can assume that there are at least two distinct vertices in H adjacent to at least three leaves each. Let $u, u' \in S$ correspond to these vertices, and let $s, s' \in N(u) \cap S$, $s'' \in N(u') \cap S$. Since $V \setminus S$ has size at least $k-1$, it does not induce an independent set; thus there is an edge vv', $v, v' \in V \setminus S$. If v, v' are not adjacent to any vertex in S, then $G[S \setminus \{s, s', s''\} \cup \{v, v'\}]$ is a $(k-1)$-subgraph of G with an isolated edge, no isolated vertices and with $|S'|-1$ leaves. This is impossible, since each $(k-1)$-subgraph of H with an isolated edge and no isolated vertices has at least $|S'|$ leaves. If v or v' is adjacent to some vertex $q \in S$ (we can always assume that $q \notin \{s, s', s''\}$ by choosing s, s', s'' accordingly), then $G[S \setminus \{s, s', s''\} \cup \{v, v'\}]$ is a connected $(k-1)$-subgraph of G

with at most $|S'|-2$ leaves. This is impossible since each connected subgraph of H has at least $|S'|-1$ leaves. $\square$

Now, we can quickly complete the proof of the main theorem using the result about the special graph Λ proven in the next section.

Proof of Theorem 4. If $H=S_k$, then any k-coloring of S_n, $n \geq k$ induces a rainbow H. If $H=K_k$, then any k-coloring of K_n, $n \geq k$ induces a rainbow H. Using Proposition 3 for a graph Λ and the fact that $f(H)=f(\overline{H})$ we have now established that for any $H \in \mathcal{F}$, $f(H)=\infty$.

Now, assume that H is a graph on k vertices, $H \notin \mathcal{F}$. If $\alpha(H)=k-1$ or $\omega(H)=k-1$, then, by Lemma 3, $f(H) \leq k+2$. If $\alpha(H)<k-1$ and $\omega(H)<k-1$ then at least one of the following holds:
1) $2 \leq \delta(H) \leq \Delta(H) \leq k-3$, and by Lemma 2, $f(H) \leq 2k-6$,
2) $\delta(H) \leq 1$, and by Lemmas 4 and 5, $f(H) \leq 4k-2$,
3) $\Delta(H) \geq k-2$, by 2) and Lemma 1, $f(H) \leq 4k-2$. $\square$

3 Treating Λ

Definition 5. Let $G(m)=(V,E)$,

$$V=\{v(i,j): \quad 1 \leq i \leq 7, \quad 1 \leq j \leq m\},$$

$$E=\{v(i,j)v(i+1,k): \quad 1 \leq j,k \leq m,\ j \neq k,\ 1 \leq i \leq 7\} \quad \cup$$
$$\{v(i,j)v(i+3,j): \quad 1 \leq j \leq m,\ 1 \leq i \leq 7\},$$

addition is taken modulo 7.

We have $V=V_1 \cup \cdots \cup V_7=L_1 \cup \cdots \cup L_m$, where $V_i=\{v(i,j): \quad 1 \leq j \leq m\}$, $1 \leq i \leq 7$, $L_j=\{v(i,j): \quad 1 \leq i \leq 7\}$, $1 \leq j \leq m$. We shall refer to V_is as vertex parts and L_is as vertex layers. The edge-set of $G(m)$ can be constructed by first taking all the edges between consecutive (in cyclic order) V_is, $i=1,\ldots,7$ then removing the edges induced by each layer L_j, $j=1,\ldots,m$, and finally adding, for each $j=1,\ldots,m$, a new 7 cycle induced by L_j, see Figure 1. Note that $G(1)$ is isomorphic to a 7-cycle, $G(2)$ has a spanning 14-cycle, and can be drawn as in the Figure 2.

Proposition 3. *For any positive integer m and any coloring of $V(G(m))$ into 4 colors, there is a rainbow induced subgraph of G isomorphic to Λ.*

Proof. We prove the statement, for $m=1,2,3$ and for $m>3$ use induction. This is a somewhat tedious but straightforward case analysis.

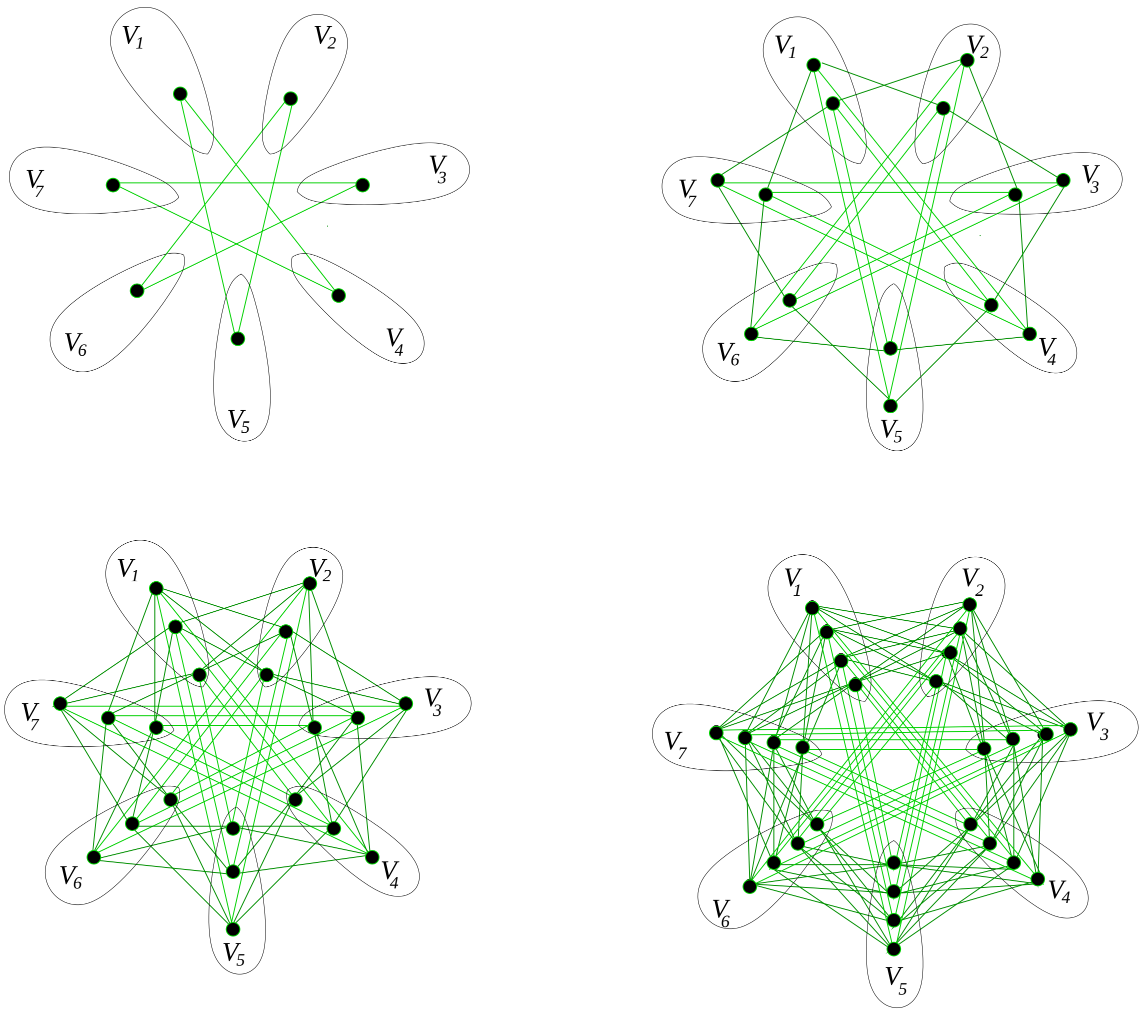

Figure 1: $G(1)$, $G(2)$, $G(3)$ and $G(4)$

Claim 1. Any coloring of $G(1)$ in 4 colors contains an induced rainbow Λ.

Let $G(1)$ have vertices $x_1, \ldots, x_7$ and edges $x_i x_{i+1}$, $i = 1, \ldots, 7$, addition taken modulo 7. Assume that there is a 4-coloring c with no induced rainbow Λ. First observe that any 4-coloring of C_7 must have three consecutive vertices with distinct colors, say $c(x_i) = i$, for $i = 1, 2, 3$. Then $c(x_5) \neq 4$, $c(x_6) \neq 4$, thus, without loss of generality $c(x_4) = 4$. Note that then $c(x_7) \neq 1$, $c(x_7) \neq 3$. If $c(x_7) = 4$ then x_6 must have color 3, and there is no color available for x_5. If $c(x_7) = 2$ then $c(x_6) = 2$ and there is no available color for x_5.

Claim 2. Any coloring of $G(2)$ in 4 colors contains an induced rainbow Λ.

Note that $G(2)$ can be drawn as C_{14} with chords as in Figure 2. Let the vertices of $G(2)$ be $x_1, \ldots, x_{14}$ in order on the cycle and let the edges be x_i, x_{i+1}, x_{i+4}, $i = 1, \ldots, 14$, where addition is taken modulo 14. We shall use the fact that the following sets of vertices

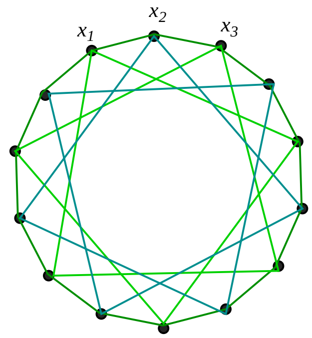

Figure 2: Different drawing of $G(2)$

induce C_7 and thus cannot use all 4 colors:

$$\{x_i, \quad x_{i+2}, x_{i+3}, x_{i+4}, \quad x_{i-2}, x_{i-3}, x_{i-4}\},$$

$i = 1, \ldots, 14$ and addition is taken modulo 14. We shall also use an easy fact that it is impossible to have a 4-colored C_4 in $G(2)$.

Case 1. There are three consecutive vertices, using distinct colors, say $c(x_i) = i$, $i = 1, 2, 3$.

Then, considering all induced cycles of length 7 containing these three vertices, we see that the only vertices which could have color 4 are x_4, x_6, x_{14} or x_{12}.

Case 1.1. $c(x_4) = 4$.

Consider vertex x_8. If $c(x_8) = 1$ then $\{x_2, x_3, x_4, x_6, x_8, x_9, x_{10}\}$ induces a C_7 using 4 colors. If $c(x_8) = 2$ then $\{x_1, x_3, x_4, x_8\}$ induces a rainbow Λ. If $c(x_8) = 3$ then $\{x_{14}, x_1, x_2, x_4, x_6, x_7, x_8\}$ induces a C_7 using 4 colors. Thus x_8 cannot be assigned any color and this case is impossible.

Case 1.2. $c(x_6) = 4$.

Consider vertex x_7. If $c(x_7) = 1$ then $\{x_2, x_3, x_6, x_7\}$ is a 4-colored C_4. If $c(x_7) = 2$ then $\{x_1, x_3, x_7, x_6\}$ induces a rainbow Λ. If $c(x_7) = 3$ then $\{x_{14}, x_1, x_2, x_4, x_6, x_7, x_8\}$ induces a C_7 using 4 colors. Therefore x_7 cannot be assigned a color and this case is impossible as well.

By symmetry $c(x_{14}) \neq 4$ and $c(x_{12}) \neq 4$, so there is no vertex colored 4, a contradiction.

Case 2. There are no three consecutive vertices using distinct colors.

Then, without loss of generality, there are consecutive vertices $x_i, x_{i+1}, \ldots, x_j$ such that $c(x_i) = a, c(x_j) = b$ and $c(x_m) = c$, for $i < m < j$, such that a, b, c are distinct. Consider smallest such set of vertices and assume that $i = 1$, $a = 2$, $b = 3$, $c = 1$. Then clearly, $j \geq 4$, moreover $j \leq 5$ since otherwise there is a smaller such set.

Case 2.1. $j = 4$.

By considering all induced C_7 containing vertices of colors $1, 2, 3$ from $\{x_1, x_2, x_3, x_4\}$, and using the fact that x_{14} and x_5 cannot have color 4 without creating three consecutive vertices of distinct colors, we see that the only vertices which could have color 4 are x_9 and x_{10}. If $c(x_{10}) = 4$ then consider vertex x_{14}. If $c(x_{14}) = 3$ or 4 then x_{14}, x_1, x_2 are three consecutive vertices using distinct colors. If $c(x_{14}) = 2$ then $\{x_{14}, x_{10}, x_4, x_2\}$ induces a rainbow Λ. Thus $c(x_{14}) = 1$. Consider x_5: $c(x_5) \neq 4$ and $c(x_5) \neq 2$ since otherwise there are three consecutive vertices of distinct colors. If $c(x_5) = 3$ then $\{x_2, x_1, x_5, x_9\}$ induces a rainbow Λ. If $c(x_5) = 1$ then $\{x_4, x_5, x_1, x_9\}$ induces a rainbow Λ. Thus this case is impossible. If $c(x_9) = 4$ we arrive at a contradiction by symmetry.

Case 2.2. $j = 5$.

By considering all induced C_7 containing vertices of colors $1, 2, 3$ from $\{x_1, \ldots, x_5\}$ we see that the only vertex which might, and thus must have color 4 is x_{10}. But then $\{x_{10}, x_1, x_2, x_5\}$ induces a rainbow Λ, a contradiction.

Claim 3. Any coloring of $G(3)$ in 4 colors contains an induced rainbow Λ.

Let c be a coloring of $G(3)$ using colors $1, 2, 3, 4$ and containing no induced rainbow copy of Λ. If there is a subgraph of $G(3)$ isomorphic to $G(2)$ and using four colors, there is a rainbow induced Λ by Claim 2. Therefore, we can assume that each vertex layer of $G(3)$ has a color used only on its vertices and on no vertex of any other layer. In particular, assume that color i is used only in L_i, $i = 1, 2, 3$. So, L_1 uses colors from $\{1, 4\}$, L_2 uses colors from $\{2, 4\}$, and L_3 uses colors from $\{3, 4\}$.

If there is a part, say V_1, using colors $1, 2, 3$, then it is easy to see that none of the vertices of V_2 could have color 4 and moreover V_2 must use all three colors $1, 2, 3$ again, in respective layers. This shows that in this case all sets V_i, $i = 1, \ldots, 7$ must use only colors $1, 2, 3$ and there is no vertex of color 4, a contradiction. Since there is no part V_i, $i = 1, \ldots, 7$ using all colors $1, 2, 3$, each part must have color 4 on some vertex.

Assume that there is a part, say V_1, having exactly one vertex of color 4. Without loss of generality, we have $c(v(1, 1)) = 4, c(v(1, 2)) = 2, c(v(1, 3)) = 3$, then $c(v(7, 1)) = c(v(2, 1)) = 4$. Moreover, $c(v(i, 1)) \neq 1$ for $i = 3, 4, 5, 6$, otherwise one of these vertices together with either $\{v(2, 1), v(1, 2), v(1, 3)\}$ or with $\{v(7, 1), v(1, 2), v(1, 3)\}$ induces a rainbow Λ. Therefore, there is no vertex of color 1 in the graph, a contradiction.

Thus, each part V_i has at least two vertices of color 4. Then, it is easy to see that there is always a rainbow induced Λ in such a coloring of $G(3)$, a contradiction.

Induction step. Assume that $m \geq 4$. If there is a vertex layer L_i such that $G[V - L_i]$ uses all 4 colors, then, since $G[V - L_i]$ is isomorphic to $G(m - 1)$, there is a rainbow induced subgraph isomorphic to Λ. Thus we can assume that each layer $L_1, L_2, \ldots, L_m$ uses a color not present in other layers. It is possible only if $m = 4$, in which case all vertices of each layer have the same color. We can assume that all vertices of layer L_i have color i, $i = 1, 2, 3, 4$. But then it is easy to see that there is an induced rainbow Λ in this coloring. $\square$

It is interesting to see that if G is a bipartite graph then there is always a coloring of $V(G)$ in 4 colors avoiding induced rainbow Λ. Indeed, if G is a complete bipartite graph, it does not have any induced copies of Λ, so any 4-coloring will work. Thus, we can assume that there are two nonadjacent vertices from different partite sets A and B, $x \in A$ and $y \in B$. Let $c(x) = 3$, $c(y) = 4$, $c(N(x)) = 1, c(N(y)) = 2$, $c(A \setminus (N(y) \cup \{x\})) = 1$ and $c(B \setminus (N(x) \cup \{y\})) = 2$. It is easy to see that this coloring does not have a rainbow induced Λ.

Concluding Remark: We have proven that for any graph $H \notin \mathcal{F}$ on k vertices and any graph G on $4k-1$ vertices there is a coloring of G in k colors avoiding rainbow induced subgraph isomorphic to H. Together with definition of f, this implies that

$$k \leq \max\{f(H) : |V(H)| = k, H \notin \mathcal{F}\} \leq 4k - 2.$$

There are many classes of graphs for which $f(H) = k$, which follows, for example, from Proposition 2. We believe that the above upper bound could be improved to $2k-1$ with a more careful analysis, and, perhaps to $k+c$, where c is a constant. As far as the lower bound is concerned, we have only one example when $f(H) = k+2$ for $k = 4$, provided by Lemma 3. It will be very interesting to see constructions of graphs giving better lower bounds on f.

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