A Gessel–Viennot-Type Method for Cycle Systems in a Directed Graph

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Abstract

We introduce a new determinantal method to count cycle systems in a directed graph that generalizes Gessel and Viennot's determinantal method on path systems. The method gives new insight into the enumeration of domino tilings of Aztec diamonds, Aztec pillows, and related regions.

1 Introduction

In this article, we present an analogue of the Gessel–Viennot method for counting cycle systems on a type of directed graph we call a hamburger graph. A *hamburger graph* H is made up of two acyclic graphs G_1 and G_2 and a connecting edge set E_3 with the following properties. The graph G_1 has k distinguished vertices $\{v_1, \ldots, v_k\}$ with directed paths from v_i to v_j only if $i < j$. The graph G_2 has k distinguished vertices $\{w_{k+1}, \ldots, w_{2k}\}$ with directed paths from w_i to w_j only if $i > j$. The edge set E_3 connects the vertices v_i and w_{k+i} by way of edges $e_i : v_i \to w_{k+i}$ and $e'_i : w_{k+i} \to v_i$. (See Figure 1 for a visualization.) Hamburger graphs arise naturally in the study of Aztec diamonds, as explained in Section 5.

The Gessel–Viennot method is a determinantal method to count path systems in an acyclic directed graph G with k sources $s_1, \ldots, s_k$ and k sinks $t_1, \ldots, t_k$. A *path system* $\mathcal{P}$ is a collection of k vertex-disjoint paths, each one directed from s_i to $t_{\sigma(i)}$, for some permutation $\sigma \in S_k$ (where S_k is the symmetric group on k elements). Call a path system $\mathcal{P}$ *positive* if the sign of this permutation σ satisfies $\mathrm{sgn}(\sigma) = +1$ and *negative* if $\mathrm{sgn}(\sigma) = -1$. Let p^+ be the number of positive path systems and p^- be the number of negative path systems.

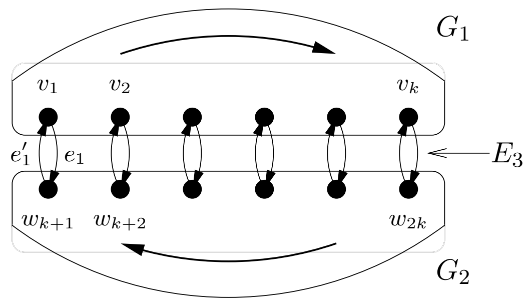

Figure 1: A hamburger graph

Corresponding to this graph G is a $k \times k$ matrix $A = (a_{ij})$, where a_{ij} is the number of paths from s_i to t_j in G. The result of Gessel and Viennot states that $\det A = p^+ - p^-$. The Gessel–Viennot method was introduced in [4, 5], and has its roots in works by Karlin and McGregor [8] and Lindström [10]. A nice exposition of the method and applications is given in the article by Aigner [1].

This article concerns a similar determinantal method for counting cycle systems in a hamburger graph H. A *cycle system* $\mathcal{C}$ is a collection of vertex-disjoint directed cycles in H. Let l be the number of edges in $\mathcal{C}$ that travel from G_2 to G_1 and let m be the number of cycles in $\mathcal{C}$. Call a cycle system *positive* if $(-1)^{l+m} = +1$ and *negative* if $(-1)^{l+m} = -1$. Let c^+ be the number of positive cycle systems and c^- be the number of negative cycle systems. Corresponding to each hamburger graph H is a $2k \times 2k$ block matrix M_H of the form

$$M_H = \begin{bmatrix} A & I_k \\ -I_k & B \end{bmatrix},$$

where in the upper triangular matrix $A = (a_{ij})$, a_{ij} is the number of paths from v_i to v_j in G_1 and in the lower triangular matrix $B = (b_{ij})$, b_{ij} is the number of paths from w_{k+i} to w_{k+j} in G_2. This matrix M_H is referred to as a *hamburger matrix.*

Theorem 1.1 (The Hamburger Theorem). *If H is a hamburger graph, then* $\det M_H = c^+ - c^-$.

A hamburger graph H is called *strongly planar* if there is a planar embedding of H that sends v_i to $(i, 1)$ and w_{k+i} to $(i, -1)$ for all $1 \leq i \leq k$, and keeps edges of E_1 in the half-space $y \geq 1$ and edges of E_2 in the half-space $y \leq -1$. This definition suggests that G_1 and G_2 are "relatively" planar in H, a stronger condition than planarity of H. Notice that when H is strongly planar, each cycle must use exactly one edge from G_2 to G_1. Hence, the sign of every cycle system is $+1$. This implies the following corollary.

Corollary 1.2. *If H is a strongly planar hamburger graph,* $\det M_H = c^+$.

The following simple example serves to guide us. Consider the two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, where $V_1 = \{v_1, v_2, v_3, v\}$, $V_2 = \{w_4, w_5, w_6, w\}$, $E_1 = \{v_1 \to v_2, v_2 \to v_3, v_1 \to v, v \to v_3\}$, and $E_2 = \{w_6 \to w_5, w_5 \to w_4, w_6 \to w, w \to w_4\}$. Our

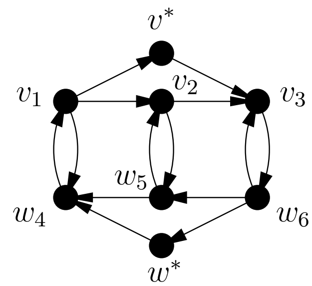

Figure 2: A simple hamburger graph H

hamburger graph H will be the union of G_1, G_2, and the edge set E_3. In this example, $k = 3$ and H is strongly planar. Figure 2 gives a graphical representation of H.

In this example, the hamburger matrix M_H equals

$$M_H = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 1 & 1 \end{bmatrix}.$$

The determinant of M_H is 17, corresponding to the seventeen cycle systems (each with sign $+1$) in Figure 3.

The graph that inspired the definition of a hamburger graph comes from the work of Brualdi and Kirkland [2], in which they give a new proof that the number of domino tilings of the Aztec diamond is $2^{n(n+1)/2}$. An *Aztec diamond*, denoted by AD_n, is the union of the $2n(n+1)$ unit squares with integral vertices (x, y) such that $|x|+|y| \leq n+1$. See Figure 4 for an example of an Aztec diamond, as well as an example of an Aztec pillow and a generalized Aztec pillow, described in the next paragraphs.

An *Aztec pillow*, as it was initially presented in [12], is also a rotationally symmetric region in the plane. On the top left boundary, however, the steps are composed of three squares to the right for every square up. Another definition is that Aztec pillows are the union of the unit squares with integral vertices (x, y) such that $|x + y| < n + 1$ and $|3y - x| < n + 3$. As with Aztec diamonds, we denote the Aztec pillow with $2n$ squares in each of the central rows by AP_n. In Section 6, we extend the notion of Aztec pillows having steps of length 3 to "odd pillows"—those that have steps that are of a constant odd length. The integral vertices (x, y) of the unit squares in q-pillows for q odd satisfy $|x + y| < n + 1$ and $|qy - x| < n + q$.

We introduce the idea of a *generalized Aztec pillow*, where the steps on all diagonals are of possibly different odd lengths. More specifically, a generalized Aztec pillow is a horizontally convex and vertically convex region such that the steps both up and down in each diagonal have an odd number of squares horizontally for every one square vertically.

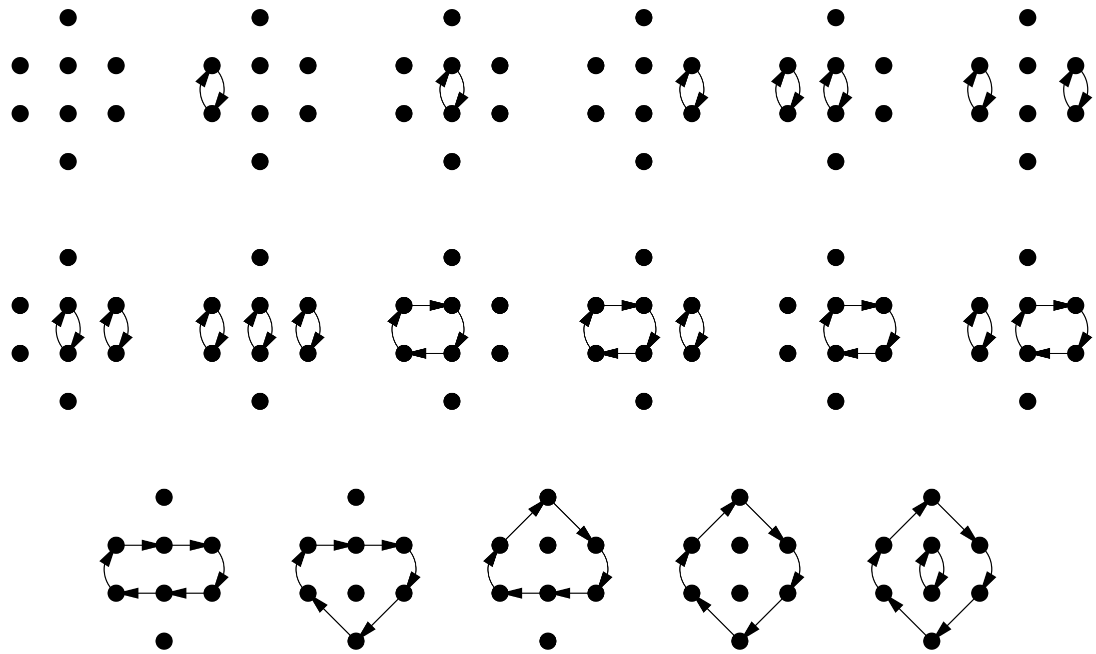

Figure 3: The seventeen cycle systems for the hamburger graph in Figure 2

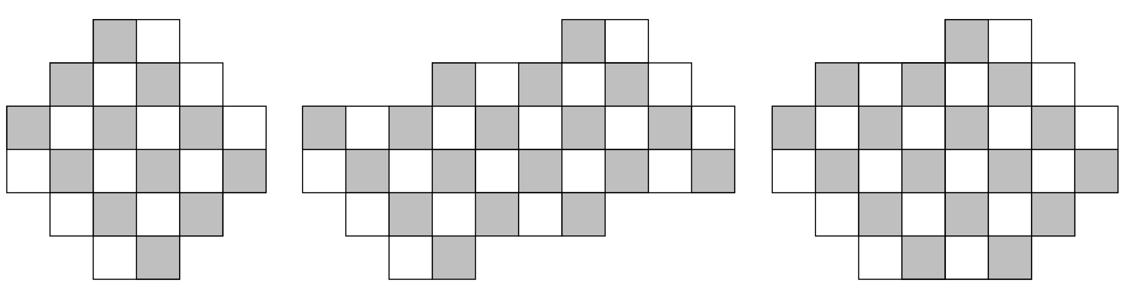

Figure 4: Examples of an Aztec diamond, an Aztec pillow, and a generalized Aztec pillow

A key fact that we will use is that any generalized Aztec pillow can be recovered from a large enough Aztec diamond by the placement of horizontal dominoes.

Brualdi and Kirkland prove the formula for the number of domino tilings of an Aztec diamond by creating an associated digraph and counting its cycle systems, manipulating the digraph's associated Kasteleyn–Percus matrix of order $n(n+1)$. To learn about Kasteleyn theory and Kasteleyn–Percus matrices, start with Kasteleyn's 1961 work [9] and Percus's 1963 work [11]. The Hamburger Theorem proves that we can count the number of domino tilings of an Aztec diamond with a much smaller determinant, of order $2n$. A Schur complement allows us to reduce the determinant calculation to one of order n. An analogous reduction in determinant size (from order $O(n^2)$ to order $O(n)$) occurs for all regions to which this theorem applies, including generalized Aztec pillows. In addition, whereas Kasteleyn theory applies only to planar graphs, there is no planarity restriction for hamburger graphs. For this reason, the Hamburger Theorem gives a new counting method for cycle systems in some non-planar graphs.

More recently, Eu and Fu present a new proof of the number of tilings of an Aztec diamond [3]. Their lattice-path-based proof also reduces to an $n \times n$ determinant but does not generalize to the case of Aztec pillows. This result is discussed further in Section 5.2.

In Section 2, we present an overview of the proof of the Hamburger Theorem, including the key lemmas involved. The necessary machinery is built up in Section 3 to complete the proof in Section 4. Section 5 presents applications of the Hamburger Theorem to Aztec diamonds, Aztec pillows, and generalized Aztec pillows. Section 6 concludes with a counterexample to the most natural generalization of the Hamburger Theorem and an extension of Propp's Conjecture on Aztec pillows.

2 Outline of the Proof of the Hamburger Theorem

2.1 The Hamburger Theorem

Like the proof of the Gessel–Viennot method, the proof of the Hamburger Theorem hinges on cancellation of terms in the permutation expansion of the determinant of M_H. In the proof, we must allow closed directed walks in addition to cycles. We must also allow *walk systems*, arbitrary collections of closed directed walks, since they can and will appear in the permutation expansion of the hamburger determinant. We call a walk system *simple* if the set of walks visits no vertex more than once. We call a cycle of the form $c : v_i \to w_{i+k} \to v_i$ a *2-cycle*.

Each signed term in the permutation expansion of the hamburger determinant is the contribution of many signed walk systems $\mathcal{W}$. Walk systems that are not cycle systems will all cancel out in the determinant expansion. We will show this in two steps. We start by considering walk systems that are not simple. If this is the case, one of the two following properties MAY hold.

Property 1. The walk system contains a walk that has a self-intersection.

Property 2. The walk system has two intersecting walks, neither of which is a 2-cycle.

The following lemma shows that the contributions of walk systems satisfying either of these two properties cancel in the permutation expansion of the determinant of M_H.

Lemma 2.1. *The set of all walk systems $\mathcal{W}$ that satisfy either Property 1 or Property 2 can be partitioned into equivalence classes, each of which contributes a net zero to the permutation expansion of the determinant of M_H.*

The proof of Lemma 2.1 uses a generalized involution principle. Walk systems cancel in families based on the their "first" intersection point.

The remainder of the cancellation in the determinant expansion is based on the concept of a minimal walk system; we motivate this definition by asking the following questions. What kind of walk systems does the permutation expansion of the hamburger determinant generate, and how is this different from our original notion of cycle systems that we wanted to count in the introduction? The key difference is that the same collection of walks can be generated by multiple terms in the determinantal expansion of M_H; whereas, we would only want to count it once as a cycle system. This redundancy arises when the walk visits three distinguished vertices in G_1 without passing via G_2 or vice versa. We illustrate this notion with the following example.

Consider the second cycle system in the third row of Figure 3, consisting of one solitary directed cycle. Since this cycle visits vertices v_1, v_2, v_3, w_6, and w_4 in that order, it contributes a non-zero weight in the permutation expansion of the determinant corresponding to the term (12364) in S_6. Notice that this cycle *also* contributes a non-zero weight in the permutation expansion of the determinant corresponding to the term (1364). We see this since our cycle follows a path from v_1 to v_3 (by way of v_2), returning to v_1 via w_6 and w_4. We must deal with this ambiguity. We introduce the idea of a *minimal* permutation cycle, one which does not include more than two successive entries with values between 1 and k or between $k+1$ and $2k$. We see that (1364) is minimal while (12364) is not.

We notice that walk systems arise from permutations, so it is natural to think of a walk system as a permutation together with a collection of walks that "follow" the permutation. This is the idea of a walk system–permutation pair (or WSP-pair for short) that is presented in Section 3.4. From the idea of a minimal permutation cycle, we define a *minimal walk* to have as its base permutation a minimal permutation cycle, and a *minimal walk system* to be composed of only minimal walks. Since our original goal was to count "cycle systems" in a directed graph, we realize we need to be precise and instead count "simple minimal walk systems". This leads to the second part of the proof of the Hamburger Theorem.

Given a walk system that is either not simple or not minimal and that satisfies neither Property 1 nor Property 2, at least one of the two following properties MUST hold.

Property 3. The walk system has two intersecting walks, one of which is a 2-cycle.

Property 4. The walk system is not minimal.

The following lemma shows that the contributions of walk systems satisfying either of these new properties cancel in the permutation expansion of the determinant of M_H.

Lemma 2.2. *The set of all walk systems $\mathcal{W}$ that satisfy neither Property 1 nor Property 2 and that satisfy Property 3 or Property 4 can be partitioned into equivalence classes, each of which contributes a net zero to the permutation expansion of the determinant of M_H.*

The proof of Lemma 2.2 is also based on involutions. Walk systems cancel in families built from an index set containing the set of all 2-cycle intersections and non-minimalities.

If a walk system satisfies none of the conditions of Properties 1 through 4, then it is indeed a simple minimal walk system, or in other words, a cycle system. The cancellation from the above sets of families gives that only cycle systems contribute to the permutation expansion of the determinant of M_H. This contribution is the signed weight of each cycle system, so the determinant of M_H exactly equals $c^+ - c^-$. Theorem 1.1 follows from Lemmas 2.1 and 2.2 in Section 4. •

2.2 The Weighted Hamburger Theorem

There is also a weighted version of the Hamburger Theorem, and it will be under this generalization that Lemmas 2.1 and 2.2 are proved. We allow *weights* $\mathrm{wt}(e)$ on the edges of the hamburger graph; the simplest weighting, which counts the number of cycle systems, assigns $\mathrm{wt}(e) \equiv 1$. We require that $\mathrm{wt}(e_i)\mathrm{wt}(e_i') = 1$ for all $2 \leq i \leq k-1$, but we do not require this condition for $i = 1$ nor for $i = k$. Define the $2k \times 2k$ *weighted hamburger matrix* M_H to be the block matrix

$$M_H = \begin{bmatrix} A & D_1 \\ -D_2 & B \end{bmatrix}. \tag{1}$$

In the upper-triangular $k \times k$ matrix $A = (a_{ij})$, a_{ij} is the sum of the products of the weights of edges over all paths from v_i to v_j in G_1. In the lower triangular $k \times k$ matrix $B = (b_{ij})$, b_{ij} is the sum of the products of the weights of edges over all paths from w_{k+i} to w_{k+j} in G_2. The diagonal $k \times k$ matrix D_1 has as its entries $d_{ii} = \mathrm{wt}(e_i)$ and the diagonal $k \times k$ matrix D_2 has as its entries $d_{ii} = \mathrm{wt}(e_i')$. Note that when the weights of the edges in E_3 are all 1, these matrices satisfy $D_1 = D_2 = I_k$.

We wish to count vertex-disjoint unions of weighted cycles in H. In any hamburger graph H, there are two possible types of cycle. There are k *2-cycles*

$$c : v_i \xrightarrow{e_i} w_{k+i} \xrightarrow{e_i'} v_i$$

and many more *general cycles* that alternate between G_1 and G_2. We can think of a general cycle as a path P_1 in G_1 connected by an edge $e_{1,1} \in E_3$ to a path Q_1 in G_2, which in turn connects to a path P_2 in G_1 by an edge $e_{1,2}'$, continuing in this fashion until arriving at a final path Q_l in G_2 whose terminal vertex is adjacent to the initial vertex of P_1. We write

$$c : P_1 \xrightarrow{e_{1,1}} Q_1 \xrightarrow{e_{1,2}'} P_2 \xrightarrow{e_{2,1}} \cdots \xrightarrow{e_{l,1}} P_l \xrightarrow{e_{l,2}'} Q_l.$$

For each cycle c, we define the weight $\mathrm{wt}(c)$ of c to be the product of the weights of all edges traversed by c:

$$\mathrm{wt}(c) = \prod_{e \in c} \mathrm{wt}(e).$$

We define a *weighted cycle system* to be a collection $\mathcal{C}$ of m vertex-disjoint cycles. We again define the *sign* of a weighted cycle system to be $\mathrm{sgn}(\mathcal{C}) = (-1)^{l+m}$, where l is the total number of edges from G_2 to G_1 in $\mathcal{C}$. We say that a weighted cycle system $\mathcal{C}$ is *positive* if $\mathrm{sgn}(\mathcal{C}) = +1$ and *negative* if $\mathrm{sgn}(\mathcal{C}) = -1$. For a hamburger graph H, let c^+ be the sum of the weights of positive weighted cycle systems, and let c^- be the sum of the weights of negative weighted cycle systems.

Theorem 2.3 (The weighted Hamburger Theorem). *The determinant of the weighted hamburger matrix M_H equals $c^+ - c^-$.*

As above, Theorem 2.3 follows from Lemmas 2.1 and 2.2. The proofs will be presented after developing the following necessary machinery.

3 Additional Definitions

3.1 Edge Cycles and Permutation Cycles

In the proof of the Hamburger Theorem, there are two distinct mathematical objects that have the name "cycle". We have already mentioned the type of cycle that appears in graph theory. There, a (simple) cycle in a directed graph is a closed directed path with no repeated vertices.

Secondly, there is a notion of cycle when we talk about permutations. If $\sigma \in S_n$ is a permutation, we can write σ as the product of disjoint cycles $\sigma = \chi_1 \chi_2 \cdots \chi_\tau$.

To distinguish between these two types of cycles when confusion is possible, we call the former kind an *edge cycle* and the latter kind a *permutation cycle*. Notationally, we use Roman letters when discussing edge cycles and Greek letters when discussing permutation cycles.

3.2 Permutation Expansion of the Determinant

We recall that the *permutation expansion* of the determinant of an $n \times n$ matrix $M = (m_{ij})$ is the expansion of the determinant as

$$\det M = \sum_{\sigma \in S_n} (\mathrm{sgn}\,\sigma) m_{1,\sigma(1)} \cdots m_{n,\sigma(n)}. \tag{2}$$

We will be considering non-zero terms in the permutation expansion of the determinant of the hamburger matrix M_H. Because of the special block form of the hamburger matrix in Equation (1), the permutations σ that make non-zero contributions to this sum are products of disjoint cycles of either of two forms—the simple transposition

$$\chi = (\varphi_{11}\ \omega_{11})$$

or the general permutation cycle

$$\chi = (\varphi_{11}\ \varphi_{12}\ \cdots\ \varphi_{1\mu_1}\ \omega_{11}\ \omega_{12}\ \cdots\ \omega_{1\nu_1}\ \varphi_{21}\ \cdots\cdots\ \varphi_{\lambda\mu_\lambda}\ \omega_{\lambda 1}\ \cdots\ \omega_{\lambda\nu_\lambda}). \tag{3}$$

In the first case, $\omega_{11} = \varphi_{11} + k$. In the second case, $1 \leq \varphi_{\iota\kappa} \leq k$, $k+1 \leq \omega_{\iota\kappa} \leq 2k$, $\varphi_{\iota\kappa} < \varphi_{\iota,\kappa+1}$, and $\omega_{\iota\kappa} > \omega_{\iota,\kappa+1}$ for all $1 \leq \iota \leq \lambda$ and relevant κ. The block matrix form also implies that $\varphi_{\iota\mu_\iota} + k = \omega_{\iota 1}$, $\omega_{\iota\nu_\iota} - k = \varphi_{\iota+1,1}$, and $\omega_{\lambda\nu_\lambda} - k = \varphi_{11}$. These last requirements along with the fact that no integers appear more than once in a permutation cycle imply that $\mu_i, \nu_i \geq 2$ for $1 \leq i \leq \lambda$. So that this permutation cycle is in standard form, we make sure that $\varphi_{11} = \min_{\iota,\kappa} \varphi_{\iota\kappa}$. In order to refer to this value later, we define a function Φ by $\Phi(\chi) = \varphi_{11}$. Each value $1 \leq \varphi_\iota \leq k$ or $k+1 \leq \omega_\iota \leq 2k$ appears at most once for any $\sigma \in S_{2k}$.

We call a permutation cycle χ *minimal* if it is a transposition or if $\mu_\iota = \nu_\iota = 2$ for all ι. Minimality implies that we can write our general permutation cycles χ in the form

$$\chi = (\varphi_{11}\ \varphi_{12}\ \omega_{11}\ \omega_{12}\ \varphi_{21}\ \cdots\ \varphi_{\lambda 2}\ \omega_{\lambda 1}\ \omega_{\lambda 2}), \tag{4}$$

with the same conditions as before. We call a permutation $\sigma = \chi_1 \cdots \chi_\tau$ *minimal* if each of its cycles χ_ι is minimal.

3.3 Walks Associated to a Permutation

To each permutation cycle $\chi \in S_{2k}$, we can associate one or more walks c_χ in H.

If χ is the transposition $\chi = (\varphi_{11}\ \omega_{11})$, then we associate the 2-cycle $c_\chi : v_{\varphi_{11}} \to w_{\omega_{11}} \to v_{\varphi_{11}}$ to χ. To any permutation cycle χ that is not a transposition, we can associate multiple walks c_χ by gluing together paths that follow χ in the following way. If χ has the form of Equation (3), then for each $1 \leq i \leq \lambda$, let P_i be any path in G_1 that visits each of the vertices $v_{\varphi_{i1}}$, $v_{\varphi_{i2}}$, all the way through $v_{\varphi_{i\mu_i}}$ in order. Similarly, let Q_i be any path in G_2 that visits each of the vertices $w_{\omega_{i1}}$, $w_{\omega_{i2}}$, through $w_{\omega_{i\nu_i}}$ in order. For each choice of paths P_i and Q_i, we have a possibility for the walk c_χ; we can set

$$c_\chi : P_1 \xrightarrow{e_{\varphi_{12}}} Q_1 \xrightarrow{e'_{\varphi_{21}}} P_2 \xrightarrow{e_{\varphi_{22}}} \cdots\cdots \xrightarrow{e'_{\varphi_{\lambda 1}}} P_\lambda \xrightarrow{e_{\varphi_{\lambda 2}}} Q_\lambda. \tag{5}$$

See Figure 5 for the choices of $c_{(12364)}$ in the hamburger graph presented in Figure 2. We call λ the *number of P-paths* in c_χ. The function Φ, defined in the previous section, defines a partial ordering on walks in a walk system—we say that the associated walk c_χ comes before the associated walk $c_{\chi'}$ if $\Phi(\chi) < \Phi(\chi')$. We call this the *initial term order*. As in Section 2.2, we define the weight of a walk c_χ to be the product of the weights of all edges traversed by c_χ.

3.4 Walk System-Permutation Pairs

We defined walk systems in Section 2, but we will see that the proof of Theorem 1.1 requires us to think of walk systems first as a permutation and second as a collection of

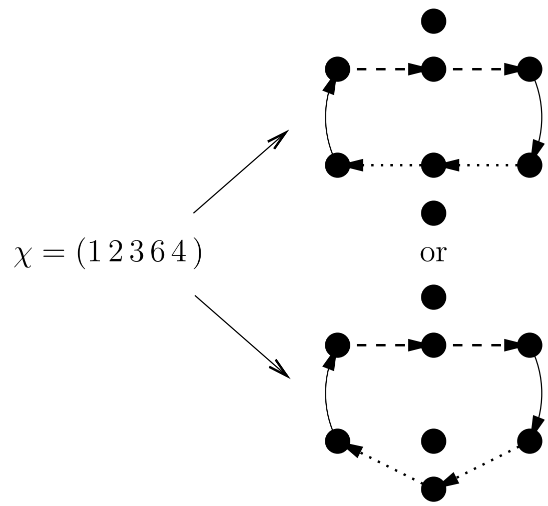

Figure 5: A permutation cycle χ and the two walks in H associated to χ

walks determined by the permutation. We will see that for cycle systems as presented initially, signs and weights are not changed by this recharacterization.

If H is a hamburger graph with k pairs of distinguished vertices, we define a walk system–permutation pair as follows.

Definition 3.1. A *walk system–permutation pair* (or *WSP-pair* for short) is a pair $(\mathcal{W}, \sigma)$, where $\sigma \in S_{2k}$ is a permutation and $\mathcal{W}$ is a collection of walks $c \in \mathcal{W}$ with the following property: if the disjoint cycle representation of σ is $\sigma = \chi_1 \cdots \chi_\tau$, then $\mathcal{W}$ is a collection of τ walks c_{χ_ι}, for $1 \leq \iota \leq \tau$, where c_{χ_ι} is a walk associated to the permutation cycle χ_ι.

We define the *weight* of a WSP-pair $(\mathcal{W}, \sigma)$ to be the product of the weights of the associated walks $c_\chi \in \mathcal{W}$.

Each permutation σ yields many collections of walks $\mathcal{W}$, collections of walks $\mathcal{W}$ may be associated to many permutations σ, but any walk system $\mathcal{W}$ corresponds to one and only one minimal permutation σ_m. This is because, given any path as in Equation (5), we can read off the initial and terminal vertices of each P_i and Q_i in order, producing a well-defined permutation cycle σ_m. We define a WSP-pair $(\mathcal{W}, \sigma)$ to be *minimal* if σ is a minimal permutation.

For a WSP-pair $(\mathcal{W}, \sigma)$, where $\sigma = \chi_1 \cdots \chi_\tau$, we define the *sign* of the WSP-pair, $\operatorname{sgn}(\mathcal{W}, \sigma)$, to be $(-1)^l \operatorname{sgn}(\sigma)$, where λ_χ is the number of P-paths in c_χ and where $l = \sum_{c_\chi \in \mathcal{W}} \lambda_\chi$. Alternatively, we could consider the sign of $(\mathcal{W}, \sigma)$ to be the product of the signs of its associated walks c_χ, where the *sign* of c_χ is $\operatorname{sgn}(c_\chi) = (-1)^{\lambda_\chi} \operatorname{sgn}(\chi)$. We say that a WSP-pair $(\mathcal{W}, \sigma)$ is *positive* if $\operatorname{sgn}(\mathcal{W}, \sigma) = +1$ and is *negative* if $\operatorname{sgn}(\mathcal{W}, \sigma) = -1$.

Note that if $(\mathcal{W}, \sigma)$ is a minimal WSP-pair, then $\operatorname{sgn}(c_\chi) = +1$ for a transposition χ and $\operatorname{sgn}(c_\chi) = (-1)^{\lambda+1}$ if χ is of the form in Equation (4). In particular, when $(\mathcal{W}, \sigma)$ is minimal and simple, its sign and weight is consistent with the definition given in the introduction.

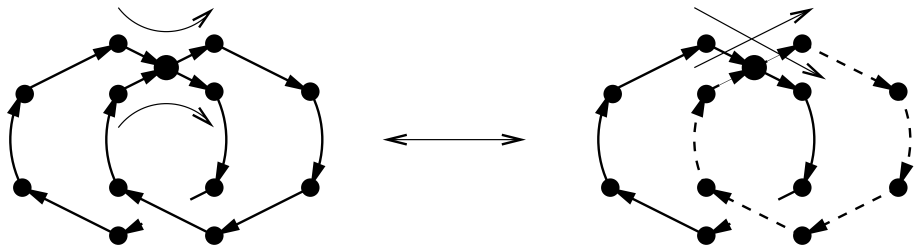

Figure 6: A self-intersecting cycle and its corresponding pair of intersecting cycles

4 Proof of the Hamburger Theorem

As mentioned in Section 2, we prove Lemmas 2.1 and 2.2 for the weighted version of the Hamburger Theorem, thereby proving Theorem 2.3. Theorem 1.1 follows as a special case of Theorem 2.3.

4.1 Proof of Lemma 2.1, Part I

Recall Properties 1 and 2 as well as Lemma 2.1.

Property 1. The walk system contains a walk that has a self-intersection.

Property 2. The walk system has two intersecting walks, neither of which is a 2-cycle.

Lemma 2.1. *The set of all walk systems $\mathcal{W}$ that satisfy either Property 1 or Property 2 can be partitioned into equivalence classes, each of which contributes a net zero to the permutation expansion of the determinant of M_H.*

The proof of Lemma 2.1 is a generalization of the involution principle, the idea of which comes from the picture presented in Figure 6. Given a self-intersecting walk, changing the order of edge traversal at the vertex of self-intersection leads to breaking the one self-intersecting walk into two walks that intersect at that same vertex. Since the edge set of the collection of walks has not changed, the weight of the two WSP-pairs is the same. We have introduced a transposition into the sign of the permutation cycle of the WSP-pair; this changes the sign of the WSP-pair, so these two WSP-pairs will cancel in the permutation expansion of the determinant of M_H.

One can imagine that this means that to every self-intersecting WSP-pair we can associate one WSP-pair with two cycles intersecting. However, more than one self-intersection may occur at this same point, and there may be additional walks that pass through that same point. Exactly what this WSP-pair would cancel with is not clear. If we decide to break all the self-intersections so that we have some number N of walks through our vertex, it is not clear how we should sew the cycles back together. One starts to get the idea that we must consider all possible ways of sewing back together. Once we do just that, we have a family $\mathcal{F}$ of WSP-pairs, all of the same weight, whose net contribution to the permutation expansion of the determinant is zero.

This idea is conceptually simple but the proof is notationally complicated.

If $\mathcal{W}$ satisfies either Property 1 or Property 2, then there is some vertex of intersection, be it either a self-intersection or an intersection of two walks. Our aim is to choose a well-defined *first* point of intersection at which we will build the family $\mathcal{F}$. The initial term order gives an order on walks associated to permutation cycles; we choose the earliest walk c_{χ_α} that has some vertex of intersection. Once we have determined the earliest walk, we start at $v_{\Phi(c_{\chi_\alpha})}$ and follow the walk

$$c_{\chi_a} : P_1 \to Q_1 \to P_2 \to \cdots \to P_m \to Q_m,$$

until we reach a vertex of intersection.

In our discussion, we make the assumption that this first vertex of intersection is a vertex v^* in G_1. A similar argument exists if the first appearance occurs in G_2. Notice that at v^* there may be multiple self-intersections or multiple intersections of walks. We will create a family $\mathcal{F}$ of WSP-pairs that takes into account each of these possibilities.

If we want to rigorously define the breaking of a self-intersecting walk at a vertex of self-intersection, we need to specify many different components of the WSP-pair (C, σ). First, we need to specify on which walk in $\mathcal{W}$ we are acting. Next, we need to specify the vertex of self-intersection. Since this self-intersection vertex may occur in multiple paths, we need to specify which two paths we interchange in the breaking process.

4.2 Definitions of Breaking and Sewing

In the following paragraphs, we define "breaking" on WSP-pairs, which takes in a WSP-pair $(\mathcal{W}, \sigma)$, one of σ's permutation cycles χ_α, the associated walk c_{χ_α}, paths P_y and P_z in c_{χ_α}, and the vertex v^* in both P_y and P_z where c_{χ_α} has a self-intersection. For simplicity, we assume that v^* is not a distinguished vertex, but the argument still holds in that case. The inverse of this operation is "sewing".

In this framework, c_{χ_α} has the form

$$c_{\chi_\alpha} : P_1 \to Q_1 \to P_2 \to \cdots \to P_y \to Q_y \to \cdots \to P_z \to Q_z \to \cdots \to P_l \to Q_l,$$

where the paths P_y and P_z in G_1 are separated into two halves as

$$P_y : P_y^{(1)} \to v^* \to P_y^{(2)}$$

and

$$P_z : P_z^{(1)} \to v^* \to P_z^{(2)}.$$

Remember that P_y and P_z are paths that stop over at various vertices depending on the permutation χ_α. The vertex v^* must have adjacent stop-over vertices in each of the two paths P_y and P_z. Let the nearest stop-over vertices on the paths P_y and P_z be v_{φ_y} and $v_{\varphi_{y+1}}$, and v_{φ_z} and $v_{\varphi_{z+1}}$, respectively.

This implies χ_α has the form

$$\chi_\alpha = (\varphi_{11} \;\cdots\; \varphi_{1\mu_1} \; \omega_{11} \;\cdots\; \varphi_y \; \varphi_{y+1} \;\cdots\; \varphi_z \; \varphi_{z+1} \;\cdots\; \varphi_{\lambda\mu_\lambda} \; \omega_{\lambda 1} \;\cdots\; \omega_{\lambda\nu_\lambda}).$$

We can now precisely define the result of breaking. We define χ_β and χ_γ by splitting χ_α as follows:

$$\chi_\beta = (\varphi_{11} \ \cdots \ \varphi_y \ \varphi_{z+1} \ \cdots \ \omega_{\lambda\nu_\lambda})$$

and

$$\chi_\gamma = (\varphi_{y+1} \ \cdots \ \varphi_z),$$

with the necessary rewriting of χ_γ to have as its initial entry the value $\Phi(\chi_\gamma)$. Define c_{χ_β} and c_{χ_γ} to be

$$c_{\chi_\beta} : P_1 \to Q_1 \to P_2 \to \cdots \to P_y^{(1)} \to v^* \to P_z^{(2)} \to Q_z \to \cdots \to P_l \to Q_l,$$

and

$$c_{\chi_\gamma} : P_z^{(1)} \to v^* \to P_y^{(2)} \to Q_y \to \cdots \to Q_{z-1},$$

again changing the starting vertex of c_{χ_γ} to $v_{\Phi(c_{\chi_\gamma})}$.

We define the breaking of the WSP-pair with the above inputs to be the WSP-pair $(\mathcal{W}', \sigma')$ such that

$$\mathcal{W}' = \mathcal{W} \cup \{c_{\chi_\beta}, c_{\chi_\gamma}\} \setminus \{c_{\chi_\alpha}\}$$

and

$$\sigma' = \sigma\chi_\alpha^{-1}\chi_\beta\chi_\gamma = \sigma \cdot (\varphi_{y+1} \ \varphi_{z+1}).$$

The edge set of $\mathcal{W}$ is equal to the edge set of $\mathcal{W}'$, so the weight of the modified cycle systems is the same as the original. Since we changed σ to σ' by multiplying only by a transposition, the sign of the modified WSP-pair is opposite to that of the original.

By only discussing the case when v^* is not distinguished, we avoid notational issues brought upon by cases when v^* is or is not one of the stop-over vertices.

4.3 Proof of Lemma 2.1, Part II

Having defined breaking and sewing, we can continue the proof.

For any WSP-pair $(\mathcal{W}, \sigma)$ satisfying either Property 1 or Property 2, let $c_{\chi_\alpha} \in \mathcal{W}$ be the first walk in the initial term order with an vertex of intersection. Let v^* be the first vertex of intersection in c_{χ_α}. Then for all walks c with one or more self-intersections at v^*, continue to break c at v^* until there are no more self-intersections. Define the resulting WSP-pair $(\mathcal{W}_u, \sigma_u)$ to be the *unlinked WSP-pair* associated to $(\mathcal{W}, \sigma)$. In $(\mathcal{W}_u, \sigma_u)$, there is some number N of general walks intersecting at vertex v^*. There may be a 2-cycle intersecting v^* as well, but this does not matter.

For any permutation $\xi \in S_N$, let $\xi = \zeta_1\zeta_2\cdots\zeta_\eta$ be its cycle representation, where each ζ_ι is a cycle. For each $1 \leq \iota \leq \eta$, sew together walks in order: if $\zeta_\iota = (\delta_{\iota 1} \ \cdots \ \delta_{\iota\varepsilon_\iota})$, sew together $c_{\chi_{\delta_{\iota 1}}}$ and $c_{\chi_{\delta_{\iota 2}}}$ at v^*. Sew this result together with $c_{\chi_{\delta_{\iota 3}}}$, and so on through $c_{\chi_{\delta_{\iota\varepsilon_\iota}}}$. Note that the result of these sewings is unique, and that every WSP-pair $(\mathcal{W}, \sigma)$ with $(\mathcal{W}_u, \sigma_u)$ as its unlinked WSP-pair can be obtained in this way, and no other WSP-pair appears. We can perform this procedure for any $\xi \in S_N$; the sign of the resulting walk system (C_ξ, σ_ξ) is $\operatorname{sgn}(C_\xi, \sigma_\xi) = \operatorname{sgn}(\xi)\operatorname{sgn}(\mathcal{W}_u, \sigma_u)$. This means that the contribution to the determinant of the weights of all WSP-pairs in the family $\mathcal{F}$ is

$$\sum_{\xi \in S_N} \operatorname{sgn}(\xi)\operatorname{sgn}(\mathcal{W}_u, \sigma_u)\operatorname{wt}(\mathcal{W}_u, \sigma_u) = \operatorname{sgn}(\mathcal{W}_u, \sigma_u)\operatorname{wt}(\mathcal{W}_u, \sigma_u)\sum_{\xi \in S_N}\operatorname{sgn}(\xi) = 0. \tag{6}$$

So elements of the same family cancel out in the determinant of M_H, giving that the contributions of all WSP-pairs satisfying either Property 1 or Property 2 cancel out in the hamburger determinant. •

4.4 Proof of Lemma 2.2

Recall Properties 3 and 4 as well as Lemma 2.2.

Property 3. The walk system has two intersecting walks, one of which is a 2-cycle.

Property 4. The walk system is not minimal.

Lemma 2.2. *The set of all walk systems $\mathcal{W}$ that satisfy neither Property 1 nor Property 2 and that satisfy Property 3 or Property 4 can be partitioned into equivalence classes, each of which contributes a net zero to the permutation expansion of the determinant of M_H.*

In the proof of Lemma 2.2, we do not base our family $\mathcal{F}$ around a singular vertex; instead, we find a set of violations that each member of the family has. If the WSP-pair satisfies the hypotheses of Lemma 2.2, then either there is some 2-cycle $c : v_\varphi \to w_\omega \to v_\varphi$ that intersects with some other walk or there is some non-minimal permutation cycle.

Define a set of indices $I \subseteq [k]$ of violations, of which an integer can become a member in one of two ways. If $(\mathcal{W}, \sigma)$ is not minimal, there is at least one permutation cycle χ_α with more than two consecutive φ's or ω's in its cycle notation. For any intermediary ι between two φ's or $\iota + k$ between two ω's, place ι in I. For example, if $\chi_\alpha = (\cdots\ \varphi'\ \iota\ \varphi''\ \cdots)$, we place $\iota \in I$. Alternatively, there may be a 2-cycle $c : v_i \to w_{k+i} \to v_i$ such that either v_i is a vertex in some other walk c_{χ_β} or w_{k+i} is a vertex in some other cycle c_{χ_γ}, or both. We declare this i to be in I as well.

Note that any WSP-pair $(\mathcal{W}, \sigma)$ satisfying either Property 3 or Property 4 has a non-empty set I. From our original WSP-pair, obtain a minimal WSP-pair $(\mathcal{W}_m, \sigma_m)$ by removing any transposition χ_α from σ and its corresponding 2-cycle c_{χ_α} from $\mathcal{W}$, and also for any non-minimal permutation cycle χ_β we remove any intermediary φ's or ω's from σ. We do not change the associated walk c_{χ_β} in $\mathcal{W}$ since it still corresponds to this minimized permutation cycle.

Let i be any element in I. Since $i \in I$, the 2-cycle $c_i : v_i \to w_{k+i} \to v_i$ intersects some walk of $\mathcal{W}_m$ either at v_i, at w_{k+i}, or both. So there are four cases:

Case 1. c_i intersects a walk c_{χ_β} at v_i and no walk at w_{k+i}.

Case 2. c_i intersects a walk c_{χ_γ} at w_{k+i} and no walk at v_i.

Case 3. c_i intersects a walk c_{χ_β} at v_i and the same walk again at w_{k+i}.

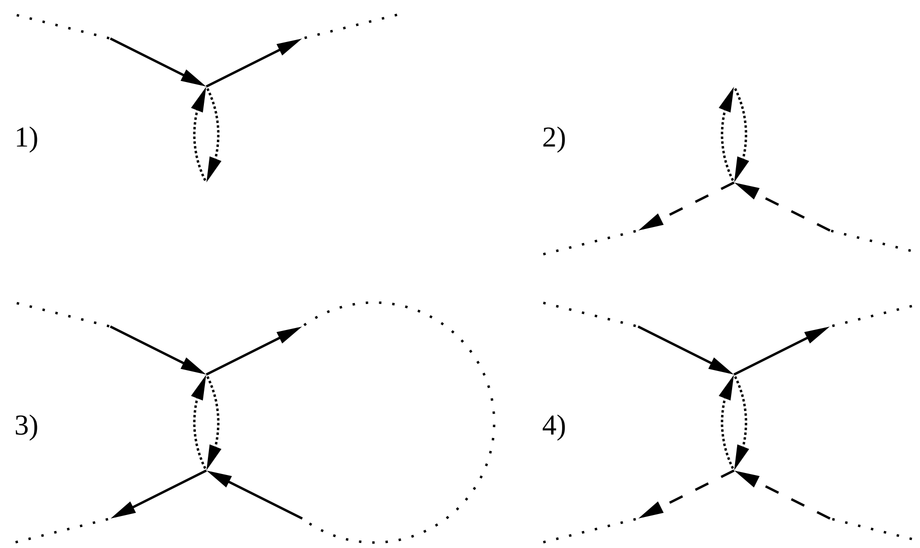

Figure 7: The four cases in which a walk can intersect with a 2-cycle

Case 4. c_i intersects a walk c_{χ_β} at v_i and another walk c_{χ_γ} at w_{k+i}.

See Figure 7 for a picture.

In order to build the family $\mathcal{F}$, we modify the minimal WSP-pair $(\mathcal{W}_m, \sigma_m)$ at each i with some choice of options, each giving a possible WSP-pair that has $(\mathcal{W}_m, \sigma_m)$ as its minimal WSP-pair. We define a set of two or four options $\mathcal{O}_i$ for each i.

In Case 1, there are two options. Let o_1 be the option of including the walk c_i in $\mathcal{W}_m$ and its corresponding transposition χ_α in σ_m. Let o_2 be the option of including the intermediary $\varphi = i$ in χ_β in the position where c_{χ_β} passes through v_i.

[Note that we could not apply both options at the same time since the resulting χ_α and χ_β would not be disjoint and therefore is not a term in the permutation expansion of the determinant.]

Similarly in Case 2, there are two options. Let o_1 be the option of including the walk c_i in $\mathcal{W}_m$ and its corresponding transposition χ_α in σ_m. Let o_2 be the option of including the intermediary $\omega = i + k$ in χ_γ in the position where c_{χ_γ} passes through w_{k+i}.

In Case 3, there are four options. Let o_1 be the option of including the walk c_i in $\mathcal{W}_m$ and its corresponding transposition χ_α in σ_m. Let o_2 be the option of including the intermediary $\varphi = i$ in χ_β in the position where c_{χ_β} passes through v_i. Let o_3 be the option of including the intermediary $\omega = i + k$ in χ_β in the position where c_{χ_β} passes through w_{k+i}. Let o_4 be the option of including both intermediaries $\varphi = i$ and $\omega = i + k$ in χ_β in the respective positions where c_{χ_β} passes through v_i and w_{k+i}.

In Case 4, there are four options. Let o_1 be the option of including the walk c_i in $\mathcal{W}_m$ and its corresponding transposition χ_α in σ_m. Let o_2 be the option of including the intermediary $\varphi = i$ in χ_β in the position where c_{χ_β} passes through v_i. Let o_3 be the option of including the intermediary $\omega = i + k$ in χ_γ in the position where c_{χ_γ} passes through w_{k+i}. Let o_4 be the option of including intermediary $\varphi = i$ in χ_β in the position where

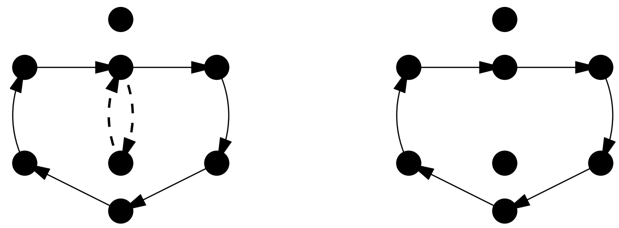

Figure 8: A family of two walk systems that cancel in the hamburger determinant

c_{χ_β} passes through v_i and intermediary $\omega = i + k$ in χ_γ in the position where c_{χ_γ} passes through w_{k+i}.

Corresponding to the associated minimal WSP-pair $(\mathcal{W}_m, \sigma_m)$ and index set I, define the family $\mathcal{F}$ to be the set of WSP-pairs $(\mathcal{W}_f, \sigma_f)$ by exercising all combinations of options $o_{j_i} \in \mathcal{O}_i$ on $(\mathcal{W}_m, \sigma_m)$ for each $i \in I$. Note that every WSP-pair derived in this fashion is a WSP-pair satisfying the hypotheses of Lemma 2.2 and is such that its associated minimal WSP-pair is $(\mathcal{W}_m, \sigma_m)$. There is also no other WSP-pair $(\mathcal{W}', \sigma')$ with $(\mathcal{W}_m, \sigma_m)$ as its minimal WSP-pair. See Figure 8 for a canceling family of two walk systems, one of which is the non-minimal walk system from Figure 5.

Every WSP-pair in $\mathcal{F}$ has the same weight since each option changes the edge set of $\mathcal{W}$ by at most a 2-cycle $v_i \to w_{k+i} \to v_i$, where $2 \leq i \leq k-1$ and each of those 2-cycles contributes a multiplier of 1 to the weight of the WSP-pair. The peculiar bounds in this restriction are due to the structure of the hamburger graph. A walk through the vertices v_1 and w_{k+1} (or v_k and w_{2k}) must use edge e_1' (or e_k), and include in its associated permutation cycle 1 and $k+1$ (or k and $2k$). This implies that we can not have multiple walks through any of these four special vertices, as that would yield two non-disjoint permutation cycles in σ.

The sign of every WSP-pair in $\mathcal{F}$ is determined by the set of options applied to $(\mathcal{W}_m, \sigma_m)$. Each option o_j contributes a multiplicative ± 1 depending on j—$\text{sgn}(o_1) = +1$, $\text{sgn}(o_2) = -1$, and $\text{sgn}(o_3) = -1$ and $\text{sgn}(o_4) = +1$ if they exist.

Each family $\mathcal{F}$ contributes a cumulative weight of 0. This is because there are the same number of positive $(\mathcal{W}_f, \sigma_f) \in \mathcal{F}$ as negative $(\mathcal{W}_f, \sigma_f) \in F$. We see this since if $i^* \in I$ then any set of WSP-pairs that exercise the same options o_{j_i} for all $i \neq i^*$ has either two or four members (depending on the case of i^*), which split evenly between positive and negative WSP-pairs. Since each family contributes a net zero to the hamburger determinant, the total contribution from WSP-pairs satisfying the hypotheses of Lemma 2.2 is zero. •

Since every WSP-pair appears once in the permutation expansion of $\det M_H$, the only WSP-pairs that contribute to the sum are those that are simple and minimal. Therefore $\det M_H$ is the sum over such WSP-pairs of $\text{sgn}(\mathcal{W}, \sigma) \cdot \text{wt}(\mathcal{W}, \sigma)$. This establishes Theorem 2.3 and gives Theorem 1.1 as a special case.

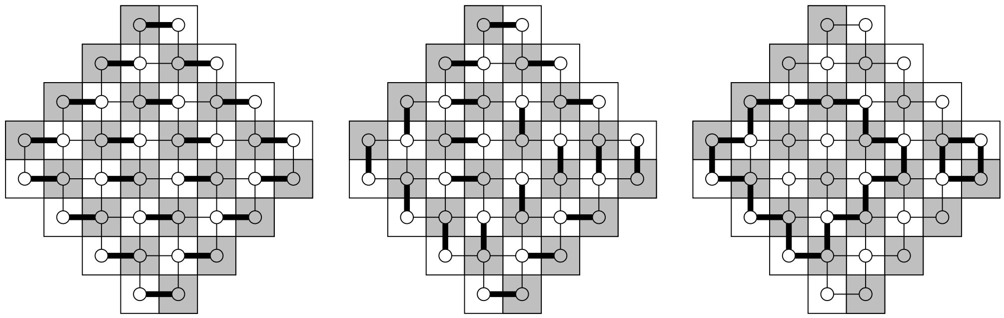

Figure 9: The symmetric difference of two matchings gives a cycle system

5 Applications of the Hamburger Theorem

We discuss first the application of the Hamburger Theorem in the case where the region is an Aztec diamond, mirroring results of Brualdi and Kirkland. Then we discuss the results from the case where the region is an Aztec pillow, and along the way, we explain how to implement the Hamburger Theorem when our region is any generalized Aztec pillow.

5.1 Domino Tilings and Digraphs

The Hamburger Theorem applies to counting the number of domino tilings of Aztec diamonds and generalized Aztec pillows. To illustrate this connection, we count domino tilings of the Aztec diamond by counting an equivalent quantity, the number of matchings on the dual graph G of the Aztec diamond. We use the natural matching N of horizontal neighbors in G, as exemplified in Figure 9a on AD_4, as a reference point. Given any other matching M on G, such as in Figure 9b, its symmetric difference with the natural matching is a collection of cycles in the graph, such as in Figure 9c.

When we orient the edges in N from black vertices to white vertices and in M from white vertices to black vertices, the symmetric difference becomes a collection of directed cycles. Notice that edges in the upper half of G all go from left to right and the edges in the bottom half of G go from right to left.

Since the edges of N always appear in the cycles, we can contract the edges of N to vertices of a new graph which retains its structure in terms of the cycle systems it produces. This new graph H is of the form in Figure 10a. This argument shows that the number of domino tilings of an Aztec diamond equals the number of cycle systems of this new condensed graph, called the region's *digraph*.

We wish to concretize this notion of a *digraph* of the Aztec diamond AD_n. Given the natural tiling of an Aztec diamond consisting solely of horizontal dominoes, we place a vertex in the center of every domino. The edges of this digraph fall into three families. From every vertex in the top half of the diamond, create edges to the east, to the northeast, and to the southeast whenever there is a vertex there. From every vertex in the bottom half of the diamond, form edges to the west, to the southwest, and to the northwest

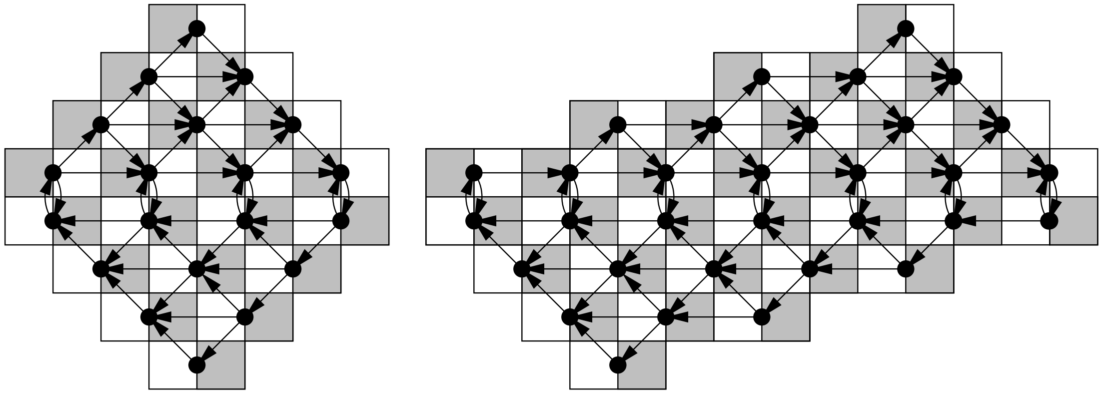

Figure 10: The hamburger graph for an Aztec diamond and an Aztec pillow

whenever there is a vertex there. Additionally, label the bottom vertices in the top half v_1 through v_n from west to east and the top vertices in the bottom half w_{n+1} through w_{2n}, also west to east. For all i between 1 and n, create a directed edge from v_i to w_{n+i} and from w_{n+i} to v_i. The result when this construction is applied to AD_4 is a graph of the form in Figure 10a. By construction, this graph is a hamburger graph.

Theorem 5.1. *The digraph of an Aztec diamond is a hamburger graph.*

Since both the upper half of the digraph and the lower half of the digraph are strongly planar, there are no negative cycle systems. This implies that the determinant of the corresponding hamburger matrix counts exactly the number of cycle systems in the digraph.

Corollary 5.2. *The number of domino tilings of an Aztec diamond is the determinant of its hamburger matrix.*

5.2 Applying the Hamburger Theorem to Aztec Diamonds

In order to apply Theorem 1.1 to count the number of tilings of AD_n, we need to find the number of paths in the upper half of D from v_i to v_j and the number of paths in the lower half of D from w_{n+j} to w_{n+i}. The key observation is that by the equivalence in Figure 11, we are in effect counting the number of paths from (i, i) to (j, j) using steps of size $(0, 1)$, $(1, 0)$, or $(1, 1)$ that do not pass above the line $y = x$. This is a combinatorial interpretation for the $(j - i)$-th large Schröder number. The large Schröder numbers $s_0, \ldots, s_5$ are $1, 2, 6, 22, 90, 394$. This is sequence number A006318 in the Encyclopedia of Integer Sequences [13].

Corollary 5.3. *The number of domino tilings of the Aztec diamond AD_n is equal to*

$$\#AD_n = \det \begin{bmatrix} S_n & I_n \\ -I_n & S_n^T \end{bmatrix},$$

where S_n is an upper triangular matrix with the (i, j)-th entry equal to s_{j-i} for $j \geq i$.

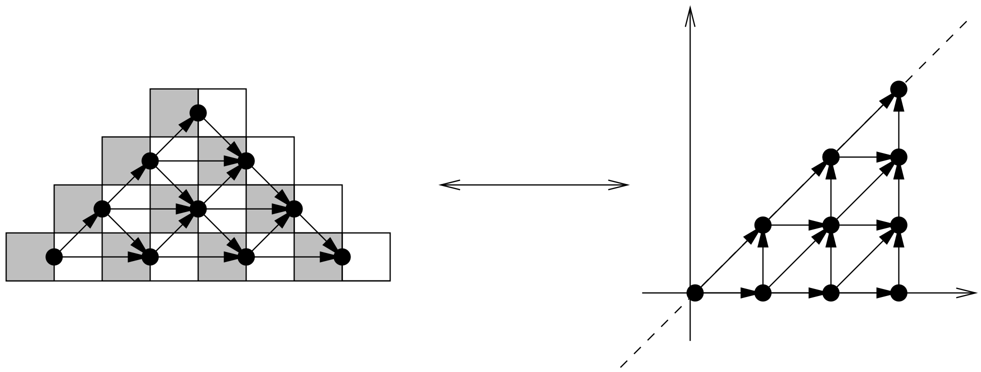

Figure 11: The equivalence between paths in D and lattice paths in the first quadrant

For example, when $n = 6$, the matrix S_6 is

$$S_6 = \begin{bmatrix} 1 & 2 & 6 & 22 & 90 & 394 \\ 0 & 1 & 2 & 6 & 22 & 90 \\ 0 & 0 & 1 & 2 & 6 & 22 \\ 0 & 0 & 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Brualdi and Kirkland prove a similar determinant formula for the number of tilings of an Aztec diamond in a matrix-theoretical fashion based on the $n(n+1) \times n(n+1)$ Kasteleyn matrix of the graph H and a Schur complement calculation. The Hamburger Theorem gives a purely combinatorial way to reduce the calculation of the number of tilings of an Aztec diamond to the calculation of a $2n \times 2n$ Hamburger determinant.

Following cues from Brualdi and Kirkland, we can reduce this to an $n \times n$ determinant via a Schur complement. For uses and the history of the Schur Complement, see the new book [14] by Zhang. In the case of the block matrix M_H in Equation (1), taking the Schur complement of B in M_H gives

$$\begin{aligned} \det M_H &= \det\begin{pmatrix} A & D_1 \\ -D_2 & B \end{pmatrix} \cdot \det\begin{pmatrix} I & 0 \\ B^{-1}D_2 & I \end{pmatrix} \\ &= \det\begin{pmatrix} A + D_1B^{-1}D_2 & D_1 \\ 0 & B \end{pmatrix} \\ &= \det(A + D_1B^{-1}D_2) \cdot \det B \\ &= \det(A + D_1B^{-1}D_2), \end{aligned}$$

since B is a lower triangular matrix with 1's on the diagonal. In this way, every hamburger determinant can be reduced to a smaller determinant of a Schur complement matrix. In the case of a simple hamburger graph where $D_2 = D_1 = I$, the determinant reduces further to $\det(A + B^{-1})$. Lastly, in the case where the hamburger graph is rotationally symmetric, $B = JAJ$ where J is the exchange matrix, which consists of 1's down the main skew-diagonal and 0's elsewhere. This implies that we can write the determinant in terms of just the submatrix A, i.e., $\det(A + JA^{-1}J)$. (Note that $J^{-1} = J$.)

Corollary 5.4. *The number of domino tilings of the Aztec diamond AD_n is equal to $\det(S_n + J_n S_n^{-1} J_n)$, where J_n is the $n \times n$ exchange matrix.*

In the case of a hamburger graph H, we call this Schur complement, $A+B^{-1}$, a *reduced hamburger matrix*. In the Aztec diamond graph example above, we can thus calculate the number of tilings of the Aztec diamond AD_6 as follows. The inverse of S_6 is

$$S_6^{-1} = \begin{bmatrix} 1 & -2 & -2 & -6 & -22 & -90 \\ 0 & 1 & -2 & -2 & -6 & -22 \\ 0 & 0 & 1 & -2 & -2 & -6 \\ 0 & 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which implies that the determinant of the reduced hamburger matrix

$$M_6 = \begin{bmatrix} 2 & 2 & 6 & 22 & 90 & 394 \\ -2 & 2 & 2 & 6 & 22 & 90 \\ -2 & -2 & 2 & 2 & 6 & 22 \\ -6 & -2 & -2 & 2 & 2 & 6 \\ -22 & -6 & -2 & -2 & 2 & 2 \\ -90 & -22 & -6 & -2 & -2 & 2 \end{bmatrix}$$

gives the number of tilings of AD_6.

Brualdi and Kirkland were the first to find such a determinantal formula for the number of tilings of an Aztec diamond [2]. Their matrix was different only in the fact that each entry was multiplied by (-1) and there was a multiplicative factor of $(-1)^n$. Brualdi and Kirkland were able to calculate the sequence of determinants, $\{\det M_n\}$, using a J-fraction expansion, which only works when matrices are Toeplitz or Hankel.

Eu and Fu also found an $n \times n$ determinant that calculated the number of tilings of an Aztec diamond [3]. Their matrix also involves large Schröder numbers; when $n = 6$, their matrix is

$$\begin{bmatrix} 1 & 2 & 6 & 22 & 90 & 394 \\ 2 & 6 & 22 & 90 & 394 & 1806 \\ 6 & 22 & 90 & 394 & 1806 & 8558 \\ 22 & 90 & 394 & 1806 & 8558 & 41586 \\ 90 & 394 & 1806 & 8558 & 41586 & 206098 \\ 394 & 1806 & 8558 & 41586 & 206098 & 1037718 \end{bmatrix}.$$

Their proof hinges on the relationship between domino tilings in an Aztec diamond and path systems in a lattice derived from the Aztec diamond's structure. The key observation is that these path systems in the lattice can be extended uniquely outside the Aztec

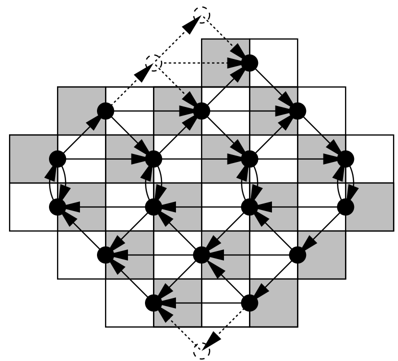

Figure 12: The digraph of a generalized Aztec pillow from the digraph of an Aztec diamond

diamond to form Schröder paths. Applying the Gessel–Viennot method gives the above matrix of Schröder numbers. Unfortunately, Eu and Fu's method does not generalize further, as explained at the end of the next section.

5.3 Applying the Hamburger Theorem to Aztec Pillows

A generalized Aztec pillow can be produced by restricting the placement of certain dominoes in an Aztec diamond in order to generate the desired boundary. We define the digraph of a generalized Aztec pillow to be the restriction of the digraph of an Aztec diamond to the vertices on the interior of the pillow. An example is presented Figure 12. Since the generalized Aztec pillow's digraph is a restriction of the Aztec diamond's digraph, we have the following corollary.

Corollary 5.5. *The digraph of an Aztec pillow or a generalized Aztec pillow is a hamburger graph.*

Aztec pillows were introduced in part because of an intriguing conjecture for the number of their tilings given by Propp [12].

Conjecture 5.6 (Propp's Conjecture). *The number of tilings of an Aztec pillow AP_n is a larger number squared times a smaller number. We write $\#AP_n = l_n^2 s_n$. In addition, depending on the parity of n, the smaller number s_n satisfies a simple generating function. For AP_{2m}, the generating function is*

$$\sum_{m=0}^{\infty} s_{2m}x^m = (1+3x+x^2-x^3)/(1-2x-2x^2-2x^3+x^4).$$

For AP_{2m+1}, the generating function is

$$\sum_{m=0}^{\infty} s_{2m+1}x^m = (2+x+2x^2-x^3)/(1-2x-2x^2-2x^3+x^4).$$

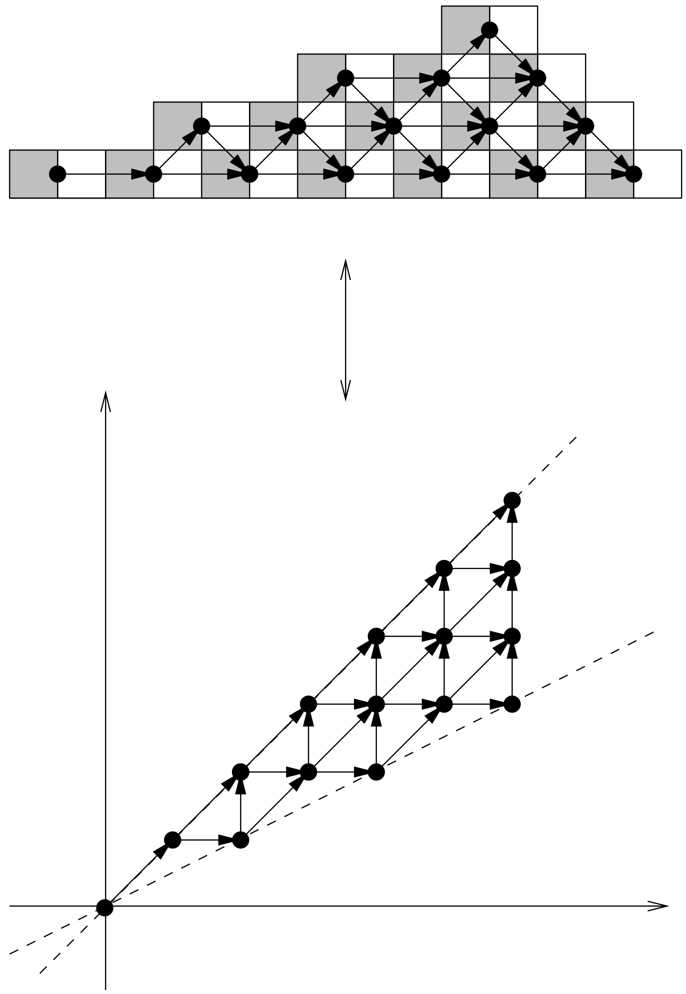

Figure 13: The equivalence between paths in D and lattice paths in the first quadrant

My hope for the Hamburger Theorem was that it would give a proof of Propp's Conjecture. We shall see that although we achieve a faster determinantal method to calculate $\#AP_n$, the sequence of determinants is not calculable by a J-factor expansion, as was the case in Brualdi and Kirkland's work.

Using the same method as for Aztec diamonds, creating the hamburger graph H for an Aztec pillow gives Figure 10b. Counting the number of paths from v_i to v_j and from w_{n+j} to w_{n+i} in successively larger Aztec pillows gives us the infinite upper-triangular array $\widetilde{S} = (\widetilde{s}_{i,j})$ of *modified Schröder numbers* defined by the following combinatorial interpretation. Let $\widetilde{s}_{i,j}$ be the number of paths from (i,i) to (j,j) using steps of size $(0,1)$, $(1,0)$, and $(1,1)$, not passing above the line $y = x$ nor below the line $y = x/2$. The equivalence between these paths and paths in the upper half of the hamburger graph is shown in Figure 13. The principal 7×7 minor matrix $\widetilde{S}_7$ of $\widetilde{S}$ is

$$\widetilde{S}_7 = \begin{bmatrix} 1 & 1 & 2 & 5 & 16 & 57 & 224 \\ 0 & 1 & 2 & 5 & 16 & 57 & 224 \\ 0 & 0 & 1 & 2 & 6 & 21 & 82 \\ 0 & 0 & 0 & 1 & 2 & 6 & 22 \\ 0 & 0 & 0 & 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This table of numbers is sequence number A114292 in the Encyclopedia of Integer Sequences [13].

Theorem 5.7. *The number of domino tilings of an Aztec pillow of order n is equal to*

$$\#AP_n = \det \begin{bmatrix} \widetilde{S}_n & I_n \\ -I_n & J_n\widetilde{S}_nJ_n \end{bmatrix},$$

where $\widetilde{S}_n$ is the $n \times n$ principal submatrix of $\widetilde{S}$ and J_n is the $n \times n$ exchange matrix.

As in the case of Aztec diamonds, we can calculate the reduced hamburger matrix through a Schur calculation. The inverse of $\widetilde{S}_6$ is

$$\widetilde{S}_6^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & -1 & -2 & -5 \\ 0 & 0 & 1 & -2 & -2 & -5 \\ 0 & 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and the resulting reduced hamburger matrix for AP_6 is

$$M_6 = \begin{bmatrix} 2 & 1 & 2 & 5 & 16 & 57 \\ -2 & 2 & 2 & 5 & 16 & 57 \\ -2 & -2 & 2 & 2 & 6 & 21 \\ -5 & -2 & -2 & 2 & 2 & 6 \\ -5 & -2 & -1 & -2 & 2 & 2 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

This gives us a much faster way to calculate the number of domino tilings of an Aztec pillow than was known previously. We have reduced the calculation of the $O(n^2) \times O(n^2)$ Kasteleyn–Percus determinant to an $n \times n$ reduced hamburger matrix. To be fair, the Kasteleyn–Percus matrix has -1, 0, and $+1$ entries while the reduced hamburger matrix may have very large entries, which makes theoretical running time comparisons difficult. Experimentally, when calculating the number of domino tilings of AP_{14} using Maple 8.0 on a 447 MHz Pentium III processor, the determinant of the 112×112 Kasteleyn–Percus matrix took 25.3 seconds while the determinant of the 14×14 reduced hamburger matrix took less than 0.1 seconds.

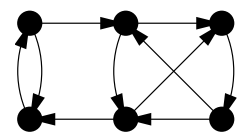

Figure 14: A simple generalized hamburger graph

Whereas we have a very understandable determinantal formula for the number of tilings of the region, this does not translate into a proof of Propp's Conjecture because we cannot calculate the determinant of the matrices M_n explicitly. We cannot apply a J-fraction expansion as Brualdi and Kirkland did since the reduced hamburger matrix is not Toeplitz or Hankel. Also, we might hope to apply Eu and Fu's method. It is not hard to construct the background lattice derived from generalized Aztec pillows; however, the path systems that are associated to domino tilings of the region cannot be extended as Eu and Fu did. A Gessel–Viennot-derived matrix still exists, but its structure does not lend itself to a simple determinant calculation as in the case of Aztec diamonds.

6 Generalizations

In this section we present a counterexample to a natural generalization of the Hamburger Theorem and discuss an intriguing generalization of Propp's Conjecture.

6.1 Counterexample to Generalizations of the Hamburger Theorem

The structure of the hamburger graph presented in Figure 1 seems restrictive, so the question naturally arises whether it is somehow necessary. Can the edge set E_3 between graphs G_1 and G_2 be an arbitrary bipartite graph? The answer in general is no.

Take for example the simple graph H in Figure 14. As one can count by hand, there are 10 distinct cycle systems in H, as enumerated in Figure 15. Creating the hamburger matrix that would correspond to this graph gives

$$M_H = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 1 \end{bmatrix},$$

where the lower left block describes paths from w_4, w_5, and w_6 to v_1, v_2, and v_3 and we have changed no other block of the matrix. This seems the most logical extension of the hamburger matrix.

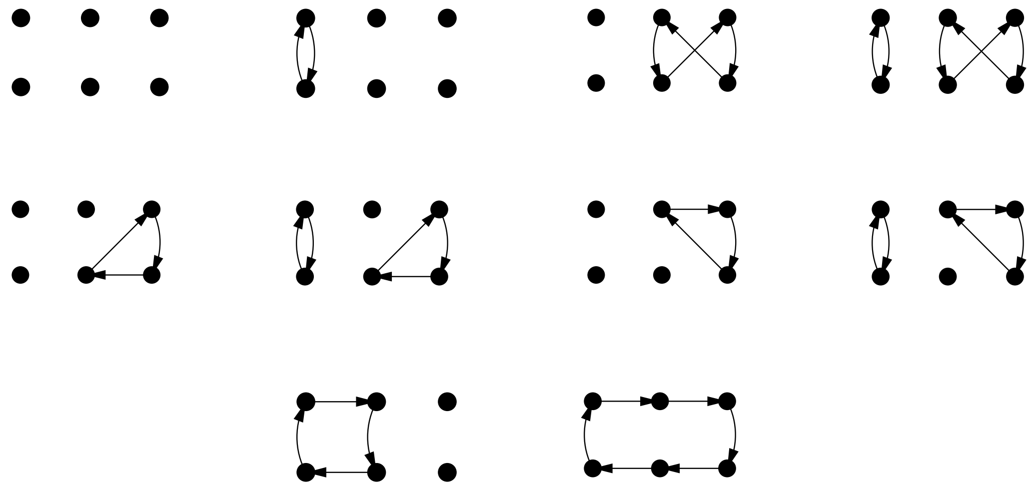

Figure 15: The ten cycle systems for the generalized hamburger graph in Figure 14

The determinant of this matrix is -5. Even though one might expect to need a new sign convention on cycle systems in generalized hamburger graphs, any sign convention would necessarily conserve the parity of the number of cycle systems. Therefore there is no $+1/-1$ labeling of the cycle systems in Figure 15 that would allow $\det M_H = \sum_{c \in \mathcal{C}} \operatorname{sgn}(c)$. This means that either the matrix for this generalized hamburger graph is not correct or that the theorem does not hold in general.

The only hope for my idea of how to generalize the hamburger matrix is what happens when one considers the difference between even and odd permutations. In the counterexample above, the edge set E_3 sends vertex w_4 to vertex v_1, w_5 to v_3, and w_6 to v_2. This can be thought of as an odd permutation of a hamburger's edge set (sending w_4 to v_1, w_5 to v_2, and w_6 to v_3). In my limited calculations so far, whenever the permutation of the natural edge set is even, the parity of the number of cycle systems, counted by hand, is equal to the parity of the determinant of the matrix. I have not found a sign convention for these graphs that explains the calculated determinant, but being equal in parity allows for hope of a possible extension to the Hamburger Theorem.

6.2 Generalizing Propp's Conjecture

A key motivation in the study of Aztec pillows is Propp's Conjecture for 3-pillows, presented in Section 5.3. Through experimental calculations, I have expanded and extended the conjecture.

For one, I extend the conjecture to odd pillows. For any odd number q, we can use the notation AP_n^q to represent the n-th q-pillow for odd q. Recall from the introduction that this is a centrally symmetric region with steps of length q and central belt of size $2 \times 2n$. Calculating the number of tilings of AP_n^5, AP_n^7, and AP_n^9 for n at least up to 40 gives strong evidence that $\#AP_n^q$ for q odd is always a larger number squared times a smaller number. As with 3-pillows, we can write $\#AP_n^q = l_{n,q}^2 s_{n,q}$, where $s_{n,q}$ is relatively

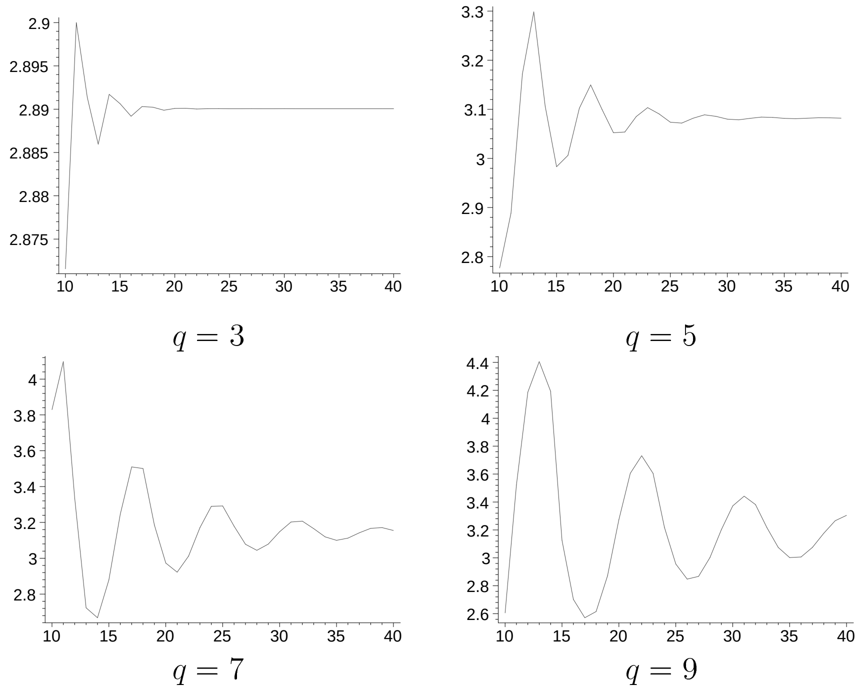

Figure 16: Plots of values of $s_{n,q}/s_{n-2,q}$ for $q = 3, 5, 7, 9$

small but is not necessarily the squarefree part of $\#AP_n^q$.

Additional structure in these smaller numbers also appears. Propp's Conjecture gives an explicit recurrence for the smaller-number values when $q = 3$. In the case $q = 5$, there is no linear, constant coefficient k-th degree recurrence for any k up to 20. However in each case ($q = 5$, $q = 7$, and $q = 9$) the structure is undeniable. Plotting the values of $s_{n,q}/s_{n-2,q}$ gives a remarkable damped sinusoidal graph converging to some number ρ. (See Figure 16.)

In addition, plotting the values of $s_{n,q}/s_{n-1,q}$ depending on whether n is even or odd gives two damped sinusoidal graphs converging to some values τ and 2τ respectively. This implies $\rho = 2\tau^2$. Approximate values for ρ are given in Table 1. We have included the values for Aztec diamonds ($q = 1$) and the original Aztec pillows ($q = 3$).

q	Approximate limit of $s_{n,q}/s_{n-2,q}$
1	2
3	2.890053636
5	3.0821372
7	3.145
9	3.18

Table 1: ρ values for various values of q

Even more appears to be true. By construction, the dual graphs to all odd pillows are 2-even-symmetric—that is, centrally symmetric so that antipodes are an even path length apart. Jockusch [7] proved that the number of matchings of such a graph G is the sum of two squares. In terms of the underlying graph, these two squares are the real and imaginary parts of a weighted matching of the graph G_2. If we consider R_2 to be the rotation by 180 degrees, the graph G_2 is derived from G by taking a branch cut through the graph, removing the vertices on one side of the branch cut, and placing an new edge between any two vertices v and w such that v and $R_2(w)$ were connected in G. The weighting gives these new edges the weight i to the power of the net number of times the original edge in G crossed the branch cut counterclockwise when traversing from the white vertex to the black vertex.

Through a different method in my doctoral dissertation [6], I also showed that the number of domino tilings of the region is a sum of two squares. This follows from the structure of the Kasteleyn–Percus matrix for a rotationally symmetric generalized Aztec pillow. After limited calculation, these two methods appear to give the same sums-of-squares representation. For a q-pillow, we will call these values $a_{n,q}$ and $b_{n,q}$; that is, for any Aztec Pillow, we can write $\#AP_n^q = a_{n,q}^2 + b_{n,q}^2$. Analysis of this set of squares yields unexpected information about the larger numbers l_n appearing in Propp's Conjecture, and the larger numbers $l_{n,q}$ in general. For any odd pillow AP_n^q, $l_{n,q}$ divides both $a_{n,q}$ and $b_{n,q}$. This gives us new insight into the structure of the previously ununderstood $l_{n,q}$, and implies that $s_{n,q}$ is also a sum of squares. Thus, I formulate a new and improved conjecture about odd pillows.

Conjecture 6.1. *The number of tilings of AP_n^q is a larger number squared times a smaller number. There exist values $l_{n,q}$ and $s_{n,q}$ such that we can write $\#AP_n^q = l_{n,q}^2 s_{n,q}$. The value s_n satisfies the following structure. The ratio of $s_{2n+1,q}/s_{2n,q}$ to $s_{2n+2,q}/s_{2n+1,q}$ is exactly 2 in the limit. In addition, for the values $a_{n,q}$ and $b_{n,q}$ in $\#AP_n^q$'s sum-of-squares representation, $l_{n,q}$ divides both $a_{n,q}$ and $b_{n,q}$.*

6.3 Conclusion

The Hamburger Theorem allows us to calculate the number of cycle systems in generalized Aztec Pillow graphs more quickly. It also allows us to count the number of cycle systems in many previously inaccessible graphs, such as non-planar hamburger graphs. Just as Gessel and Viennot's result has found many applications, I hope the Hamburger Theorem will be useful as well.

6.4 Remarks

This article is based on work in the author's doctoral dissertation [6]. A preliminary version of this article appeared as a poster at the 2005 International Conference on Formal Power Series and Algebraic Combinatorics in Taormina, Italy. The author would like to thank Henry Cohn and Tom Zaslavsky for numerous intriguing discussions and useful corrections. The author would also like to thank an anonymous referee for his or her time and suggestions for improving this article.

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