# Discrepancy of Symmetric Products of Hypergraphs

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**Abstract**

For a hypergraph $\mathcal{H} = (V, \mathcal{E})$, its $d$–fold symmetric product is defined to be $\Delta^d\mathcal{H} = (V^d, \{E^d | E \in \mathcal{E}\})$. We give several upper and lower bounds for the $c$-color discrepancy of such products. In particular, we show that the bound $\operatorname{disc}(\Delta^d\mathcal{H}, 2) \leq \operatorname{disc}(\mathcal{H}, 2)$ proven for all $d$ in [B. Doerr, A. Srivastav, and P. Wehr, Discrepancy of Cartesian products of arithmetic progressions, Electron. J. Combin. 11(2004), Research Paper 5, 16 pp.] cannot be extended to more than $c = 2$ colors. In fact, for any $c$ and $d$ such that $c$ does not divide $d!$, there are hypergraphs having arbitrary large discrepancy and $\operatorname{disc}(\Delta^d\mathcal{H}, c) = \Omega_d(\operatorname{disc}(\mathcal{H}, c)^d)$. Apart from constant factors (depending on $c$ and $d$), in these cases the symmetric product behaves no better than the general direct product $\mathcal{H}^d$, which satisfies $\operatorname{disc}(\mathcal{H}^d, c) = O_{c,d}(\operatorname{disc}(\mathcal{H}, c)^d)$.

## 1 Introduction

We investigate the discrepancy of certain products of hypergraphs. In [3], Srivastav, Wehr and the first author noted the following. For a hypergraph $\mathcal{H} = (V, \mathcal{E})$ define the $d$–fold direct product and the $d$–fold symmetric product by

$$\begin{aligned} \mathcal{H}^d &:= (V^d, \{E_1 \times \cdots \times E_d \,|\, E_i \in \mathcal{E}\}), \\ \Delta^d\mathcal{H} &:= (V^d, \{E^d \,|\, E \in \mathcal{E}\}). \end{aligned}$$

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Then for the (two-color) discrepancy

$$\operatorname{disc}(\mathcal{H}) := \min_{\chi: V \to \{-1,1\}} \max_{E \in \mathcal{E}} \left| \sum_{v \in E} \chi(v) \right|,$$

we have

$$\begin{aligned} \operatorname{disc}(\mathcal{H}^d) &\le \operatorname{disc}(\mathcal{H})^d, \\ \operatorname{disc}(\Delta^d \mathcal{H}) &\le \operatorname{disc}(\mathcal{H}). \end{aligned}$$

In this paper, we show that the situation is more complicated for discrepancies in more than two colors. In particular, it depends highly on the dimension $d$ and the number of colors, whether the discrepancy of symmetric products is more like the discrepancy of the original hypergraph or the $d$-th power thereof. Let us make this precise:

Let $\mathcal{H} = (V, \mathcal{E})$ be a *hypergraph*, that is, $V$ is some finite set and $\mathcal{E} \subseteq 2^V$. Without loss of generality, we will assume that $V = [n]$ for some $n \in \mathbb{N}$. Here and in the following we use the shorthand $[r] := \{n \in \mathbb{N} \,|\, n \le r\}$ for any $r \in \mathbb{R}$. The elements of $V$ are called *vertices*, those of $\mathcal{E}$ *(hyper)edges*. For $c \in \mathbb{N}_{\ge 2}$, a *$c$–coloring* of $\mathcal{H}$ is a mapping $\chi : V \to [c]$. The discrepancy problem asks for balanced colorings of hypergraphs in the sense that each hyperedge shall contain the same number of vertices in each color. The *discrepancy of* $\chi$ and the *$c$–color discrepancy of* $\mathcal{H}$ are defined by

$$\begin{aligned} \operatorname{disc}(\mathcal{H}, \chi) &:= \max_{E \in \mathcal{E}} \max_{i \in [c]} \left| |\chi^{-1}(i) \cap E| - \tfrac{1}{c}|E| \right|, \\ \operatorname{disc}(\mathcal{H}, c) &:= \min_{\chi: V \to [c]} \operatorname{disc}(\mathcal{H}, \chi). \end{aligned}$$

These notions were introduced in [2] extending the discrepancy problem for hypergraphs to arbitrary numbers of colors (see, e.g., the survey of Beck and Sós [1]). Note that $\operatorname{disc}(\mathcal{H}) = 2\operatorname{disc}(\mathcal{H}, 2)$ holds for all $\mathcal{H}$. In this more general setting, the product bound proven in [3] is

$$\operatorname{disc}(\mathcal{H}^d, c) \le c^{d-1} \operatorname{disc}(\mathcal{H}, c)^d. \tag{1}$$

However, as we show in this paper the relation $\operatorname{disc}(\Delta^d \mathcal{H}, c) = O(\operatorname{disc}(\mathcal{H}, c))$ does not hold in general. In Section 2, we give a characterization of those values of $c$ and $d$, for which it is satisfied for every hypergraph $\mathcal{H}$. In particular, we present for all $c, d, k$ such that $c$ does not divide $d!$ a hypergraph $\mathcal{H}$ having $\operatorname{disc}(\mathcal{H}, c) \ge k$ and $\operatorname{disc}(\Delta^d \mathcal{H}, c) = \Omega_d(k^d)$. In the light of (1), this is largest possible apart from factors depending on $c$ and $d$ only.

On the other hand, there are further situations where this worst case does not occur. We prove some in Section 3, but the complete picture seems to be complicated.

## 2 Coloring Simplices

To get some intuition of what we do in the remainder, let us regard some small examples first. For $c = 2$ colors and dimension $d = 2$, it is easy to see that $\operatorname{disc}(\Delta^d \mathcal{H}, c) \leq \operatorname{disc}(\mathcal{H}, c)$ holds for arbitrary hypergraphs $\mathcal{H} = (V, \mathcal{E})$. As mentioned above, we assume for simplicity that $V = [n]$. Now coloring the vertices above the diagonal in one color, the ones below in the other, and those on the diagonal according to an optimal coloring for the one-dimensional case does the job. More formally, let $\chi : V \to [2]$. Let $\tilde{\chi} : V^2 \to [2]$ such that $\tilde{\chi}((x, y)) = 1$, if $x < y$, $\tilde{\chi}((x, y)) = 2$, if $x > y$, and $\tilde{\chi}((x, y)) = \chi(x)$, if $x = y$. Then $\operatorname{disc}(\Delta^2 \mathcal{H}, \tilde{\chi}) = \operatorname{disc}(\mathcal{H}, \chi)$. Hence $\operatorname{disc}(\Delta^2 \mathcal{H}, 2) \leq \operatorname{disc}(\mathcal{H}, 2)$. This argument can be extended to arbitrary dimension to show $\operatorname{disc}(\Delta^d \mathcal{H}, 2) \leq \operatorname{disc}(\mathcal{H}, 2)$ for all $d \in \mathbb{N}$.

Things become more interesting if we do not restrict ourselves to 2 colors. For example, it is not clear how to extend the simple above/below diagonal approach to 3 colors (in two dimensions). In fact, as we will show in the following, such bounds do not exist for many pairs $(c, d)$, including $(3, 2)$. However, in three dimensions $\operatorname{disc}(\Delta^3 \mathcal{H}, 3) \leq \operatorname{disc}(\mathcal{H}, 3)$ follows similarly to the $(2, 2)$ proof above. Indeed, for $c = 2$ and $d = 2$ we divided the product set $V^2$ into the sets above and below the diagonal, which we want to call two-dimensional simplices of $V^2$, and the diagonal, a one-dimensional simplex of $V^2$. For $c = 3$ and $d = 3$ we divide $V^3$ into the six three-dimensional simplices in $V^3$ that we obtain from the set $\{x \in V^3 \mid x_1 < x_2 < x_3\}$ by permuting coordinates, the six two-dimensional simplices in $V^3$ that we obtain from $\{x \in V^3 \mid x_1 = x_2 < x_3\}$ by permuting coordinates and possibly changing $<$ to $>$, and finally the one-dimensional simplex $\{x \in V^3 \mid x_1 = x_2 = x_3\}$. Now with each color we color exactly two three-dimensional and two two-dimensional simplices of $V^3$. The vertices of the diagonal will again be colored according to an optimal coloring for the one-dimensional case.

We shall now give a formal definition of $l$-dimensional simplices in arbitrary dimensions. A set $\{x_1, \ldots, x_k\}$ of integers with $x_1 < \ldots < x_k$ is denoted by $\{x_1, \ldots, x_k\}_<$. For a set $S$ we put

$$\binom{S}{k} = \{T \subseteq S \mid |T| = k\}.$$

Furthermore, let $S_k$ be the symmetric group on $[k]$. For $l, d \in \mathbb{N}$ with $l \leq d$ let $P_l(d)$ be the set of all partitions of $[d]$ into $l$ non-empty subsets. Let $e_1 = (1, 0, \ldots, 0)$, …, $e_d = (0, \ldots, 0, 1)$ be the standard basis of $\mathbb{R}^d$. For $c \in \mathbb{N}$ and $\lambda \in \mathbb{N}_0$ we write $c \mid \lambda$ if there exists an $m \in \mathbb{N}_0$ with $mc = \lambda$.

**Definition 1.** *Let $d \in \mathbb{N}$, $l \in [d]$ and $T \subseteq \mathbb{N}$ finite. For $J = \{J_1, \ldots, J_l\} \in P_l(d)$ with $\min J_1 < \ldots < \min J_l$ put $f_i = f_i(J) = \sum_{j \in J_i} e_j$, $i = 1, \ldots, l$. Let $\sigma \in S_l$. We call*

$$S_J^\sigma(T) := \Big\{ \sum_{i=1}^{l} \alpha_{\sigma(i)} f_i(J) \,\Big|\, \{\alpha_1, \ldots, \alpha_l\}_< \subseteq T \Big\}$$

*an $l$-dimensional simplex in $T^d$. If $l = d$, we simply write $S^\sigma(T)$ instead of $S^\sigma_J(T)$ (as $|P_d(d)| = 1$).*

Clearly, the simplices in a $d$-dimensional grid $T^d$ form a partition of $T^d$. The next remark shows that the numbers of $l$-dimensional simplices are well-understood.

**Remark 2.** *If $S(d, l)$, $d, l \in \mathbb{N}$, denote the* Stirling numbers of the second kind*, then $|P_l(d)| = S(d, l)$ (see, e.g. [6]). We have*

$$S(d, l) = \sum_{j=0}^{l} \frac{(-1)^j (l-j)^d}{j!\,(l-j)!}\,. \tag{2}$$

*Let $T \subseteq \mathbb{N}$ finite. Furthermore, let $I, J \in P_l(d)$ and $\sigma, \tau \in S_l$. If $|T| \geq l$, we have $S^\sigma_I(T) \neq S^\tau_J(T)$ as long as $I \neq J$ or $\sigma \neq \tau$. Thus the number of $l$-dimensional simplices in $T^d$ is $l!\,S(d, l)$. If $|T| < l$, then there exists obviously no non-empty $l$-dimensional simplex in $T^d$.*

We are now able to prove the main result of this paper.

**Theorem 3.** *Let $c, d \in \mathbb{N}$.*

*(i) If $c \mid k!\,S(d, k)$ for all $k \in \{2, \dots, d\}$, then every hypergraph $\mathcal{H}$ satisfies*

$$\operatorname{disc}(\Delta^d \mathcal{H}, c) \leq \operatorname{disc}(\mathcal{H}, c)\,. \tag{3}$$

*(ii) If $c \nmid k!\,S(d, k)$ for some $k \in \{2, \dots, d\}$, then there exists a hypergraph $\mathcal{K}$ such that*

$$\operatorname{disc}(\Delta^d \mathcal{K}, c) \geq \frac{1}{3\,k!} \operatorname{disc}(\mathcal{K}, c)^k\,, \tag{4}$$

*and $\mathcal{K}$ can be chosen to have arbitrary large discrepancy* $\operatorname{disc}(\mathcal{K}, c)$.

Before proving the theorem, we state some consequences. In particular, (3) holds never for $c = 4$. For $c = 3$, it holds exactly if $d$ is odd.

**Corollary 4.** *(a) Let $d \geq 3$ be an odd number. Then* $\operatorname{disc}(\Delta^d \mathcal{H}, 3) \leq \operatorname{disc}(\mathcal{H}, 3)$ *holds for any hypergraph $\mathcal{H}$.*

*(b) Let $d \geq 2$ be an even number and $c = 3l$, $l \in \mathbb{N}$. There exists a hypergraph $\mathcal{H}$ with arbitrary large discrepancy that satisfies* $\operatorname{disc}(\Delta^d \mathcal{H}, c) \geq \frac{1}{6} \operatorname{disc}(\mathcal{H}, c)^2$.

*Proof.* Obviously $3 \mid k!$ for all $k \geq 3$. Since $S(d, 2) = 2^{d-1} - 1$, we have $3 \mid S(d, 2)$ if and only if $d$ is odd. Indeed, $2^{3-1} - 1 = 3$, $2^{4-1} - 1 = 7$ and if $d = k+2$, then $2^{d-1} - 1 = 4(2^{k-1} - 1) + 3$, hence $3 \mid (2^{d-1} - 1)$ if and only if $3 \mid (2^{k-1} - 1)$. Hence Theorem 3 proves both claims. $\square$

**Corollary 5.** *Let $l \in \mathbb{N}$ and $c = 4l$. For all $d \geq 2$ there exists a hypergraph $\mathcal{H}$ with arbitrary large discrepancy such that* $\operatorname{disc}(\Delta^d \mathcal{H}, c) \geq \frac{1}{6} \operatorname{disc}(\mathcal{H}, c)^2$.

*Proof.* As $S(d,2) = 2^{d-1} - 1$ is an odd number, we have $4 \nmid 2!\, S(d,2)$. Applying Theorem 3 concludes the proof. $\square$

**Corollary 6.** *Let $c \geq 3$ be an odd number and $d \geq 2$. We have*

$$\operatorname{disc}(\Delta^d \mathcal{H}, c) \leq \operatorname{disc}(\mathcal{H}, c) \quad \textit{for all hypergraphs } \mathcal{H} \tag{5}$$

*if and only if we have*

$$\operatorname{disc}(\Delta^d \mathcal{H}, 2c) \leq \operatorname{disc}(\mathcal{H}, 2c) \quad \textit{for all hypergraphs } \mathcal{H}\,. \tag{6}$$

*Proof.* According to Theorem 3, (5) is equivalent to the statement that $c|\, k!\, S(d,k)$ for all $k \in \{2, \ldots, d\}$. But, since $2|\, k!$ for all $k \geq 2$ and $c$ is odd, this is equivalent to $2c|\, k!\, S(d,k)$ for all $k \in \{2, \ldots, d\}$, which is equivalent to (6). $\square$

We now prove the upper bound Theorem 3(i). The main idea is that each hyperedge of the symmetric product intersects all $l$-dimensional simplices with same cardinality. Hence we may color the simplices monochromatically if we can use each color equally often for each $l \geq 2$.

*Proof of Theorem 3(i).* Let $c, d$ be such that $c \,|\, k!\, S(d,k)$ for all $k \in \{2, \ldots, d\}$. Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph and let $\psi : V \to [c]$ such that $\operatorname{disc}(\mathcal{H}, \psi) = \operatorname{disc}(\mathcal{H}, c)$. For $X \subseteq V$, put $D(X) = \{(x, \ldots, x) \mid x \in X\}$. We define the following $c$-coloring $\chi : V^d \to [c]$. For $(v, \ldots, v) \in D(V)$, set $\chi(v, \ldots, v) = \psi(v)$. For the remaining vertices, let $\chi$ be such that all simplices are monochromatic, and for each $k$ there are exactly $\frac{1}{c} k! S(d,k)$ monochromatic $k$-dimensional simplices in each color.

Let $E \in \mathcal{E}$ and put $R(E) := E^d \setminus D(E)$. For any $k \in \{2, \ldots, d\}$ and any two $k$-dimensional simplices $S, S'$ we have $|S \cap R(E)| = |S' \cap R(E)|$. Therefore, our choice of $\chi$ implies $|\chi^{-1}(i) \cap R(E)| = \frac{1}{c} |R(E)|$ for all $i \in [c]$. Hence

$$\begin{aligned}
&\max_{i \in [c]} \left| |\chi^{-1}(i) \cap E^d| - \frac{|E^d|}{c} \right| \\
&= \max_{i \in [c]} \left| |\chi^{-1}(i) \cap R(E)| - \frac{|R(E)|}{c} + |\chi^{-1}(i) \cap D(E)| - \frac{|D(E)|}{c} \right| \\
&= \max_{i \in [c]} \left| |\chi^{-1}(i) \cap D(E)| - \frac{|D(E)|}{c} \right| = \max_{i \in [c]} \left| |\psi^{-1}(i) \cap E| - \frac{|E|}{c} \right|.
\end{aligned}$$

This calculation establishes $\operatorname{disc}(\Delta^d \mathcal{H}, c) \leq \operatorname{disc}(\mathcal{H}, c)$. $\square$

To prove the lower bound in Theorem 3, we use the following Ramsey theoretic approach.

**Lemma 7.** *Let $c, d \in \mathbb{N}$. For all $m \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ having the following property: For each c-coloring $\chi : [n]^d \to [c]$ we find a subset $T \subseteq [n]$ with $|T| = m$ such that for all $l \in [d]$ each l-dimensional simplex in $T^d$ is monochromatic with respect to $\chi$.*

*Proof of Lemma 7.* The proof is based on an argument from Ramsey theory. First we verify the statement of Lemma 7 for a fixed simplex. Then, by induction over the number of all simplices, we prove the complete assertion of Lemma 7.

*Claim*: For all $m \in \mathbb{N}$, all $l \in [d]$, all $\sigma \in S_l$, and all $J \in P_l(d)$, there is an $n \in \mathbb{N}$ such that for all $N \subseteq \mathbb{N}$ with $|N| = n$ and each $c$–coloring $\chi : N^d \to [c]$ there is a subset $T \subseteq N$ with $|T| = m$ and $S_J^\sigma(T)$ is monochromatic with respect to $\chi$.

*Proof of the claim*: By Ramsey's theorem (see, e.g. [4], Section 1.2), for every $l \in [d]$ there exists an $n$ such that for each $c$-coloring $\psi : \binom{[n]}{l} \to [c]$ there is a subset $T$ of $[n]$ with $|T| = m$ and $\binom{T}{l}$ is monochromatic with respect to $\psi$. Let $N \subseteq \mathbb{N}$ with $|N| = n$. We can assume $N = [n]$ by renaming the elements of $N$ and preserving their order. Let $\chi : [n]^d \to [c]$ be an arbitrary $c$–coloring. We define $\chi_{l,\sigma,J} : \binom{[n]}{l} \to [c]$ by $\chi_{l,\sigma,J}(\{x_1, \ldots, x_l\}_<) = \chi(\sum_{i=1}^{l} x_{\sigma(i)} f_i)$, where the $f_i = f_i(J)$ are the vectors corresponding to the partition $J$ introduced in Definition 1. By the Ramsey theory argument there is a $T \subseteq N$ with $|T| = m$ and $\chi_{l,\sigma,J}$ is constant on $\binom{T}{l}$. Hence, $S_J^\sigma(T)$ is monochromatic with respect to $\chi$. This proves the claim.

Now we derive Lemma 7 from the claim. Each simplex is uniquely determined by a pair

$$(\sigma, J) \in \bigcup_{l=1}^{d} (S_l \times P_l(d)) \,.$$

Let $(\sigma_i, J_i)_{i \in [s]}$ be an enumeration of all these pairs. Put $n_0 := m$. We proceed by induction. Let $i \in [s]$ be such that $n_{i-1}$ is already defined and has the property that for any $N \subseteq \mathbb{N}$, $|N| = n_{i-1}$ and any coloring $\chi : N^d \to [c]$ there is a $T \subseteq N$, $|T| = m$ such that for all $j \in [i-1]$, $S_{J_j}^{\sigma_j}(T)$ is monochromatic. Using the claim, we choose $n_i$ large enough such that for each $N \subseteq \mathbb{N}$ with $|N| = n_i$ and for each $c$–coloring $\varphi : N^d \to [c]$ there exists a subset $T$ of $N$ with $|T| = n_{i-1}$ and $S_{J_i}^{\sigma_i}(T)$ is monochromatic with respect to $\varphi$. Note that there is a $T' \subseteq T$, $|T| = m$ such that $S_{J_j}^{\sigma_j}(T')$ is monochromatic for all $j \in [i]$. Choosing $n := n_s$ proves the lemma. □

Related to Lemma 7 is a result of Gravier, Maffray, Renault and Trotignon [5]. They have shown that for any $m \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that any collection of $n$ different sets contains an induced subsystem on $m$ points such that one of the following holds: (a) each vertex forms a singleton, (b) for each vertex there is a set containing all $m$ points except this one, or (c) by sufficiently ordering the points $p_1, \ldots, p_m$ we have that all sets $\{p_1, \ldots, p_\ell\}, \ell \in [m]$, are contained in the system.[1]

In our language, this means that any $0, 1$ matrix having $n$ distinct rows contains a $m \times m$ submatrix that can be transformed through row and column permutations into a matrix that is (a) a diagonal matrix, (b) the inverse of a diagonal matrix, or (c) a triangular matrix.

Hence this result is very close to the assertion of Lemma 7 for dimension $d = 2$ and $c = 2$ colors. It is stronger in the sense that not only monochromatic simplices are guaranteed, but also a restriction to 3 of the 8 possible color combinations for the 3 simplices is given. Of course, this stems from the facts that (a) column and row permutations are allowed, (b) not a submatrix with index set $T^2$ is provided but only one of type $S \times T$, and (c) the assumption of having different sets ensures sufficiently many entries in both colors.

We are now in the position to prove the second part of Theorem 3.

*Proof of Theorem 3(ii).* Let $c$ and $d$ be such that $c \not| \, k! \, S(d, k)$ for some $k \in \{2, \ldots, d\}$. Let $m$ be large enough to satisfy

$$\frac{1}{2}\binom{m}{\kappa} - \sum_{l=0}^{\kappa-1} l! \, S(d, l) \binom{m}{l} \geq \frac{1}{3\, k!} m^k$$

for all $\kappa \in \{k, \ldots, d\}$. (This can obviously be done, since the left hand side of the last inequality is of the form $m^\kappa / 2\kappa! + O(m^{\kappa-1})$ for $m \to \infty$.) Using Lemma 7, we choose $n \in \mathbb{N}$ such that for any $c$-coloring $\chi : [n]^d \to [c]$ there is an $m$-point set $T \subseteq [n]$ with all simplices in $T^d$ being monochromatic with respect to $\chi$.

We show that $\mathcal{K} = \left([n], \binom{[n]}{m}\right)$ satisfies our claim. Let $\chi$ be any $c$–coloring of $\mathcal{K}$, choose $T$ as in Lemma 7. Let $\kappa \in \{k, \ldots, d\}$ be such that for each $l \in \{\kappa + 1, \ldots, d\}$ there is the same number of $l$-dimensional simplices in $T$ in each color but not so for the $\kappa$-dimensional simplices. With

$$\mathcal{S} := \bigcup_{l=\kappa}^{d} \bigcup_{J \in P_l(d)} \bigcup_{\sigma \in S_l} S_J^\sigma(T)$$

[1] To be precise, the authors also have the empty set contained in cases (a) and (c) and the whole set in case (b). It is obvious that by altering $m$ by one, one can transform one result into the other.

we obtain

$$\begin{aligned}
\operatorname{disc}(\Delta^d\mathcal{K},\chi) &\geq \max_{i\in[c]}\left||\chi^{-1}(i)\cap T^d| - \frac{|T^d|}{c}\right| \\
&\geq \max_{i\in[c]}\left\{\left||\chi^{-1}(i)\cap \mathcal{S}| - \frac{|\mathcal{S}|}{c}\right| - \left||\chi^{-1}(i)\cap (T^d\setminus\mathcal{S})| - \frac{|T^d\setminus\mathcal{S}|}{c}\right|\right\} \\
&\geq \max_{i\in[c]}\left|\sum_{J\in P_\kappa(d),\sigma\in S_\kappa}|\chi^{-1}(i)\cap S_J^\sigma(T)| - \frac{\kappa!\,S(d,\kappa)}{c}\binom{m}{\kappa}\right| \\
&\quad -\frac{c-1}{c}\left(m^d - \sum_{l=\kappa}^{d} l!\,S(d,l)\binom{m}{l}\right) \\
&\geq \frac{1}{2}\binom{m}{\kappa} - \sum_{l=0}^{\kappa-1} l!\,S(d,l)\binom{m}{l} \geq \frac{1}{3\,k!}m^k\,.
\end{aligned}$$

This establishes $\operatorname{disc}(\Delta^d\mathcal{K},c) \geq \frac{1}{3\,k!}m^k$. Note that our choice of $n$ implies $\operatorname{disc}(\mathcal{K},c) = \left(1-\frac{1}{c}\right)m$. □

## 3 Further Upper Bounds

Besides the first part of Theorem 3, there are more ways to obtain upper bounds.

**Theorem 8.** *Let $\mathcal{H}=(V,\mathcal{E})$ be a hypergraph. Let $p$ be a prime number, $q\in\mathbb{N}$ and $c=p^q$. Furthermore, let $d\geq c$ and $s=d-(p-1)p^{q-1}$. Then* $\operatorname{disc}(\Delta^d\mathcal{H},c)\leq\operatorname{disc}(\Delta^s\mathcal{H},c)$.

**Corollary 9.** *Let $\mathcal{H}=(V,\mathcal{E})$ be a hypergraph.*

*(a) If $c$ is a prime number, $q\in\mathbb{N}$ and $d=c^q$, then* $\operatorname{disc}(\Delta^d\mathcal{H},c)\leq\operatorname{disc}(\mathcal{H},c)$.

*(b) For arbitrary $d\in\mathbb{N}$ there holds* $\operatorname{disc}(\Delta^d\mathcal{H},2)\leq\operatorname{disc}(\mathcal{H},2)$.

Statement (a) of the corollary follows from the identity $c^q = 1+(c-1)\sum_{j=0}^{q-1}c^j$ and the (repeated) use of Theorem 8. Conclusion (b) follows also from Theorem 8. Note that Theorem 3 implies that in both parts of Corollary 9 we have $c\,|\,k!\,S(d,k)$ for all $k\in\{2,\ldots,d\}$. Hence Corollary 9 could also have been proven by analysing the Stirling numbers.

*Proof of Theorem 8.* As always, we assume without loss of generality that $V=[n]$. Let us define the shift operator $S:[n]^d\to[n]^d$ by

$$S(x_1,\ldots,x_c,x_{c+1},\ldots,x_d) = (x_2,\ldots,x_c,x_1,x_{c+1},\ldots,x_d)\,.$$

It induces an equivalence relation $\sim$ on $[n]^d$ by $x \sim y$ if and only if there exists a $k \in [c]$ with $S^k x = y$. Now let $x \in [n]^d$ and denote its equivalence class by $\langle x \rangle$. Put $k = |\langle x \rangle|$. Obviously $k$ is the minimal integer in $[c]$ with $S^k x = x$. A standard argument from elementary group theory ("group acting on a set") shows that $k \,|\, c$. Thus either $k = c$ or $S^{p^{q-1}} x = x$. Define $D = \{y \in [n]^d \,|\, |\langle y \rangle| < c\}$. Then

$$\psi : D \to [n]^s \,,\; y \mapsto (y_1, \ldots, y_{p^{q-1}}, y_{c+1}, \ldots, y_d)$$

is a bijection. For a given $c$-coloring $\chi$ of $[n]^s$, we define a $c$-coloring $\tilde{\chi}$ of $[n]^d$ in the following way: We choose a system of representatives $R$ for $\sim$. If $x \in R$ with $|\langle x \rangle| = c$, we put $\tilde{\chi}(S^i x) = i$ for all $i \in [c]$. If $|\langle x \rangle| < c$, then $\tilde{\chi}(y) = (\chi \circ \psi)(y)$ for all $y \in \langle x \rangle$.

Let $E \in \mathcal{E}$. Notice, that $x \in E^d$ implies $\langle x \rangle \subseteq E^d$, and $x \in D$ implies $\langle x \rangle \subseteq D$. Furthermore, the restriction of $\psi$ to $E^d \cap D$ is a bijection onto $E^s$. Thus

$$\begin{aligned}
\max_{i \in [c]} \left| |\tilde{\chi}^{-1}(i) \cap E^d| - \frac{|E^d|}{c} \right| &\leq \max_{i \in [c]} \left| |\tilde{\chi}^{-1}(i) \cap (E^d \cap D)| - \frac{|E^d \cap D|}{c} \right| \\
&\quad + \max_{i \in [c]} \left| |\tilde{\chi}^{-1}(i) \cap (E^d \setminus D)| - \frac{|E^d \setminus D|}{c} \right| \\
&\leq \max_{i \in [c]} \left| |\chi^{-1}(i) \cap E^s| - \frac{|E^s|}{c} \right| + 0 \,.
\end{aligned}$$

Hence $\operatorname{disc}(\Delta^d \mathcal{H}, c) \leq \operatorname{disc}(\Delta^s \mathcal{H}, c)$. □

The following is an extension of the first statement of Theorem 3.

**Theorem 10.** *Let $c$, $d \in \mathbb{N}$, and let $d' \in \{2. \ldots, d\}$. If $c \,|\, k!\, S(d', k)$ for all $k \in \{2, \ldots, d'\}$, then*

$$\operatorname{disc}(\Delta^d \mathcal{H}, c) \leq \operatorname{disc}(\Delta^{d-d'+1} \mathcal{H}, c) \tag{7}$$

*holds for every hypergraph $\mathcal{H}$.*

*Proof of Theorem 10.* Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with $V = [n]$. Let $\chi : [n]^{d-d'+1} \to [c]$ be an arbitrary $c$–coloring. We define a $c$–coloring $\widetilde{\chi} : [n]^d \to [c]$. Let $z \in [n]^d$, $x = (z_1, \ldots, z_{d'})$, and $y = (z_{d'+1}, \ldots, z_d)$. If $z_1 = \ldots = z_{d'} =: \zeta$, put $\widetilde{\chi}(z) = \chi(\zeta, z_{d'+1}, \ldots, z_d)$. Otherwise we find $k \in \{2, \ldots, d'\}$, $J \in P_k(d')$ and $\sigma \in S_k$ with $x \in S^\sigma_J([n])$. Since $c \,|\, k!\, S(d', k)$, we can color the set $D := \big\{(z_{\tau(1)}, \ldots, z_{\tau(d')}, y) \,|\, \tau \in S_{d'}\big\}$ of cardinality $k!\, S(d', k)$ evenly by our coloring $\widetilde{\chi} : [n]^d \to [c]$. A similar calculation as the one at the end of the proof of Theorem 8 establishes $\operatorname{disc}(\Delta^d \mathcal{H}, \widetilde{\chi}) \leq \operatorname{disc}(\Delta^{d-d'+1} \mathcal{H}, \chi)$. □

**Remark 11.** *The condition in Theorem 10 is only sufficient but not necessary for the validity of (7), as the following example shows:*

*Let $c = 4$, $d \geq c$ and $d' = 3$. According to Theorem 8, we get for each hypergraph $\mathcal{H}$ that* $\operatorname{disc}(\Delta^d \mathcal{H}, c) \leq \operatorname{disc}(\Delta^{d-2} \mathcal{H}, c) = \operatorname{disc}(\Delta^{d-d'+1} \mathcal{H}, c)$. *But we have* $2!\, S(d', 2) = 6 = 3!\, S(d', 3)$ *and* $4 \nmid 6$.

*This example shows also, that the methods used in the proofs of Theorem 8 and Theorem 10 are different.*

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