A New Statistic on Linear and Circular r-Mino Arrangements

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Abstract

We introduce a new statistic on linear and circular r-mino arrangements which leads to interesting polynomial generalizations of the r-Fibonacci and r-Lucas sequences. By studying special values of these polynomials, we derive periodicity and parity theorems for this statistic.

1 Introduction

In what follows, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{P}$ denote, respectively, the integers, the nonnegative integers, and the positive integers. Empty sums take the value 0 and empty products the value 1, with $0^0 := 1$. If q is an indeterminate, then $0_q := 0$, $n_q := 1 + q + \cdots + q^{n-1}$ for $n \in \mathbb{P}$, $0_q^! := 1$, $n_q^! := 1_q 2_q \cdots n_q$ for $n \in \mathbb{P}$, and

$$\binom{n}{k}_q := \begin{cases} \frac{n_q^!}{k_q^!(n-k)_q^!}, & \text{if } 0 \leqslant k \leqslant n; \\ 0, & \text{if } k < 0 \text{ or } 0 \leqslant n < k. \end{cases} \tag{1.1}$$

A useful variation of (1.1) is the well known formula [8, p. 29]

$$\binom{n}{k}_q = \sum_{\substack{d_0+d_1+\cdots+d_k=n-k \\ d_i \in \mathbb{N}}} q^{0d_0+1d_1+\cdots+kd_k} = \sum_{t \geqslant 0} p(k, n-k, t) q^t, \tag{1.2}$$

where $p(k, n-k, t)$ denotes the number of partitions of the integer t with at most $n-k$ parts, each no larger than k.

If $r \geqslant 2$, the *r-Fibonacci numbers* $F_n^{(r)}$ are defined by $F_0^{(r)} = F_1^{(r)} = \cdots = F_{r-1}^{(r)} = 1$, with $F_n^{(r)} = F_{n-1}^{(r)} + F_{n-r}^{(r)}$ if $n \geqslant r$. The *r-Lucas numbers* $L_n^{(r)}$ are defined by $L_1^{(r)} = L_2^{(r)} = \cdots = L_{r-1}^{(r)} = 1$ and $L_r^{(r)} = r+1$, with $L_n^{(r)} = L_{n-1}^{(r)} + L_{n-r}^{(r)}$ if $n \geqslant r+1$. If $r = 2$, the $F_n^{(r)}$ and $L_n^{(r)}$ reduce, respectively, to the classical Fibonacci and Lucas numbers (parametrized as in [10], by $F_0 = F_1 = 1$, etc., and $L_1 = 1$, $L_2 = 3$, etc.).

Polynomial generalizations of F_n and/or L_n have arisen as generating functions for statistics on binary words [1], lattice paths [4], and linear and circular domino arrangements [6]. Generalizations of $F_n^{(r)}$ and/or $L_n^{(r)}$ have arisen similarly in connection with statistics on Morse code sequences [2], [3].

In the present paper, we study the polynomial generalizations

$$F_n^{(r)}(q,t) := \sum_{0 \leqslant k \leqslant \lfloor n/r \rfloor} q^{\binom{n-rk+1}{2}} \binom{n-(r-1)k}{k}_{q^r} t^k \tag{1.3}$$

of $F_n^{(r)}$ and

$$L_n^{(r)}(q,t) := \sum_{0 \leqslant k \leqslant \lfloor n/r \rfloor} q^{\binom{n-rk+1}{2}} \left[\frac{k_{q^r} \sum_{i=1}^{r} q^{i(n-rk)} + (n-rk)_{q^r}}{(n-(r-1)k)_{q^r}} \right] \binom{n-(r-1)k}{k}_{q^r} t^k \tag{1.4}$$

of $L_n^{(r)}$. We present both algebraic and combinatorial evaluations of $F_n^{(r)}(-1,t)$ and $L_n^{(r)}(-1,t)$, as well as determine when the sequences $F_n^{(r)}(1,-1)$, $F_n^{(r)}(-1,1)$, $L_n^{(r)}(1,-1)$, and $L_n^{(r)}(-1,1)$ are periodic. Our algebraic proofs make frequent use of the identity [9, pp. 201–202]

$$\sum_{n \geqslant 0} \binom{n}{k}_q x^n = \frac{x^k}{(1-x)(1-qx)\cdots(1-q^k x)}, \qquad k \in \mathbb{N}. \tag{1.5}$$

Our combinatorial proofs are based on the fact that $F_n^{(r)}(q,t)$ and $L_n^{(r)}(q,t)$ are, respectively, bivariate generating functions for a pair of statistics on linear and circular r-mino arrangements.

2 Linear r-Mino Arrangements

Consider the problem of finding the number of ways to place k indistinguishable non-overlapping r-minos on the numbers $1, 2, \ldots, n$, arranged in a row, where an *r-mino*, $r \geqslant 2$, is a rectangular piece capable of covering r numbers. It is useful to place *squares* (pieces covering a single number) on each number not covered by an r-mino. The original problem then becomes one of enumerating $\mathcal{R}_{n,k}^{(r)}$, the set of coverings of the row of numbers $1, 2, \ldots, n$ by k r-minos and $n-rk$ squares. Since each such covering corresponds uniquely to a word in the alphabet $\{r, s\}$ comprising k r's and $n-rk$ s's, it follows that

$$|\mathcal{R}_{n,k}^{(r)}| = \binom{n-(r-1)k}{k}, \qquad 0 \leqslant k \leqslant \lfloor n/r \rfloor, \tag{2.1}$$

for all $n \in \mathbb{P}$. (In what follows, we will identify coverings with such words.) If we set $\mathcal{R}_{0,0}^{(r)} = \{\emptyset\}$, the "empty covering," then (2.1) holds for $n = 0$ as well. With

$$\mathcal{R}_n^{(r)} := \bigcup_{0 \leqslant k \leqslant \lfloor n/r \rfloor} \mathcal{R}_{n,k}^{(r)}, \qquad n \in \mathbb{N}, \tag{2.2}$$

it follows that

$$|\mathcal{R}_n^{(r)}| = \sum_{0 \leqslant k \leqslant \lfloor n/r \rfloor} \binom{n-(r-1)k}{k} = F_n^{(r)}, \tag{2.3}$$

where $F_0^{(r)} = F_1^{(r)} = \cdots = F_{r-1}^{(r)} = 1$, with $F_n^{(r)} = F_{n-1}^{(r)} + F_{n-r}^{(r)}$ if $n \geqslant r$. Note that

$$\sum_{n \geqslant 0} F_n^{(r)} x^n = \frac{1}{1-x-x^r}. \tag{2.4}$$

Given $c \in \mathcal{R}_n^{(r)}$, let $v(c) :=$ the number of r-minos in the covering c, let $s(c) :=$ the sum of the numbers covered by the squares in c, and let

$$F_n^{(r)}(q,t) := \sum_{c \in \mathcal{R}_n^{(r)}} q^{s(c)} t^{v(c)}, \qquad n \in \mathbb{N}. \tag{2.5}$$

The statistic v is well known and has occurred in several contexts (see, e.g., [2], [4], [6]). On the other hand, the statistic s does not seem to have appeared in the literature.

Categorizing covers of $1, 2, \ldots, n$ according as n is covered by a square or r-mino yields the recurrence relation

$$F_n^{(r)}(q,t) = q^n F_{n-1}^{(r)}(q,t) + t F_{n-r}^{(r)}(q,t), \qquad n \geqslant r, \tag{2.6}$$

with $F_i^{(r)}(q,t) = q^{\binom{i+1}{2}}$ for $0 \leqslant i \leqslant r-1$. The following theorem gives an explicit formula for $F_n^{(r)}(q,t)$.

Theorem 2.1. *For all $n \in \mathbb{N}$,*

$$F_n^{(r)}(q,t) = \sum_{0 \leqslant k \leqslant \lfloor n/r \rfloor} q^{\binom{n-rk+1}{2}} \binom{n-(r-1)k}{k}_{q^r} t^k. \tag{2.7}$$

Proof. It clearly suffices to show that

$$\sum_{c \in \mathcal{R}_{n,k}^{(r)}} q^{s(c)} = q^{\binom{n-rk+1}{2}} \binom{n-(r-1)k}{k}_{q^r}.$$

Each $c \in \mathcal{R}_{n,k}^{(r)}$ corresponds uniquely to a sequence $(d_0, \ldots, d_{n-rk})$, where d_0 is the number of r-minos following the $(n-rk)^{th}$ square in the covering c, d_{n-rk} is the number of r-minos preceding the first square, and, for $0 < i < n-rk$, d_{n-rk-i} is the number of r-minos between squares i and $i+1$. Then $s(c) = (rd_{n-rk}+1) + (rd_{n-rk}+rd_{n-rk-1}+2) + \cdots + (rd_{n-rk}+rd_{n-rk-1}+\cdots+rd_1+n-rk) = \binom{n-rk+1}{2} + r(0d_0+1d_1+2d_2+\cdots+(n-rk)d_{n-rk})$, so that

$$\begin{aligned}\sum_{c\in\mathcal{R}^{(r)}_{n,k}} q^{s(c)} &= q^{\binom{n-rk+1}{2}} \sum_{\substack{d_0+d_1+\cdots+d_{n-rk}=k\\ d_i\in\mathbb{N}}} q^{r(0d_0+1d_1+\cdots+(n-rk)d_{n-rk})}\\ &= q^{\binom{n-rk+1}{2}} \binom{n-(r-1)k}{k}_{q^r},\end{aligned}$$

by (1.2). □

Remark 1. The occurrence of a q^r-binomial coefficient in (2.7), and in (3.6) below, supports Knuth's contention [5] that Gaussian coefficients should be denoted by $\binom{n}{k}_q$, rather than by the traditional notation $\left[{n \atop k}\right]$.

Remark 2. Cigler [3] has studied the generalized Carlitz-Fibonacci polynomials given by

$$F_n(j,x,t,q) = \sum_{0\leqslant kj\leqslant n-j+1} q^{j\binom{k}{2}} \binom{n-(j-1)(k+1)}{k}_q t^k x^{n-(k+1)j+1},$$

to which the $F^{(r)}_n(q,t)$ are related by

$$F^{(r)}_n(q,t) = q^{\binom{n+1}{2}} F_{n+r-1}(r,1,t/q^{\binom{r+1}{2}},1/q^r).$$

Theorem 2.2. *The ordinary generating function of the sequence* $(F^{(r)}_n(q,t))_{n\geqslant 0}$ *is given by*

$$\sum_{n\geqslant 0} F^{(r)}_n(q,t)x^n = \sum_{k\geqslant 0} \frac{q^{\binom{k+1}{2}}x^k}{(1-x^rt)(1-q^rx^rt)\cdots(1-q^{rk}x^rt)}. \tag{2.8}$$

Proof. By (2.7),

$$\begin{aligned}\sum_{n\geqslant 0} F^{(r)}_n(q,t)x^n &= \sum_{n\geqslant 0} x^n \sum_{0\leqslant k\leqslant \lfloor n/r\rfloor} q^{\binom{n-rk+1}{2}} \binom{n-(r-1)k}{k}_{q^r} t^k\\ &= \sum_{j=0}^{r-1}\sum_{m\geqslant 0} x^{mr+j} \sum_{0\leqslant k\leqslant m} q^{\binom{(m-k)r+j+1}{2}} \binom{(m-k)(r-1)+m+j}{k}_{q^r} t^k\\ &= \sum_{j=0}^{r-1}\sum_{m\geqslant 0} x^{mr+j} \sum_{0\leqslant k\leqslant m} q^{\binom{kr+j+1}{2}} \binom{k(r-1)+m+j}{m-k}_{q^r} t^{m-k}\\ &= \sum_{j=0}^{r-1}\sum_{k\geqslant 0} q^{\binom{kr+j+1}{2}} x^{-(r-1)(kr+j)} t^{-(kr+j)} \sum_{m\geqslant k} \binom{k(r-1)+m+j}{kr+j}_{q^r} (x^rt)^{k(r-1)+m+j}\\ &= \sum_{j=0}^{r-1}\sum_{k\geqslant 0} q^{\binom{kr+j+1}{2}} \frac{x^{kr+j}}{(1-x^rt)(1-q^rx^rt)\cdots(1-q^{(kr+j)r}x^rt)},\end{aligned}$$

by (1.5), which yields (2.8), upon replacing $kr + j$ by $k \geqslant 0$. $\square$

Note that $F_n^{(r)}(1,1) = F_n^{(r)}$, whence (2.8) generalizes (2.4). Setting $q = 1$ and $q = -1$ in (2.8) yields

Corollary 2.2.1. *The ordinary generating function of the sequence* $(F_n^{(r)}(1,t))_{n\geqslant 0}$ *is given by*

$$\sum_{n\geqslant 0} F_n^{(r)}(1,t)x^n = \frac{1}{1 - x - tx^r}. \tag{2.9}$$

and

Corollary 2.2.2. *The ordinary generating function of the sequence* $(F_n^{(r)}(-1,t))_{n\geqslant 0}$ *is given by*

$$\sum_{n\geqslant 0} F_n^{(r)}(-1,t)x^n = \begin{cases} \dfrac{1 - x - tx^r}{1 + x^2 - 2tx^r + t^2x^{2r}}, & \textit{if } r \textit{ is even;} \\[2ex] \dfrac{1 - x + tx^r}{1 + x^2 - t^2x^{2r}}, & \textit{if } r \textit{ is odd.} \end{cases} \tag{2.10}$$

When $r = 2$ and $t = -1$ in (2.9), we get

$$\sum_{n\geqslant 0} F_n^{(2)}(1,-1)x^n = \frac{1}{1 - x + x^2} = \frac{(1+x)(1-x^3)}{1 - x^6}, \tag{2.11}$$

so that $(F_n^{(2)}(1,-1))_{n\geqslant 0}$ is periodic with period 6 (we'll call a sequence $(a_n)_{n\geqslant 0}$ *periodic with period* d if $a_{n+d} = a_n$ for all $n \geqslant m$ for some $m \in \mathbb{N}$). However, this behavior is restricted to the case $r = 2$:

Theorem 2.3. *The sequence* $(F_n^{(r)}(1,-1))_{n\geqslant 0}$ *is never periodic for* $r \geqslant 3$.

Proof. By (2.9) at $t = -1$, it suffices to show that $1 - x + x^r$ divides $x^m - 1$ for some $m \in \mathbb{P}$, only if $r = 2$.

We first describe the roots of unity that are zeros of $1 - x + x^r$. If z is such a root of unity, let $y = z^{r-1}$. Since $z(1 - z^{r-1}) = 1$ and z is a root of unity, it follows that both y and $1 - y$ are roots of unity. In particular, $|y| = |1 - y| = 1$. Therefore, $1 - 2\mathrm{Re}(y) + |y|^2 = 1$, so $\mathrm{Re}(y) = 1/2$. This forces y, and hence $1 - y$, to be primitive 6^{th} roots of unity. But $1 - y = 1/z$, so z is also a primitive 6^{th} root of unity.

This implies that the only possible roots of unity which are zeros of $1 - x + x^r$ are the primitive 6^{th} roots of unity. Since the derivative of $1 - x + x^r$ has no roots of unity as zeros, these 6^{th} roots of unity can only be simple zeros of $1 - x + x^r$. In particular, if every root of $1 - x + x^r$ is a root of unity, then $r = 2$. $\square$

If r is even, then by (2.7),

$$\begin{aligned} F_n^{(r)}(-1,t) &= \sum_{0\leqslant k\leqslant \lfloor n/r\rfloor} (-1)^{\binom{n-rk+1}{2}} \binom{n-(r-1)k}{k} t^k \\ &= (-1)^{\binom{n+1}{2}} \sum_{0\leqslant k\leqslant \lfloor n/r\rfloor} (-1)^{rk/2} \binom{n-(r-1)k}{k} t^k \\ &= (-1)^{\binom{n+1}{2}} F_n^{(r)}\left(1,(-1)^{r/2}t\right). \end{aligned} \tag{2.12}$$

Setting $t=1$ in (2.12) gives for $n\in\mathbb{N}$,

$$F_n^{(4j)}(-1,1) = (-1)^{\binom{n+1}{2}} F_n^{(4j)} \text{ and } F_n^{(4j+2)}(-1,1) = (-1)^{\binom{n+1}{2}} F_n^{(4j+2)}(1,-1). \tag{2.13}$$

Substituting $q=-1$ in (2.7) (and in (3.6) below) when r is odd gives a -1, instead of a 1, for the subscript of the q-binomial coefficients occurring in that formula. This may account in part for the difference in behavior seen in the following theorem for $F_n^{(r)}(-1,t)$ when r is odd (and in Theorem 3.4 below for $L_n^{(r)}(-1,t)$). Iterating (2.6) yields $F_{-i}^{(r)}(q,t)=0$ if $1\leqslant i\leqslant r-1$, which we'll take as a convention.

Theorem 2.4. *For r odd and all $m\in\mathbb{N}$,*

$$F_{2m}^{(r)}(-1,t) = (-1)^m F_m^{(r)}(1,-t^2) \tag{2.14}$$

and

$$F_{2m+1}^{(r)}(-1,t) = (-1)^{m+1}\left(F_m^{(r)}(1,-t^2) + (-1)^{\frac{r+1}{2}} t F_{m-(\frac{r-1}{2})}^{(r)}(1,-t^2)\right). \tag{2.15}$$

Proof. Taking the even and odd parts of both sides of (2.10) when r is odd followed by replacing x with $ix^{1/2}$, where $i=\sqrt{-1}$, yields

$$\sum_{m\geqslant 0}(-1)^m F_{2m}^{(r)}(-1,t)x^m = \frac{1}{1-x+t^2x^r}$$

and

$$\sum_{m\geqslant 0}(-1)^m F_{2m+1}^{(r)}(-1,t)x^m = \frac{-1+(-1)^{\frac{r-1}{2}} t x^{\frac{r-1}{2}}}{1-x+t^2x^r},$$

from which (2.14) and (2.15) now follow from (2.9).

For a combinatorial proof of (2.14) and (2.15), we first assign to each r-mino arrangement $c\in\mathcal{R}_n^{(r)}$ the weight $w_c := (-1)^{s(c)}t^{v(c)}$, where t is an indeterminate. Let $\mathcal{R}_n^{(r)'}$ consist of those $c=x_1x_2\cdots x_p$ in $\mathcal{R}_n^{(r)}$ satisfying the conditions $x_{2i-1}=x_{2i}$, $1\leqslant i\leqslant \lfloor p/2\rfloor$. Suppose $c=x_1x_2\cdots x_p\in\mathcal{R}_n^{(r)}-\mathcal{R}_n^{(r)'}$, with i_0 being the smallest value of i for which $x_{2i-1}\neq x_{2i}$. Exchanging the positions of x_{2i_0-1} and x_{2i_0} within c produces an s-parity changing involution of $\mathcal{R}_n^{(r)}-\mathcal{R}_n^{(r)'}$ which preserves $v(c)$.

If $n=2m$, then

$$\begin{aligned}
F_{2m}^{(r)}(-1,t) &= \sum_{c\in\mathcal{R}_{2m}^{(r)}} w_c = \sum_{c\in\mathcal{R}_{2m}^{(r)\prime}} w_c = \sum_{c\in\mathcal{R}_{2m}^{(r)\prime}} (-1)^{(2m-rv(c))/2} t^{v(c)} \\
&= (-1)^m \sum_{c\in\mathcal{R}_{2m}^{(r)\prime}} (-1)^{v(c)/2} t^{v(c)} = (-1)^m \sum_{z\in\mathcal{R}_{m}^{(r)}} (-1)^{v(z)} t^{2v(z)} \\
&= (-1)^m F_m^{(r)}(1,-t^2),
\end{aligned}$$

since each pair of consecutive squares in $c\in\mathcal{R}_{2m}^{(r)\prime}$ contributes an odd amount towards $s(c)$. If $n=2m+1$, then

$$\begin{aligned}
F_{2m+1}^{(r)}(-1,t) &= \sum_{c\in\mathcal{R}_{2m+1}^{(r)}} w_c = \sum_{c\in\mathcal{R}_{2m+1}^{(r)\prime}} w_c = \sum_{\substack{c\in\mathcal{R}_{2m+1}^{(r)\prime}\\ v(c)\text{ even}}} w_c + \sum_{\substack{c\in\mathcal{R}_{2m+1}^{(r)\prime}\\ v(c)\text{ odd}}} w_c \\
&= -\sum_{c\in\mathcal{R}_{2m}^{(r)\prime}} (-1)^{(2m-rv(c))/2} t^{v(c)} + t \sum_{c\in\mathcal{R}_{2m-(r-1)}^{(r)\prime}} (-1)^{(2m-(r-1)-rv(c))/2} t^{v(c)} \\
&= (-1)^{m+1} \sum_{z\in\mathcal{R}_{m}^{(r)}} (-1)^{v(z)} t^{2v(z)} + (-1)^{m-\left(\frac{r-1}{2}\right)} t \sum_{z\in\mathcal{R}^{(r)}_{m-\left(\frac{r-1}{2}\right)}} (-1)^{v(z)} t^{2v(z)} \\
&= (-1)^{m+1} F_m^{(r)}(1,-t^2) + (-1)^{m-\left(\frac{r-1}{2}\right)} t F^{(r)}_{m-\left(\frac{r-1}{2}\right)}(1,-t^2),
\end{aligned}$$

since members of $\mathcal{R}_{2m+1}^{(r)\prime}$ end in either a single square or in a single r-mino. $\square$

Setting $t=1$ in Theorem 2.4 gives

$$F_{2m}^{(r)}(-1,1) = (-1)^m F_m^{(r)}(1,-1) \tag{2.16}$$

and

$$F_{2m+1}^{(r)}(-1,1) = (-1)^{m+1}\left(F_m^{(r)}(1,-1) + (-1)^{\frac{r+1}{2}} F^{(r)}_{m-\left(\frac{r-1}{2}\right)}(1,-1)\right) \tag{2.17}$$

for r odd and $m\in\mathbb{N}$. Formulas (2.12)–(2.17) above (and (3.15)–(3.23) below) are somewhat reminiscent of the combinatorial reciprocity theorems of Stanley [7].

When $r=2$ in (2.13), we get

$$F_n^{(2)}(-1,1) = (-1)^{\binom{n+1}{2}} F_n^{(2)}(1,-1) \tag{2.18}$$

so that $(F_n^{(2)}(-1,1))_{n\geqslant 0}$ is periodic with period 12, by (2.11). Indeed, from (2.10) when $r=2$ and $t=1$,

$$\sum_{n\geqslant 0} F_n^{(2)}(-1,1)x^n = \frac{1-x-x^2}{1-x^2+x^4} = \frac{(1-x-x^3-x^4)(1-x^6)}{1-x^{12}}. \tag{2.19}$$

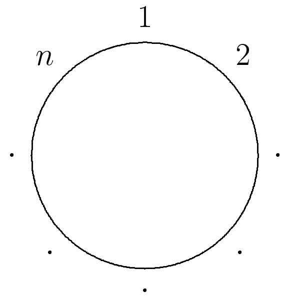
Periodicity is again restricted to the case $r=2$:

Corollary 2.4.1. *The sequence $(F_n^{(r)}(-1,1))_{n\geqslant 0}$ is never periodic for $r\geqslant 3$.*

Proof. This follows immediately from (2.13), (2.16), and Theorem 2.3. $\square$

3 Circular r-Mino Arrangements

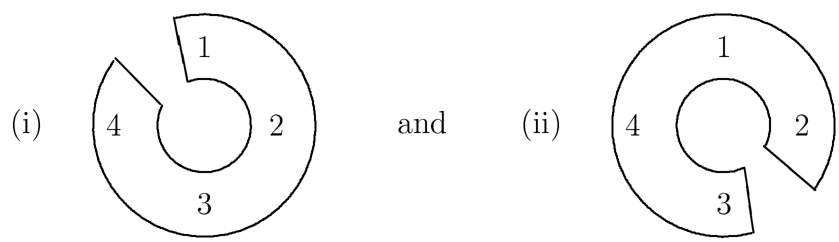
If $n \in \mathbb{P}$ and $0 \leqslant k \leqslant \lfloor n/r \rfloor$, let $\mathcal{C}_{n,k}^{(r)}$ denote the set of coverings by k r-minos and $n - rk$ squares of the numbers $1, 2, \ldots, n$ arranged clockwise around a circle:



By the *initial segment* of an r-mino occurring in such a cover, we mean the segment first encountered as the circle is traversed clockwise. Classifying members of $\mathcal{C}_{n,k}^{(r)}$ according as (i) 1 is covered by one of r segments of an r-mino or (ii) 1 is covered by a square, and applying (2.1), yields

$$\begin{aligned} \left|\mathcal{C}_{n,k}^{(r)}\right| &= r\binom{n-(r-1)k-1}{k-1} + \binom{n-(r-1)k-1}{k} \\ &= \frac{n}{n-(r-1)k}\binom{n-(r-1)k}{k}, \qquad 0 \leqslant k \leqslant \lfloor n/r \rfloor. \end{aligned} \tag{3.1}$$

Below we illustrate two members of $\mathcal{C}_{4,1}^{(4)}$:



In covering (i), the initial segment of the 4-mino covers 1, and in covering (ii), the initial segment covers 3.

With

$$\mathcal{C}_n^{(r)} := \bigcup_{0 \leqslant k \leqslant \lfloor n/r \rfloor} \mathcal{C}_{n,k}^{(r)}, \qquad n \in \mathbb{P}, \tag{3.2}$$

it follows that

$$\left|\mathcal{C}_n^{(r)}\right| = \sum_{0 \leqslant k \leqslant \lfloor n/r \rfloor} \frac{n}{n-(r-1)k}\binom{n-(r-1)k}{k} = L_n^{(r)}, \tag{3.3}$$

where $L_1^{(r)} = \cdots = L_{r-1}^{(r)} = 1$, $L_r^{(r)} = r+1$, and $L_n^{(r)} = L_{n-1}^{(r)} + L_{n-r}^{(r)}$ if $n \geqslant r+1$. Note that

$$\sum_{n \geqslant 1} L_n^{(r)} x^n = \frac{x + rx^r}{1 - x - x^r}. \tag{3.4}$$

Given $c \in \mathcal{C}_n^{(r)}$, let $v(c) :=$ the number of r-minos in the covering c, let $s(c) :=$ the sum of the numbers covered by the squares in c, and let

$$L_n^{(r)}(q,t) := \sum_{c \in \mathcal{C}_n^{(r)}} q^{s(c)} t^{v(c)}, \qquad n \in \mathbb{P}. \tag{3.5}$$

This leads to a new polynomial generalization of $L_n^{(r)}$:

Theorem 3.1. *For all $n \in \mathbb{P}$,*

$$L_n^{(r)}(q,t) = \sum_{0 \leqslant k \leqslant \lfloor n/r \rfloor} q^{\binom{n-rk+1}{2}} \left[\frac{k_{q^r} \sum_{i=1}^{r} q^{i(n-rk)} + (n-rk)_{q^r}}{(n-(r-1)k)_{q^r}} \right] \binom{n-(r-1)k}{k}_{q^r} t^k. \tag{3.6}$$

Proof. It suffices to show that

$$\sum_{c \in \mathcal{C}_{n,k}^{(r)}} q^{s(c)} = q^{\binom{n-rk+1}{2}} \left[\frac{s k_{q^r} + (n-rk)_{q^r}}{(n-(r-1)k)_{q^r}} \right] \binom{n-(r-1)k}{k}_{q^r},$$

where $s := \sum_{i=1}^{r} q^{i(n-rk)}$. Partitioning $\mathcal{C}_{n,k}^{(r)}$ into the categories employed above in deriving (3.1), and applying (2.7), yields

$$\begin{aligned}
\sum_{c \in \mathcal{C}_{n,k}^{(r)}} q^{s(c)} &= q^{\binom{n-rk+1}{2}} \binom{n-(r-1)k-1}{k-1}_{q^r} [q^{r(n-rk)} + q^{(r-1)(n-rk)} + \cdots + q^{n-rk}] \\
&\quad + q^{\binom{n-rk}{2}} \binom{n-(r-1)k-1}{k}_{q^r} q^{n-rk} \\
&= q^{\binom{n-rk+1}{2}} \binom{n-(r-1)k-1}{k-1}_{q^r} s + q^{\binom{n-rk+1}{2}} \binom{n-(r-1)k-1}{k}_{q^r} \qquad (3.7) \\
&= q^{\binom{n-rk+1}{2}} \left[\frac{s k_{q^r}}{(n-(r-1)k)_{q^r}} \binom{n-(r-1)k}{k}_{q^r} \right. \\
&\quad \left. + \frac{(n-rk)_{q^r}}{(n-(r-1)k)_{q^r}} \binom{n-(r-1)k}{k}_{q^r} \right],
\end{aligned}$$

which completes the proof. $\square$

Theorem 3.2. *The ordinary generating function of the sequence* $(L_n^{(r)}(q,t))_{n\geqslant 1}$ *is given by*

$$\sum_{n\geqslant 1} L_n^{(r)}(q,t)x^n = \frac{rx^rt}{1-x^rt} + \sum_{k\geqslant 1} \frac{q^{\binom{k+1}{2}}\left[1+x^rt\sum_{i=1}^{r-1}q^{ki}\right]x^k}{(1-x^rt)(1-q^rx^rt)\cdots(1-q^{rk}x^rt)}. \qquad (3.8)$$

Proof. By convention, we take $\binom{m}{0}_q = 1$ and $\binom{m}{-1}_q = 0$ for $m \in \mathbb{Z}$. From (3.7),

$$\begin{aligned}
\sum_{n\geqslant 1} L_n^{(r)}(q,t)x^n &= \sum_{n\geqslant 1} x^n \sum_{0\leqslant k\leqslant \lfloor n/r\rfloor} q^{\binom{n-rk+1}{2}} t^k \left[\binom{n-(r-1)k-1}{k-1}_{q^r} \cdot \sum_{i=1}^{r} q^{i(n-rk)} \right.\\
&\qquad \left. + \binom{n-(r-1)k-1}{k}_{q^r}\right]\\
&= \sum_{\substack{j=0\\ j+m\geqslant 1}}^{r-1} \sum_{m\geqslant 0} x^{mr+j} \sum_{0\leqslant k\leqslant m} q^{\binom{kr+j+1}{2}} t^{m-k} \left[\binom{k(r-1)+m+j-1}{m-k-1}_{q^r} \cdot \sum_{i=1}^{r} q^{i(kr+j)} \right.\\
&\qquad \left. + \binom{k(r-1)+m+j-1}{m-k}_{q^r}\right]\\
&= \sum_{\substack{j=0\\ j+m\geqslant 1}}^{r-1} \sum_{k\geqslant 0} q^{\binom{kr+j+1}{2}} \sum_{m\geqslant k} x^{mr+j} t^{m-k} \left[s\binom{k(r-1)+m+j-1}{kr+j}_{q^r} \right.\\
&\qquad \left. + \binom{k(r-1)+m+j-1}{kr+j-1}_{q^r}\right],
\end{aligned}$$

by symmetry, where $s := \sum_{i=1}^{r} q^{i(kr+j)}$. Separating the terms for which $k = j = 0$ gives

$$\begin{aligned}
\sum_{n\geqslant 1} L_n^{(r)}(q,t)x^n &= \frac{rx^rt}{1-x^rt} + \sum_{\substack{j=0\\ j+m\geqslant 1\\ j+k\geqslant 1}}^{r-1} \left(\sum_{k\geqslant 0} s q^{\binom{kr+j+1}{2}} \sum_{m\geqslant k} \binom{k(r-1)+m+j-1}{kr+j}_{q^r} x^{mr+j}t^{m-k} \right.\\
&\qquad \left. + \sum_{k\geqslant 0} q^{\binom{kr+j+1}{2}} \sum_{m\geqslant k} \binom{k(r-1)+m+j-1}{kr+j-1}_{q^r} x^{mr+j}t^{m-k} \right)\\
&= \frac{rx^rt}{1-x^rt} + \sum_{\substack{j=0\\ j+k\geqslant 1}}^{r-1} \left(\sum_{k\geqslant 0} s q^{\binom{kr+j+1}{2}} \frac{x^{kr+j+r}t}{(1-x^rt)(1-q^rx^rt)\cdots(1-q^{(kr+j)r}x^rt)} \right.
\end{aligned}$$

$$+\sum_{k\geqslant 0} q^{\binom{kr+j+1}{2}} \frac{x^{kr+j}}{(1-x^rt)(1-q^rx^rt)\cdots(1-q^{(kr+j-1)r}x^rt)}\Bigg)$$

$$=\frac{rx^rt}{1-x^rt}+\sum_{\substack{j=0\\ j+k\geqslant 1}}^{r-1}\sum_{k\geqslant 0} q^{\binom{kr+j+1}{2}} \frac{\left(1+x^rt\sum_{i=1}^{r-1}q^{i(kr+j)}\right)x^{kr+j}}{(1-x^rt)(1-q^rx^rt)\cdots(1-q^{(kr+j)r}x^rt)},$$

by (1.5), which yields (3.8), upon replacing $kr+j$ by $k\geqslant 1$. □

Note that $L_n^{(r)}(1,1)=L_n^{(r)}$, whence (3.8) generalizes (3.4). The $L_n^{(r)}(q,t)$ are related to the $F_n^{(r)}(q,t)$ by the formula

$$L_n^{(r)}(1,t)=F_{n-1}^{(r)}(1,t)+rtF_{n-r}^{(r)}(1,t), \qquad n\geqslant 1, \tag{3.9}$$

which reduces to

$$L_n^{(r)}=F_{n-1}^{(r)}+rF_{n-r}^{(r)}, \qquad n\geqslant 1, \tag{3.10}$$

when $t=1$, though there do not appear to be such formulas for $L_n^{(r)}(q,t)$ or $L_n^{(r)}(q,1)$. Furthermore, the $L_n^{(r)}(q,t)$ do not seem to satisfy a simple recursion like (2.6). Setting $q=1$ and $q=-1$ in (3.8) yields

Corollary 3.2.1. *The ordinary generating function of the sequence* $(L_n^{(r)}(1,t))_{n\geqslant 1}$ *is given by*

$$\sum_{n\geqslant 1} L_n^{(r)}(1,t)x^n=\frac{x+rtx^r}{1-x-tx^r}, \tag{3.11}$$

and

Corollary 3.2.2. *The ordinary generating function of the sequence* $(L_n^{(r)}(-1,t))_{n\geqslant 1}$ *is given by*

$$\sum_{n\geqslant 1} L_n^{(r)}(-1,t)x^n=\begin{cases}\dfrac{-x-x^2+rtx^r+tx^{r+1}-rt^2x^{2r}}{1+x^2-2tx^r+t^2x^{2r}}, & \textit{if } r \textit{ is even;}\\[2ex] \dfrac{-x-x^2+rtx^r+rt^2x^{2r}}{1+x^2-t^2x^{2r}}, & \textit{if } r \textit{ is odd.}\end{cases} \tag{3.12}$$

When $r=2$ and $t=-1$ in (3.11), we get

$$\sum_{n\geqslant 1} L_n^{(2)}(1,-1)x^n=\frac{x-2x^2}{1-x+x^2}=\frac{(x-x^2-2x^3)(1-x^3)}{1-x^6}, \tag{3.13}$$

so that $(L_n^{(2)}(1,-1))_{n\geqslant 1}$ is periodic with period 6. Again, no such periodicity occurs for $r\geqslant 3$:

Theorem 3.3. *The sequence $(L_n^{(r)}(1,-1))_{n\geqslant 1}$ is never periodic for $r \geqslant 3$.*

Proof. By (3.11) at $t=-1$, we must show that $1-x+x^r$ does not divide the product $(1-x^m)(x-rx^r)$ for any $m \in \mathbb{P}$ whenever $r \geqslant 3$. Note that the polynomials $1-x+x^r$ and $x-rx^r$ cannot share a zero; for if t_0 is a common zero, then $t_0^r = \frac{t_0}{r}$ and $0 = 1-t_0+t_0^r = 1-t_0+\frac{t_0}{r}$, i.e., $t_0 = \frac{r}{r-1}$, which isn't a zero of either polynomial. From (2.9) and Theorem 2.3, the polynomial $1-x+x^r$ doesn't divide $1-x^m$ when $r \geqslant 3$, which completes the proof. $\square$

When r is even, the $L_n^{(r)}(-1,t)$ can be expressed as a linear combination of the $F_n^{(r)}(-1,t)$ by the relation

$$\begin{aligned} L_n^{(r)}(-1,t) = \tfrac{-r}{2}F_{n+1}^{(r)}(-1,t) + F_n^{(r)}(-1,t) &- \tfrac{r}{2}F_{n-1}^{(r)}(-1,t) + \tfrac{rt}{2}F_{n-r+1}^{(r)}(-1,t) \\ &+ (\tfrac{r}{2}-1)tF_{n-r}^{(r)}(-1,t), \quad n \geqslant 1, \quad (3.14) \end{aligned}$$

which follows from (3.12) and (2.10). We were unable to find a relation comparable to (3.14) when r is odd.

If r is even, then by (3.6), (2.7), and (3.9),

$$\begin{aligned} L_{2m}^{(r)}(-1,t) &= \sum_{0\leqslant k\leqslant \lfloor 2m/r \rfloor} (-1)^{\binom{2m-rk+1}{2}} \frac{2m}{2m-(r-1)k}\binom{2m-(r-1)k}{k}t^k \\ &= (-1)^m \sum_{0\leqslant k\leqslant \lfloor 2m/r \rfloor} (-1)^{rk/2} \frac{2m}{2m-(r-1)k}\binom{2m-(r-1)k}{k}t^k \\ &= (-1)^m L_{2m}^{(r)}\left(1,(-1)^{r/2}t\right) \qquad (3.15) \end{aligned}$$

and

$$\begin{aligned} L_{2m-1}^{(r)}(-1,t) &= \sum_{0\leqslant k\leqslant \lfloor (2m-1)/r \rfloor} (-1)^{\binom{2m-rk}{2}} \frac{2m-1-rk}{2m-1-(r-1)k}\binom{2m-1-(r-1)k}{k}t^k \\ &= (-1)^m \Bigg[\sum_{0\leqslant k\leqslant \lfloor (2m-1)/r \rfloor} (-1)^{rk/2} \frac{2m-1}{2m-1-(r-1)k}\binom{2m-1-(r-1)k}{k}t^k \\ &\qquad - (-1)^{r/2}rt \sum_{0\leqslant k\leqslant \lfloor (2m-r-1)/r \rfloor} (-1)^{rk/2}\binom{2m-r-1-(r-1)k}{k}t^k\Bigg] \\ &= (-1)^m \left(L_{2m-1}^{(r)}\left(1,(-1)^{r/2}t\right) - (-1)^{r/2}\, rtF_{2m-r-1}^{(r)}\left(1,(-1)^{r/2}t\right)\right) \\ &= (-1)^m F_{2m-2}^{(r)}\left(1,(-1)^{r/2}t\right). \qquad (3.16) \end{aligned}$$

Setting $t=1$ in (3.15) and (3.16) gives for $m \in \mathbb{P}$,

$$L_{2m}^{(4j)}(-1,1) = (-1)^m L_{2m}^{(4j)} \text{ and } L_{2m-1}^{(4j)}(-1,1) = (-1)^m F_{2m-2}^{(4j)} \qquad (3.17)$$

and

$$L_{2m}^{(4j+2)}(-1,1)=(-1)^m L_{2m}^{(4j+2)}(1,-1) \text{ and } L_{2m-1}^{(4j+2)}(-1,1)=(-1)^m F_{2m-2}^{(4j+2)}(1,-1). \quad (3.18)$$

The following theorem gives analogues of (3.15) and (3.16) when r is odd. Recall that $F_{-i}^{(r)}(q,t)=0$ for $1\leqslant i\leqslant r-1$, by convention.

Theorem 3.4. *For r odd and all $m\in\mathbb{P}$,*

$$L_{2m}^{(r)}(-1,t)=(-1)^m L_m^{(r)}(1,-t^2) \quad (3.19)$$

and

$$L_{2m-1}^{(r)}(-1,t)=(-1)^m\left(F_{m-1}^{(r)}(1,-t^2)+(-1)^{\frac{r+1}{2}}rtF_{m-(\frac{r+1}{2})}^{(r)}(1,-t^2)\right). \quad (3.20)$$

Proof. Taking the even and odd parts of both sides of (3.12) when r is odd followed by replacing x with $ix^{1/2}$, where $i=\sqrt{-1}$, yields

$$\sum_{m\geqslant 1}(-1)^m L_{2m}^{(r)}(-1,t)x^m=\frac{x-rt^2x^r}{1-x+t^2x^r}$$

and

$$\sum_{m\geqslant 1}(-1)^m L_{2m-1}^{(r)}(-1,t)x^m=\frac{x+(-1)^{\frac{r+1}{2}}rtx^{\frac{r+1}{2}}}{1-x+t^2x^r},$$

from which (3.19) and (3.20) now follow from (3.11) and (2.9).

For a combinatorial proof of (3.19) and (3.20), we first assign to each r-mino arrangement $c\in\mathcal{C}_n^{(r)}$ the weight $w_c:=(-1)^{s(c)}t^{v(c)}$. Associate to each $c\in\mathcal{C}_n^{(r)}$ a word $v_c:=v_1v_2\cdots$ in the alphabet $\{r,s\}$, where

$$v_i:=\begin{cases} r, & \text{if the } i^{th} \text{ piece of } c \text{ is an } r\text{-mino;}\\ s, & \text{if the } i^{th} \text{ piece of } c \text{ is a square,}\end{cases}$$

and one determines the i^{th} piece of c by starting with the piece covering 1 and proceeding clockwise from that piece. Note that for each word starting with r, there are exactly r associated members of $\mathcal{C}_n^{(r)}$, while for each word starting with s, there is only one associated member.

Let $\mathcal{C}_n^{(r)'}$ consist of those c in $\mathcal{C}_n^{(r)}$ for which $v_c=v_1v_2\cdots$ satisfies $v_{2i}=v_{2i+1}$ for all i. Let $c\in\mathcal{C}_n^{(r)}-\mathcal{C}_n^{(r)'}$ with $v_c=v_1v_2\cdots$, and let i_0 be the smallest index i for which $v_{2i}\neq v_{2i+1}$. Interchanging the $(2i_0)^{th}$ and $(2i_0+1)^{st}$ pieces of c furnishes an s-parity changing, v-preserving involution of $\mathcal{C}_n^{(r)}-\mathcal{C}_n^{(r)'}$.

If $n=2m-1$, then

$$\begin{aligned} L_{2m-1}^{(r)}(-1,t) &= \sum_{c\in\mathcal{C}_{2m-1}^{(r)}} w_c=\sum_{c\in\mathcal{C}_{2m-1}^{(r)'}} w_c=\sum_{\substack{c\in\mathcal{C}_{2m-1}^{(r)'}\\ v(c)\text{ even}}} w_c+\sum_{\substack{c\in\mathcal{C}_{2m-1}^{(r)'}\\ v(c)\text{ odd}}} w_c\\ &= -\sum_{c\in\mathcal{R}_{2m-2}^{(r)'}}(-1)^{(2m-2-rv(c))/2}t^{v(c)}+rt\sum_{c\in\mathcal{R}_{2m-r-1}^{(r)'}}(-1)^{(2m-r-1-rv(c))/2}t^{v(c)}\end{aligned}$$

$$= (-1)^m \sum_{z\in\mathcal{R}^{(r)}_{m-1}} (-1)^{v(z)}t^{2v(z)} + (-1)^{m-\left(\frac{r+1}{2}\right)} rt \sum_{z\in\mathcal{R}^{(r)}_{m-\left(\frac{r+1}{2}\right)}} (-1)^{v(z)}t^{2v(z)}$$

$$= (-1)^m F^{(r)}_{m-1}(1,-t^2) + (-1)^{m-\left(\frac{r+1}{2}\right)} rtF^{(r)}_{m-\left(\frac{r+1}{2}\right)}(1,-t^2),$$

which gives (3.20), where $\mathcal{R}^{(r)'}_n$ is as in the proof of Theorem 2.4, since members c of $\mathcal{C}^{(r)'}_{2m-1}$ have a square as the first piece iff $v(c)$ is even.

Now suppose that $n = 2m$. Let $\mathcal{C}^{(r)^*}_{2m}$ consist of those $c \in \mathcal{C}^{(r)'}_{2m}$ for which the first and last letters of v_c are the same. Consider the r members of $\mathcal{C}^{(r)'}_{2m} - \mathcal{C}^{(r)^*}_{2m}$ associated with the same word v_c starting with r (and thus ending in s) along with the arrangement resulting when v_c is read backwards, denoting the set consisting of these $r+1$ arrangements by $\mathcal{S}_{v_c}$. Note that $\mathcal{C}^{(r)'}_{2m} - \mathcal{C}^{(r)^*}_{2m}$ is partitioned by the $\mathcal{S}_{v_c}$ as v_c ranges over all possible associated words. The $\frac{r+1}{2}$ members of $\mathcal{S}_{v_c}$ whose first piece is an r-mino with initial segment covering an odd number have s-parity opposite the remaining $\frac{r+1}{2}$ members of $\mathcal{S}_{v_c}$, with each arrangement in $\mathcal{S}_{v_c}$ possessing the same number of r-minos. Hence, the contribution of each $\mathcal{S}_{v_c}$ towards $L^{(r)}_{2m}(-1,t)$ is zero, which implies the net weight of $\mathcal{C}^{(r)'}_{2m} - \mathcal{C}^{(r)^*}_{2m}$ is zero.

Therefore,

$$\begin{aligned} L^{(r)}_{2m}(-1,t) &= \sum_{c\in\mathcal{C}^{(r)}_{2m}} w_c = \sum_{c\in\mathcal{C}^{(r)^*}_{2m}} w_c = \sum_{c\in\mathcal{C}^{(r)^*}_{2m}} (-1)^{(2m-rv(c))/2}t^{v(c)} \\ &= (-1)^m \sum_{c\in\mathcal{C}^{(r)^*}_{2m}} (-1)^{v(c)/2}t^{v(c)} = (-1)^m \sum_{z\in\mathcal{C}^{(r)}_{m}} (-1)^{v(z)}t^{2v(z)} \\ &= (-1)^m L^{(r)}_m(1,-t^2), \end{aligned}$$

which gives (3.19), since the first and last pieces of $c \in \mathcal{C}^{(r)^*}_{2m}$ are the same. Note that each pair of consecutive squares in $c \in \mathcal{C}^{(r)^*}_{2m}$ corresponding to either $v_{2i} = v_{2i+1} = s$ for some i or to (possibly) $v_p = v_1 = s$ in $v_c = v_1v_2\cdots v_p$ contributes an odd amount towards $s(c)$. $\square$

Setting $t = 1$ in Theorem 3.4 gives

$$L^{(r)}_{2m}(-1,1) = (-1)^m L^{(r)}_m(1,-1) \tag{3.21}$$

and

$$L^{(r)}_{2m-1}(-1,1) = (-1)^m \left(F^{(r)}_{m-1}(1,-1) + (-1)^{\frac{r+1}{2}} rF^{(r)}_{m-(\frac{r+1}{2})}(1,-1)\right) \tag{3.22}$$

for r odd and $m \in \mathbb{P}$.

When $r = 2$ in (3.18), we get

$$L^{(2)}_{2m}(-1,1) = (-1)^m L^{(2)}_{2m}(1,-1) \text{ and } L^{(2)}_{2m-1}(-1,1) = (-1)^m F^{(2)}_{2m-2}(1,-1) \tag{3.23}$$

so that $(L_n^{(2)}(-1,1))_{n\geqslant 1}$ is periodic with period 12, by (3.13) and (2.11). Indeed, from (3.12) when $r=2$ and $t=1$,

$$\begin{aligned}\sum_{n\geqslant 1} L_n^{(2)}(-1,1)x^n &= \frac{-x+x^2+x^3-2x^4}{1-x^2+x^4} \\ &= \frac{(-x+x^2-x^4+x^5-2x^6)(1-x^6)}{1-x^{12}}. \end{aligned} \tag{3.24}$$

When $r\geqslant 3$, we have

Corollary 3.4.1. *The sequence* $(L_n^{(r)}(-1,1))_{n\geqslant 1}$ *is never periodic for* $r\geqslant 3$.

Proof. This follows immediately from (3.17), (3.18), (3.21), and Theorem 3.3. □

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