Domino Fibonacci Tableaux

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Abstract

In 2001, Shimozono and White gave a description of the domino Schensted algorithm of Barbasch, Vogan, Garfinkle and van Leeuwen with the "color-to-spin" property, that is, the property that the total color of the permutation equals the sum of the spins of the domino tableaux. In this paper, we describe the poset of domino Fibonacci shapes, an isomorphic equivalent to Stanley's Fibonacci lattice $Z(2)$, and define domino Fibonacci tableaux. We give an insertion algorithm which takes colored permutations to pairs of tableaux (P, Q) of domino Fibonacci shape. We then define a notion of spin for domino Fibonacci tableaux for which the insertion algorithm preserves the color-to-spin property. In addition, we give an evacuation algorithm for standard domino Fibonacci tableaux which relates the pairs of tableaux obtained from the domino insertion algorithm to the pairs of tableaux obtained from Fomin's growth diagrams.

1 Introduction

The Fibonacci lattice $Z(r)$ was introduced by Stanley in 1975 [10], and like Young's lattice Y^r, it is one of the prime examples of an r-differential poset. In 1988, Stanley showed that for any r-differential poset P

$$\sum_{\lambda \in P_n} e(\lambda)^2 = r^n n! \tag{1}$$

where λ is a partition of n and $e(\lambda)$ is the number of chains in P from $\hat{0}$ to λ. (Corollary 3.9, [10]) In the case of Young's lattice with $r = 1$, the Schensted insertion algorithm provides a bijective proof of this identity by taking a permutation $\pi \in S_n$ to a pair of standard Young tableaux (P, Q) of the same shape λ. Given $\pi \in S_n$, Fomin's growth diagram [2] provides another method for obtaining the same pair of standard Young tableaux provided by the Schensted insertion algorithm.

In addition to Young's lattice, Fomin's growth diagrams can be used to give a bijection between a permutation in S_n and a pair of chains in the Fibonacci poset $Z(1)$ which can be represented as a pair of Fibonacci path tableaux $(\hat{P}, \hat{Q})$. Roby [6] described an insertion algorithm which provides a bijection between a permutation in S_n and a pair of tableaux (P, Q) of the same shape where P is a Fibonacci insertion tableau and Q is a Fibonacci path tableau. Unlike Young's lattice, the pairs of tableaux obtained from these two methods are not the same. While $\hat{Q} = Q$, $\hat{P}$ is not equal to P. Killpatrick [4] defined an evacaution method for Fibonacci tableaux and proved that $ev(P) = \hat{P}$.

The poset of 2-ribbon (or domino) shapes is isomorphic to Y^2 and thus 2-differential. For the domino poset, the Barbasch-Vogan [1] and Garfinkle [3] domino insertion algorithms provide a bijective proof of (1) with $r = 2$ by taking colored permutations to pairs (P, Q) of standard domino tableaux of the same shape. Shimozono and White [8] gave a description of this algorithm and noted the property that the total color of the permutation is the sum of the spins of P and Q.

The motivation of this paper is to describe a reasonable notion of domino Fibonacci tableaux for which there is a "spin-preserving" bijection between pairs of chains in the poset and colored permutations. The poset of domino Fibonacci tableaux is naturally isomorphic to $Z(2)$. We describe an insertion algorithm for colored permutations which gives a pair (P, Q) for which P is a standard domino Fibonacci tableau and Q is a domino Fibonacci path tableau. As in the case of $Z(1)$, Fomin's growth diagrams can be used to give a bijection between a colored permutation in S_n and a pair of chains in $Z(2)$ which we show can be represented as a pair of domino Fibonacci path tableaux $(\hat{P}, \hat{Q})$. We prove that $Q = \hat{Q}$ and define an evacuation algorithm that gives $ev(P) = \hat{P}$.

Section 2 gives the necessary background and definitions for the rest of the paper, and in Section 3 we describe Fomin's chain theoretic approach to differential posets. In Sections 4 and 5 we define domino Fibonacci tableaux and give the domino Fibonacci insertion algorithm. Sections 6 and 7 describe the evacuation algorithm and a geometric interpretation of Fomin's growth diagrams. In these sections we give a relation between the tableaux resulting from the insertion algorithm and the tableaux resulting from Fomin's growth diagrams. Finally the "color-to-spin" property of the domino insertion algorithm is proved in Section 8.

2 Background and Definitions

In this section we give the necessary background and definitions for the theorems in this paper. The interested reader is encouraged to read Chapter 5 of *The Symmetric Group, 2nd Edition* by Bruce Sagan [7] for general reference.

The general definition of a Fibonacci r-differential poset was given by Richard Stanley in [11] (Definition 5.2).

Definition 1. *An r-differential poset P is a poset which satisfies the following three conditions:*

1. P has a $\hat{0}$ element, is graded and is locally finite.

2. If $x \neq y$ and there are exactly k elements in P which are covered by x and by y, then there are exactly k elements in P which cover both x and y.

3. For $x \in P$, if x covers exactly k elements of P, then x is covered by exactly $k+r$ elements of P.

The classic example of a 1-differential poset is Young's lattice Y, which is the poset of partitions together with the binary relation $\lambda \leq \mu$ if and only if $\lambda_i \leq \mu_i$ for all i.

A generalization of Young's lattice is the domino poset, which is 2-differential. A *domino* is a skew shape consisting of two adjacent cells in the same row or column. If the two adjacent cells are in the same column, the domino is considered *vertical*. Otherwise, it is considered *horizontal*. A *domino shape* is a partition (or Ferrers diagram) which can be completely covered (or tiled) by dominos. The *domino poset* $\mathcal{D}$ is the set of domino shapes together with the following binary relation. For two domino shapes λ and μ, we say that λ *covers* μ, $\lambda \gtrdot \mu$, if λ/μ is a domino. In general, $\lambda \geq \mu$ if λ/μ can be tiled by dominos, i.e., we can obtain μ by successively removing dominos from λ, or we can obtain λ by successively adding dominos to μ.

From a domino shape, a *domino tableau* D can be created by tiling the shape with dominos and then filling the dominos with the numbers $1, 1, 2, 2, \ldots, n, n$ so that (i) the numbers appearing in a single domino are identical and (ii) the numbers weakly increase across rows and down columns. The number of vertical dominos in D is denoted $vert(D)$. The *spin* of D, $sp(D)$, is defined as $\frac{1}{2}vert(D)$.

Shimozono and White [8] describe the domino insertion algorithm which takes colored permutations π (i.e., permutations where each element can be either barred or unbarred) to pairs of domino tableaux (P, Q) of the same shape and prove that this insertion has the property that if $tc(\pi)$ is the total color of π (i.e, the number of barred elements in π), then $tc(\pi) = sp(P) + sp(Q)$.

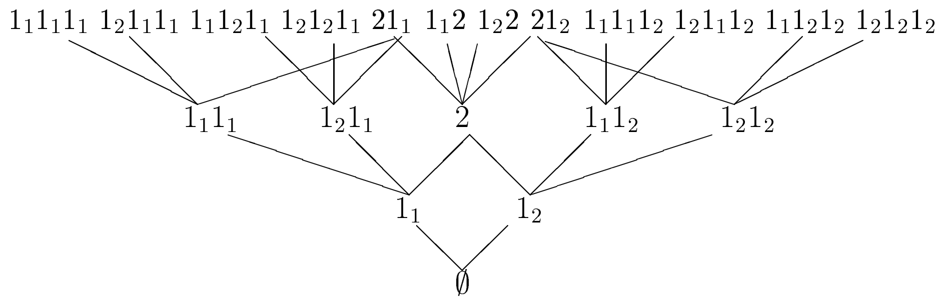
A second type of r-differential poset is the Fibonacci differential poset $Z(r)$ first described by Richard Stanley [11]. Let $A = \{1_1, 1_2, \ldots, 1_r, 2\}$ and let A^* be the set of all finite words $a_1 a_2 \cdots a_k$ of elements of A (including the empty word).

Definition 2. *The Fibonacci differential poset $Z(r)$ has as its elements the set of words in A^*. For $w \in Z(r)$, we say z is covered by w (i.e. $z \lessdot w$) in $Z(r)$ if either:*

1. z is obtained from w by changing a 2 to 1_k for some k if the only letters to the left of this 2 are also 2's, or

2. z is obtained from w by deleting the leftmost 1 of any type.

In this paper we will focus on $Z(2)$. The first four rows of the Fibonacci lattice $Z(2)$ are shown below:



3 A Chain Theoretic Approach

Fomin [2] gave a general method for representing a permutation with a square diagram and then using a growth function to create a pair of saturated chains in a differential poset. In particular, Fomin's method can be applied to the square diagram of a colored permutation to create a pair of saturated chains in $Z(2)$, giving a proof for $Z(2)$ of Stanley's result [11] that for any 2-differential poset,

$$\sum_{\lambda \in P_n} e(\lambda)^2 = 2^n n! \tag{2}$$

where λ is a partition of n and $e(\lambda)$ is the number of chains in P from $\hat{0}$ to λ.

Given a permutation in S_n, we can create a colored permutation by assigning each element to be either colored or uncolored. We will denote colored elements by a bar. For a colored permutation written in two line notation:

$$\pi \;=\; \begin{array}{cccc} 1 & 2 & \cdots & n \\ x_1 & x_2 & \cdots & x_n \end{array}$$

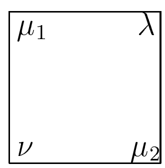
with each x_i either barred or unbarred, we create a square diagram by placing an X in column i and row x_i (indexed from left to right, bottom to top) if $\overset{i}{x_i}$ is a column in the permutation π and by placing a $\bar{X}$ in column i and row x_i if $\overset{i}{\bar{x}_i}$ is a column in π. For example, for the permutation

$$\pi = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \bar{2} & 7 & 1 & \bar{5} & \bar{6} & \bar{4} & 3 \end{array}$$

we obtain the following square diagram:

	X					
				$\bar{X}$		
			$\bar{X}$			
					$\bar{X}$	
						X
$\bar{X}$						
		X				

Fomin's method gives a way to translate this square diagram into a pair of saturated chains in $Z(2)$ in the following manner. Begin by placing $\emptyset$'s along the lower edge and the left edge at leach corner. Label the remaining corners in the diagram by following the rules given below (called a *growth function*). If we have



with each side of the square representing a cover relation in the $Z(2)$ or an equality, then:

1. If $\mu_1 \gtrdot \nu$ and $\mu_2 = \nu$ then $\lambda = \mu_1$ (and similarly for μ_1 and μ_2 interchanged).

2. If $\mu_1 \gtrdot \nu$, $\mu_2 \gtrdot \nu$ then λ is obtained from ν by prepending a 2.

3. If $\mu_1 = \nu = \mu_2$ and the box does contain an X or an $\bar{X}$, then obtain λ from ν by prepending a 1_1 if the box contains an X and by prepending a 1_2 if the box contains an $\bar{X}$.

4. If $\mu_1 = \nu = \mu_2$ and the box does not contain an X or an $\bar{X}$, then $\lambda = \nu$.

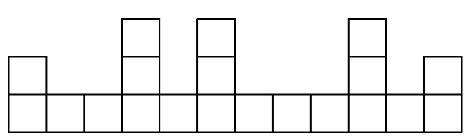
By following this procedure on our previous example, we obtain the complete growth diagram:

$\emptyset$	1_2	$1_1 1_2$ X	21_2	22	$21_2 2$	$21_2 1_2 2$	$221_2 2$
$\emptyset$	1_2	1_2	2	$1_2 2$	$1_2 1_2 2$ $\bar{X}$	$21_2 2$	222
$\emptyset$	1_2	1_2	2	$1_2 2$ $\bar{X}$	$1_2 2$	22	$21_2 2$
$\emptyset$	1_2	1_2	2	2	2	$1_2 2$ $\bar{X}$	22
$\emptyset$	1_2	1_2	2	2	2	2	12 X
$\emptyset$	1_2 $\bar{X}$	1_2	2	2	2	2	2
$\emptyset$	$\emptyset$	$\emptyset$	1_1 X	1_1	1_1	1_1	1_1
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Fomin [2] proved that this growth function produces a pair of saturated chains in $Z(2)$ by following the right edge and the top edge of the diagram.

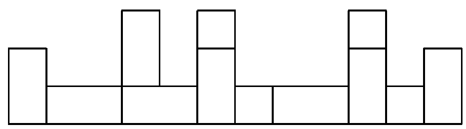
4 Domino Fibonacci Tableaux

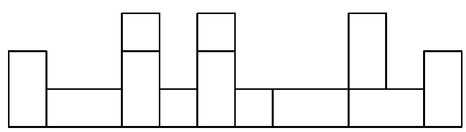
An element of $Z(2)$ can be represented by a *domino Fibonacci shape* by letting 1_1 correspond to two adjacent squares in the first row, a 1_2 correspond to two adjacent squares, one on top of the other, and a 2 correspond to a column of 3 squares followed by an adjacent single square in the first row. For example, the element $1_2 1_1 2 2 1_1 2 1_2$ is represented by

S = 

Define a *vertical domino* to be a rectangle containing two squares in the same column, one on top of the other. Define a *horizontal domino* to be a rectangle containing two adjacent squares in the first row of the domino Fibonacci shape and define a *split horizontal domino* to be the top square of a column of height 3 and the single square in the column immediately to the right of the column of height 3.

A *domino tiling* is a placement of vertical and horizontal dominos into a domino Fibonacci shape such that all squares are covered. A domino Fibonacci shapes may have more than one domino tiling. For example, each of the following is a valid domino tiling of the shape corresponding to $1_2 1_1 2 2 1_1 2 1_2$:

T_1= 

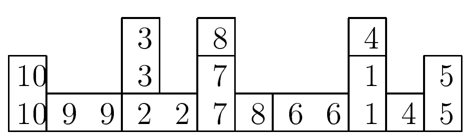
T_2= 

We define the poset *DomFib* to be the set of domino Fibonacci shapes together with cover relations inherited from $Z(2)$. *DomFib* is naturally isomorphic to $Z(2)$.

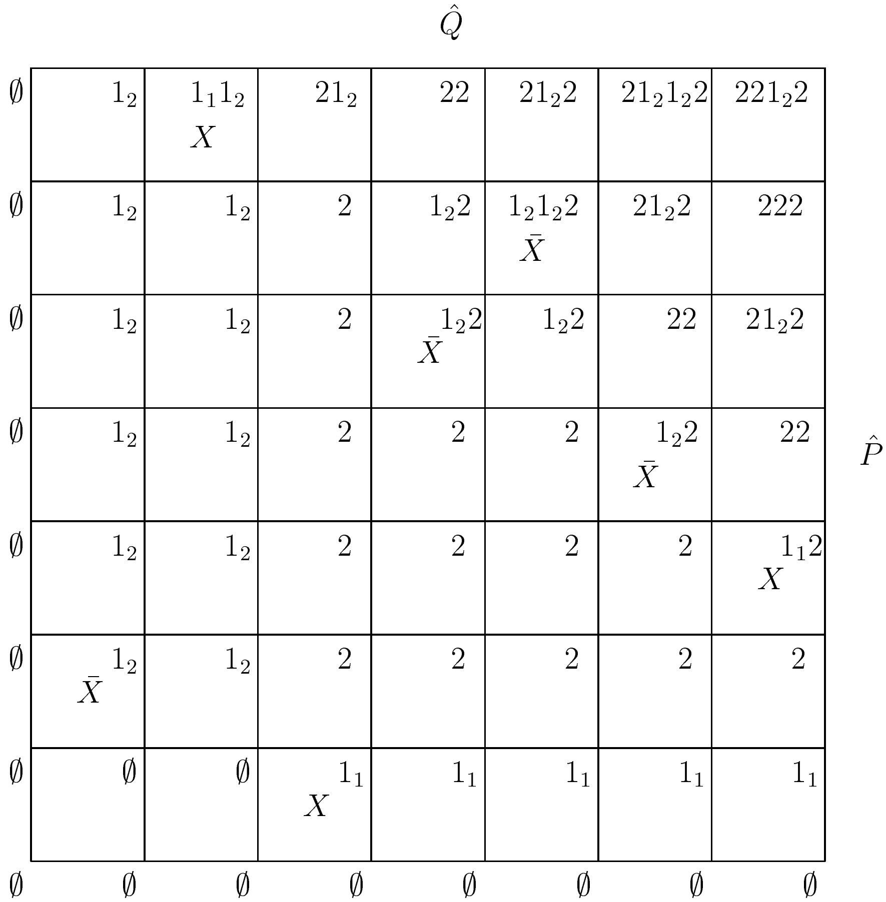
A saturated chain $(\emptyset, \nu_1, \nu_2, \cdots, \nu_k = \nu)$ in $Z(2)$ can be translated into a *domino Fibonacci path tableau* by placing i's in ν_i/ν_{i-1}, i.e. in each of the two new squares created at the ith step. For example, the chain

$$(\emptyset, 1_2, 1_1 1_2, 2 1_2, 22, 221_2, 21_1 21_2, 21_2 1_1 21_2, 221_1 21_2, 1_1 221_1 21_2, 1_2 1_1 221_1 21_2)$$

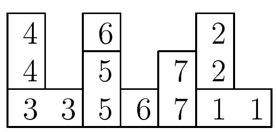
corresponds to the domino Fibonacci path tableau

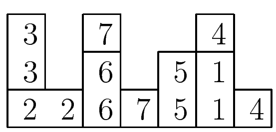
T_1= 

As seen in Section 3, Fomin's method gives a bijection between a colored permutation and a pair of chains in $Z(2)$, each of which can be represented by a domino Fibonacci path tableau. We will call the domino Fibonacci path tableau obtained from the right edge of the diagram $\hat{P}$ and the one obtained from the top edge of the diagram $\hat{Q}$. From our previous growth diagram:



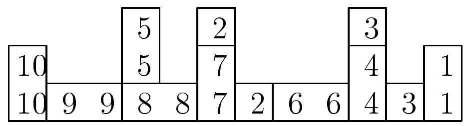
we have

$\hat{P} =$ 

$\hat{Q} =$ 

We define a *domino Fibonacci tableau* as a filling of the dominos in a tiling of a domino Fibonacci shape with the numbers $\{1, 1, 2, 2, \ldots, n, n\}$ such that each number appears in exactly one domino and each domino contains two of the same number.

A *standard* domino Fibonacci tableau has two additional properties. First, the domino containing the leftmost square in the first row is the domino containing n. Second, for every k, the domino containing k is either appended as a horizontal or vertical domino to the shape of the dominos containing i's for $k < i \leq n$ or is placed as a vertical or split horizontal domino on top of a single domino containing i's for $k < i \leq n$. For example, the following is a standard domino Fibonacci tableau:

$T_1 =$ 

One can also think of a standard domino Fibonacci tableau in terms of a chain in a partial order. Define $S(2)$ to be a new partial order on the set of Fibonacci words in the alphabet $\{1_1, 1_2, 2\}$ in which an element z is covered by an element w if w is obtained from z by appending a 1_i for $i = 1$ or $i = 2$ or if w is obtained from z by replacing 1_1 or 1_2 by a 2. A standard domino Fibonacci tableau of shape w is then just a path tableau representing a maximal chain from $\emptyset$ to w in $S(2)$, but with i's placed in the domino created at the $n - i + 1$st step.

The evacuation method described in Section 6 can be used to prove that the number of standard domino Fibonacci tableaux is equal to the number of domino Fibonacci path tableaux.

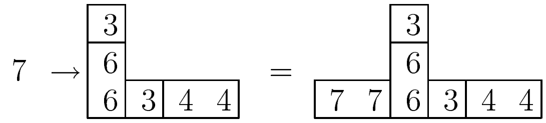
5 Domino Fibonacci Insertion

We now give a domino insertion algorithm which gives a bijection between a colored permutation and a pair of tableaux (P, Q) of domino Fibonacci shape. In the domino insertion algorithm, the P tableau that is created will be a standard domino Fibonacci tableau and the Q tableau that is created will be a domino Fibonacci path tableau. To apply our algorithm to a colored permutation $\pi = x_1 x_2 \ldots x_n$, we will construct a sequence $\{(P_i, Q_i)\}_{i=0}^n$ where $(P_0, Q_0) = (\emptyset, \emptyset)$ and (P_i, Q_i) are the tableaux obtained from the insertion of x_i (which may be barred or unbarred) into P_{i-1}. To begin with, if x_1 is barred then both P_1 and Q_1 are horizontal dominos containing 1's. If x_1 is unbarred then both P_1 and Q_1 are vertical dominos containing 1's. Now continue the insertion process for each x_i:

1. If x_i is unbarred then x_i will be inserted as a horizontal domino in the following manner:

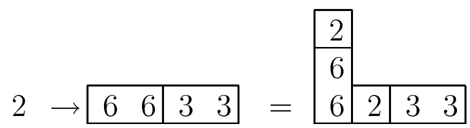
   (a) Compare the value of x_i to the value t_1 in the domino containing the leftmost square in the bottom row of P_{i-1}.

   (b) If $x_i > t_1$, add a horizontal domino containing x_i's to the left of the square containing t_1 in the bottom row. Call this new tableau P_i. For example,



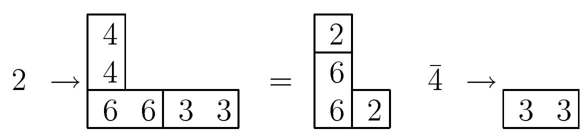
To form Q_i, a tableau of the same shape as P_i, place i's in this newly created horizontal domino.

(c) If $x_i < t_1$ and the domino d_1 containing t_1 is horizontal then change d_1 to a vertical domino in the first column and place a split horizontal domino containing the value of x_i into the square in the third row of the first column and the single square in the first row of the second column. If there were no domino on top of d_1 in P_{i-1}, then this new tableau is P_i. For example,

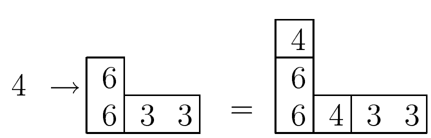


Obtain Q_i by placing i's into the vertical domino created in the second and third rows of the first column.

If there were a vertical domino containing b's on top of d_1 in P_{i-1}, then the vertical domino containing b's is bumped out of the first column as $\bar{b}$. Continue inductively inserting $\bar{b}$ into the tableau to the right of the first two columns by comparing b to the element t_2 in the domino in the bottom row of the third column and repeating steps (a), (b), (c) and (d) of Case 2. For example,

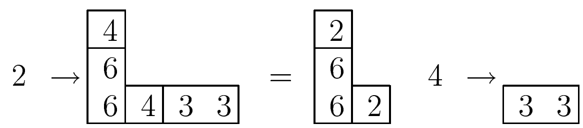


(d) If $x_i < t_1$ and d_1 is vertical, then if there were no domino on top of d_1 in P_{i-1}, create a new split horizontal domino by placing x_i in a new square in the third row of the first column and in a new square in the first row of the second column. For example,



Obtain Q_i by placing i's into this newly created split horizontal domino.

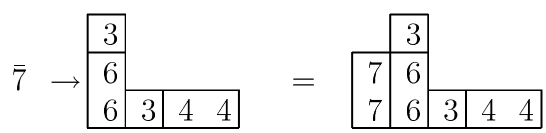
If there were a split horizontal domino containing b's on top of d_1 in P_{i-1} then replace the values in this split horizontal domino with x_i's and bump a horizontal domino containing b's out of the first stack of dominos as b. Now insert b into the tableau to the right of the first two columns by comparing b to the element t_2 in the domino in the bottom row of the third column and repeating steps (a), (b), (c), and (d) of Case 1. For example,



2. If x_i is barred then x_i will be inserted as a vertical domino in the following manner:

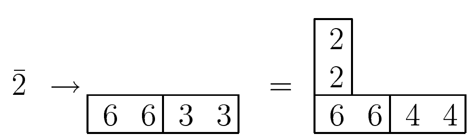
   (a) Compare the value of x_i to the value t_1 in the domino containing the leftmost square in the bottom row of P_{i-1}.

   (b) If $x_i > t_1$, add a vertical domino containing x_i's to the left of the square containing t_1 in the bottom row. Call this new tableau P_i. For example,



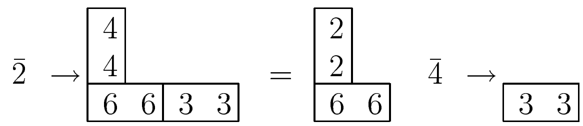
   To form Q_i, a tableau of the same shape as P_i, place i's in this newly created vertical domino.

   (c) If $x_i < t_1$ and d_1 is horizontal then place a vertical domino containing the value of x_i into the squares in the second and third rows of the first column. If there were no domino on top of d_1 in P_{i-1}, then this new tableau is P_i. For example,

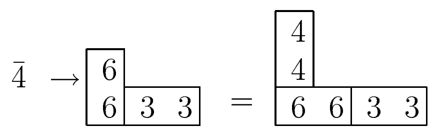


   Obtain Q_i by placing i's into the vertical domino created in the second and third rows of the first column.

   If there were a vertical domino containing b's on top of d_1 in P_{i-1}, then the vertical domino containing b's is bumped out of the first column as $\bar{b}$. Continue by inductively inserting $\bar{b}$ into the tableau to the right of the first two columns by comparing b to the element t_2 in the domino in the bottom row of the third column and repeating steps (a), (b), (c) and (d) of Case 2. For example,

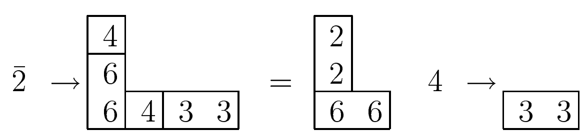


(d) If $x_i < t_1$ and d_1 is vertical, then if there were no domino on top of d_1 in P_{i-1} make d_1 into a horizontal domino by creating a new square in the first row of the second column. Place a domino containing x_i in the second and third rows of the first column and call this new tableau P_i. For example,



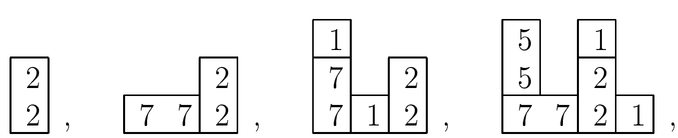
Obtain Q_i by placing i's into the new square created in the third row of the first column and the new square in the second column.

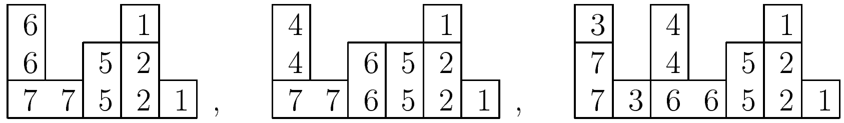
If there were a split horizontal domino contaning b's on top of d_1 then make d_1 into a horizontal domino in the first row of the first and second columns. Place a vertical domino containing x_i's in the second and third rows of the first column and bump the horizontal domino containing b's into the tableau to the right of the first two columns by comparing b to the element t_2 in the domino in the bottom row of the third column and repeating steps (a), (b), (c), and (d) of Case 1. For example,

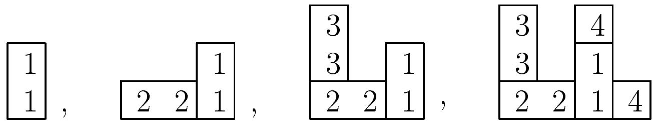


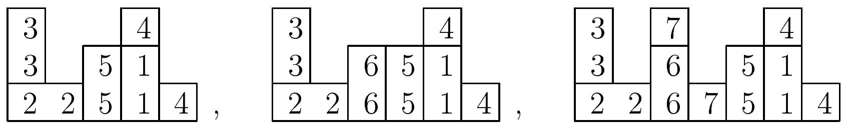
Example 1. *When applying the insertion algorithm to the permutation* $\pi = \bar{2}71\bar{5}\bar{6}\bar{4}3$ *that was used to form the square diagram in Section 2, we obtain the following:*

P_i :

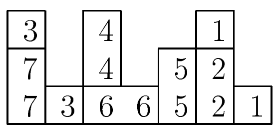
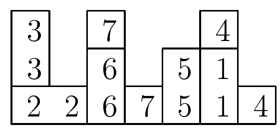




Q_i : 



From this example we have

$P =$:  $Q =$: 

Theorem 1. *The domino insertion algorithm is a bijection between colored permutations and pairs (P, Q) where P is a standard domino Fibonacci tableau and Q is a domino Fibonacci path tableau.*

Proof. We claim that the insertion procedure defined above is invertible. At the kth stage of the insertion, the Q tableau tells us which domino was the most recently created in the tableau P_k. If this domino was added on top of another domino, then the shape of P_k must have had a shape bijectively equivalent to $2^i\omega$ for some word ω of 1_1's, 1_2's and 2's.

When reversing the insertion algorithm, each domino in the top row will then bump to the left, preserving their horizontal or vertical shape, until the leftmost domino in the top row is bumped out of the tableau as either a vertical or horizontal domino. If this domino is vertical and contained x_i's, then $\bar{x_i}$ is the element that was inserted at this step. If the domino is horizontal and contained x_i's, then x_i is the element that was inserted as this step.

If the newly created domino was not added on top of another domino, then the shape of P_k is bijectively equivalent to either $2^{i-1}1_1\omega$ or $2^{i-1}1_2\omega$ depending on whether or not the newly created domino is horizontal or vertical, respectively. In both cases, the element inside the newly created domino, say t_i, is smaller than the element inside the bottom domino of the stack to the left of it. When we reverse the bumping algorithm, the domino containing t_i will bump the top domino of the stack to the left of it and each domino in the top row will bump to the left, preserving their horizontal or vertical shape until the leftmost domino in the top row is bumped out of the tableau as either a vertical or horizontal domino. If $i = 1$, then the newly created domino in the first stack is itself bumped out of the tableau. If this domino that is bumped out is vertical and contains x_i's, then $\bar{x}_i$ is the element that was inserted at this step. If the domino is horizontal and contains x_i's, then x_i is the element that was inserted as this step.

In either case, we obtain the originally inserted element, either barred or unbarred, and P_{k-1}. □

6 Evacuation

In the case of $Z(1)$, Killpatrick [4] gave an evacuation method for standard Fibonacci tableau. The evacuation given below is the generalization of that method.

Compute the evacuation of standard domino Fibonacci tableau P in the following manner.

1. Erase the number in the domino containing the leftmost square in the bottom row. This will necessarily be the largest number in P.

2. As long as there is a domino, either split horizontal or vertical, above the empty domino, compare the numbers in the domino above and the domino to the right of the empty domino, ignoring the latter if it does not exist.

   (a) Suppose the number in the domino on top is larger than the number in the domino on the right. Place the number in the top domino in a vertical domino (that starts on the bottom row) if the domino on top was vertical and place the number in the top domino in a horizontal domino if the domino on top was a split horizontal domino. This leaves an empty split horizontal domino in the first case and an empty vertical domino in the second and third rows in the second case.

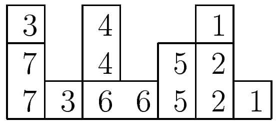
   (b) If the number in the domino to the right (if there is one) is larger then place that number in the empty domino leaving a new empty domino.

3. Continue in this manner until reaching a domino that has no domino immediately above it. At this point, remove the empty domino from the tableau and if this results in an empty column or columns in the middle of the tableau, slide all remaining columns to the left so that the result has the shape of a Fibonacci tableau. Call this remaining tableau $P^{(1)}$.

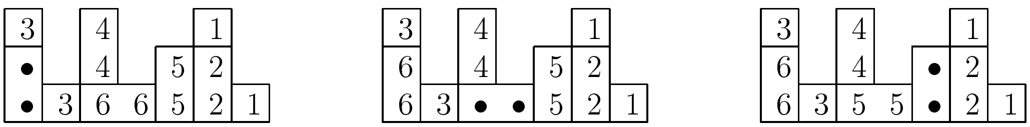
4. In a new tableau of the same shape as P, denoted by $\tilde{P}$, put n's in the position of the last empty domino.

5. Create $P^{(2)}$ by repeating the above procedure on $P^{(1)}$. At step 4, label the position of the last empty domino with $n-1$'s in the tableau $\tilde{P}$. Continue until $P^{(n)} = \emptyset$ and $\tilde{P}$ is a standard domino Fibonacci tableau containing dominos numbered 1 through n. The final tableau $\tilde{P}$ is called the evacuation tableau $ev(P)$.

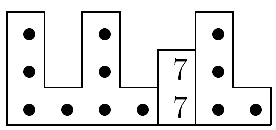
For example, using

$P(\pi)=$ 

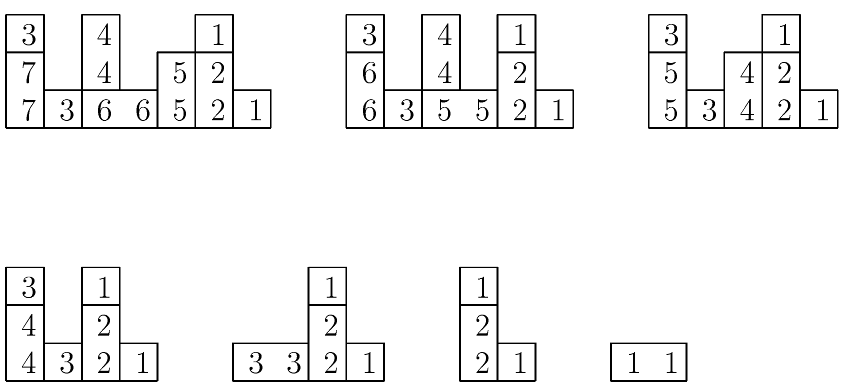
the first sequence of steps is



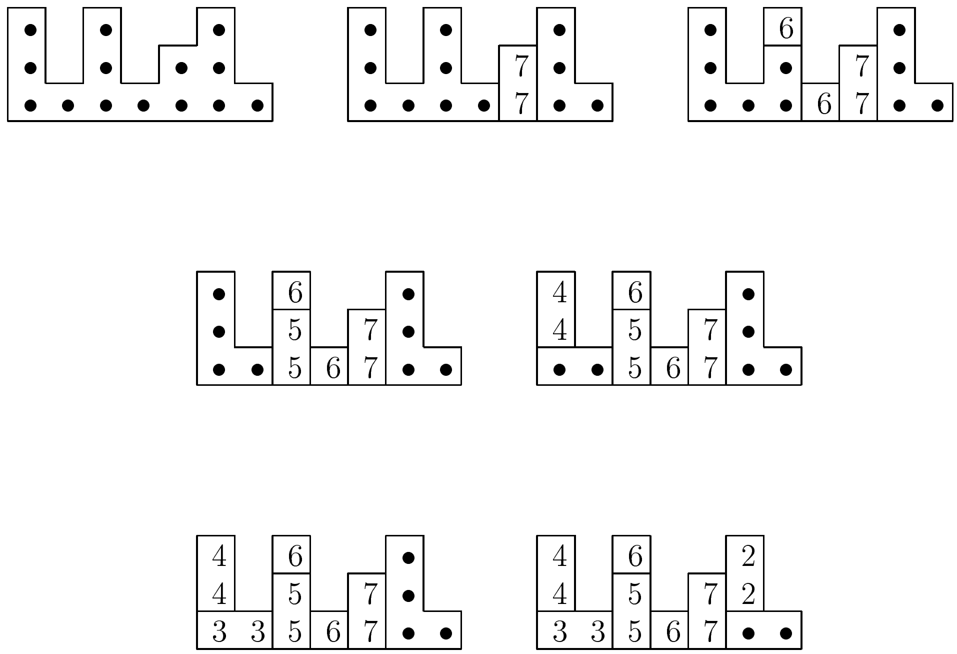
and thus after one step of the evacuation procedure, $\tilde{P}$ looks like



All of the steps in the evacuation of P and the development of $\tilde{P}$ are shown in the following example:

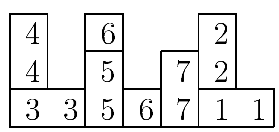
$P^{(k)}$: 

$\tilde{P}$:



Completing the last slide, we have

$ev(P(\pi))$ =



In the following section, we show that this evacuation method can be used to give a relation between the pair of tableau (P, Q) obtained from the domino insertion algorithm and the pair $(\hat{P}, \hat{Q})$ obtained from Fomin's growth diagrams and we prove that evacuation is a bijection between standard domino Fibonacci tableaux and domino Fibonacci path tableaux. Here we describe the inverse of the evacuation map.

To begin, think of a Fibonacci domino tableau as a sequence of "columns" that each contain one or two dominos. Given a path tableau of shape λ, denote the column with the domino containing 1's as column c. Remove the domino containing 1's from the tableau. Decrease all remaining values in the tableau by 1. If there is no domino in column c, then stop and place 1's in a domino in column c in an empty tableau of shape λ.

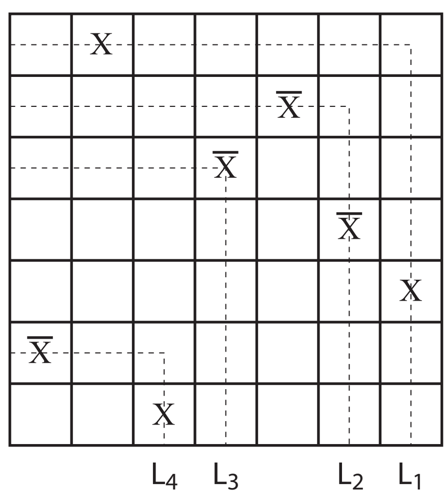
If a domino is present in column c, then (leaving all orientations of dominos fixed) cycle the values in column c and all columns to the right of c so that the largest cycled value is in column c. That is, if $a_1 < a_2 < \cdots < a_k$ are the values remaining in column c and all columns to the right of c, then replace a_1 with a_k, a_2 with a_1, a_3 with a_2, and so on. This creates a path tableau that is one domino smaller than λ and leaves an empty domino that was either at the top of a column or a singleton on the right end of the shape. Place 1's in this empty domino in an empty tableau of shape λ.

Repeat the above process on the smaller path tableau. At the ith step, place a domino containing i's into the empty tableau of shape λ. This sequence of steps defines a Fibonacci standard domino tableaux. One should note that the tiling of the standard Fibonacci domino tableau and the tiling of the evacuation of that tableau are related by swapping the shape of the dominos in the columns of height 2.

7 A Geometric Interpretation

For the case of $Z(1)$, Killpatrick [4] gave a description of shadow lines for the square diagram of a permutation in S_n that can be used to directly determine the standard Fibonacci tableau obtained through the insertion algorithm. We will use the same definition of shadow lines for the square diagram of a colored permutation and will show that these can also be used to directly determine the P tableau obtained through the domino insertion algorithm.

To draw the shadow lines, $L_1, L_2 \ldots$, for the square diagram of a colored permutation π, begin at the top row and draw a broken line L_1 through the X (barred or unbarred) in the top row and the X (barred or unbarred) in the rightmost column. The second broken line L_2 will be drawn through the row containing the highest X not already on a line and the rightmost column containing an X not already on a line. Continue in this manner until there are no more X's available. For example, for the permutation $\pi = \bar{2}71\bar{5}\bar{6}\bar{4}3$, the lines look like:



Theorem 2. *Given a colored permutation π, the tableau obtained by drawing shadow lines is the same tableau P obtained by the insertion algorithm. That is, the shadow lines $L_1, L_2, \ldots$ in the square diagram of π have the following properties:*

1. *The row numbers of the X's on each L_i give the numbers in the dominos in the ith column of the insertion tableau P.*

2. *If there is a single X on the line L_i, then the domino in the ith column of P is a vertical domino if the X is barred and a horizontal domino if the X is unbarred.*

3. *If there are two X's on the line L_i, then the larger row number is in the bottom domino and the smaller row number is in the top domino. The rightmost X on the line L_i determines the shape of the two dominos in column i: if the X is unbarred then the column contains a vertical domino with a split horizontal domino on top of it; if the X is barred, then the column contains a horizontal domino with a vertical domino on top of it.*

For the example above, the shadow lines give the P tableau:

$$P = : \quad \begin{array}{ccccccc} 3 & & 4 & & & 1 & \\ 7 & & 4 & & 5 & 2 & \\ 7 & 3 & 6 & 6 & 5 & 2 & 1 \end{array}$$

Proof. We will prove this result by induction on the size of π. Throughout the proof, any permutation π is understood to be a colored permutation. If $\pi \in S_1$ then either $\pi = 1$, in which case both the shadow lines and the insertion algorithm give the P tableau:

$$P = \boxed{1 \quad 1}$$

or $\pi = \bar{1}$, in which case both the shadow lines and the insertion algorithm give the P tableau:

$$P = \boxed{\begin{array}{c} 1 \\ 1 \end{array}}$$

If $\pi \in S_2$ then there are eight colored permutations and one can easily check that in each case the P tableau obtained by the shadow lines is equal to the P tableau obtained through the insertion algorithm.

Now assume that the tableau determined by the shadow lines for the colored permutation $\sigma \in S_k$ with $k < n$ is equal to the insertion tableau $P(\sigma)$. Let $\pi \in S_n$ be a colored permutation. Represent the permutation π with a square diagram and draw L_1.

<u>*Case 1:*</u> If there is an X (barred or unbarred) in the upper right corner of the square diagram, then L_1 only passes through one X. Since an X in the upper right corner implies that either n or $\bar{n}$ is the last number in the permutation π, we can write $\pi = \pi_{n-1}n$ in the first case or $\pi = \pi_{n-1}\bar{n}$, where π_{n-1} represents the first $n-1$ digits in the colored

permutation π. Since n, barred or unbarred, is the last number in the permutation, when we apply the insertion algorithm to π, n is the last number inserted into the tableau. Thus the insertion tableau P is either a horizontal domino or a vertical domino containing n followed by P_{n-1}, where P_{n-1} is the insertion tableaux for π_{n-1}. Thus the fact that the line L_1 drawn in the nth row and nth column only passes through one X corresponds to the fact that there is only one domino at the beginning of the P tableau and the shape of that domino is determined by whether or not X is barred or unbarred. Then the insertion tableau P and the tableau obtained from the shadow lines agree in the domino in the first column and by induction, they agree in the remaining positions.

<u>*Case 2:*</u> If there is no X in the upper right square, then L_1 passes through two X's, one in row n and one in column n and row a (counting from the bottom) with $a < n$. Since this means that a or $\bar{a}$ is the last element in the permutation π, then a or $\bar{a}$ is the last element inserted into the P tableau. Due to the method of insertion, the element n, which corresponds to the X in the top row, is always in a domino of some shape in the lower left position of P. Thus when a or $\bar{a}$ is inserted into the tableau, it is inserted as a domino above the domino containing n, possibly bumping an element b or $\bar{b}$ to the second column. The resulting P tableau has a domino containing a on top of a domino containing n in the first column, corresponding to the fact that L_1 passes through two X's, one in row n and one in row a. if $\bar{a}$ is the last element of π (i.e. the X in row a is barred), then the domino containing n is horizontal with a vertical domino containing a on top of it. If a is the last element of π (i.e. the X in row a is unbarred), then the domino containing n is vertical with a split horizontal domino containing a on top of it.

It remains to show that the rest of the P tableau can be determined by removing the nth row and the nth column from the square diagram, since these elements are in the first column of P, and applying the inductive hypothesis to the remaining diagram. Let the permutation π be written as

$$\pi = \begin{array}{cccccccc} 1 & 2 & \cdots & i-1 & i & i+1 & \cdots & n-1 & n \\ x_1 & x_2 & \cdots & x_{i-1} & n & x_{i+1} & \cdots & x_{n-1} & a \end{array}$$

where the elements in the bottom row of π can be barred or unbarred.

Recall that P_i is the insertion tableau of the first i elements $x_1 x_2 \cdots x_{i-1} n$. By definition of the insertion algorithm, P_i is either a vertical or horizontal domino containing n, depending on whether n is barred or unbarred, followed by P_{i-1}. Since $x_k < n \ \forall k \neq i$ then $x_{i+1} < n$. If x_{i+1} is unbarred then P_{i+1} has a split domino containing x_{i+1} on top of a vertical domino containing n, followed by P_{i-1}. If x_{i+1} is barred then P_{i+1} has a vertical domino containing x_{i+1} on top of a horizontal domino containing n followed by P_{i-1}.

When x_{i+2} is inserted into P_{i+1}, x_{i+1} is bumped out of the first stack of dominos and inserted into the tableau to the right, which is P_{i-1}, and the shape of the domino containing x_{i+1} is preserved. When x_{i+3} is inserted, x_{i+2} is bumped out of the first stack of dominos and inserted into the tableau to the right. At the last step, a bumps x_{n-1} from the first stack of dominos and x_{n-1} is then inserted into the tableau to the right. In the insertion algorithm, the shape of each stack of two dominos is determined by whether or not the element in the top domino is barred or unbarred. The resulting tableau is thus the same as the tableau obtained by placing the domino containing a on top of the domino containing n, with the shape determined by whether or not a is barred or unbarred, in front of the tableau obtained from the insertion of

$$\sigma \;=\; \begin{array}{cccccccc} 1 & 2 & \cdots & i-1 & i & \cdots & n-2 \\ x_1 & x_2 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_{n-1} \end{array},$$

the permutation in S_{n-2} obtained by removing n and a from π. The square diagram for σ is the same as the square diagram for π with the top row and rightmost column removed and any empty rows and columns removed (since empty rows and empty columns do not affect the growth diagram). Inductively, we can now apply the above conditions to this new square diagram and continue to determine the complete insertion tableau $P(\pi)$. $\square$

Theorem 3. *For π a colored permutation of length n, $ev(P(\pi)) = \hat{P}(\pi)$.*

Proof. We will prove that $ev(P(\pi)) = \hat{P}(\pi)$ by induction. If the length of π is 1, then the path tableau $\hat{P}$ is a single horizontal domino or a single vertical domino and the insertion tableau P is the same, so $\hat{P}(1) = ev(P(1))$.

Assume that for σ a colored permutation of length k with $k < n$, $ev(P(\sigma)) = \hat{P}(\sigma)$ and let π be a colored permutation of length n.

<u>*Case 1:*</u> Suppose the square in the uppermost, rightmost corner of the square diagram for π contains an X.

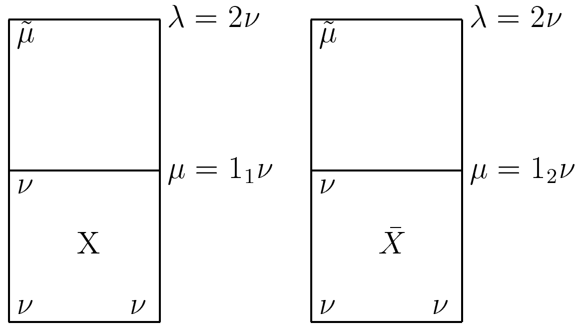
An X in this square, barred or unbarred, implies that $\bar{n}$ or n is the last element in the permutation π, so $\pi = \pi_{n-1}\bar{n}$ or $\pi = \pi_{n-1}n$ where π_{n-1} represents the first $n-1$ digits in the colored permutation π. From the square diagram, we have that $\hat{P} = \bar{n}\hat{P}_{n-1}$ or $\hat{P} = n\hat{P}_{n-1}$ where $\hat{P}_{n-1}$ is the path tableau of shape ν obtained from π_{n-1}. Since n, barred or unbarred, is the last number in the permutation π, when we apply the insertion algorithm, n is the last number inserted into the tableau. Thus the insertion tableau P is a vertical or horizontal domino containing n's followed by P_{n-1} where P_{n-1} is the insertion tableaux for π_{n-1}. Following the evacuation procedure, the domino containing n's is simply removed from P and $ev(P)$ is either a vertical or horizontal domino followed by $ev(P_{n-1})$. Since $\pi = \pi_{n-1}n$ or $\pi = \pi_{n-1}\bar{n}$, π_{n-1} is a colored permutation of length $n-1$ and $\hat{P}_{n-1}$ is the path tableau obtained from π_{n-1}, then the inductive hypothesis implies that $\hat{P}_{n-1} = ev(P_{n-1})$. Thus

$$ev(P) = \hat{P}.$$

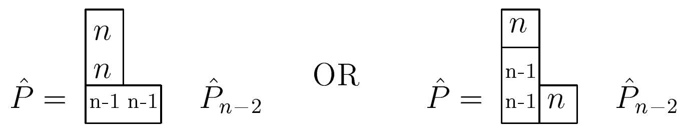
<u>*Case 2:*</u> Suppose the X, barred or unbarred, in the nth column of the square diagram is in row $n-1$. In this case, the permutation π looks like:

$$\pi \;=\; \begin{array}{cccccccc} 1 & 2 & \cdots & i & i+1 & \cdots & n-1 & n \\ x_1 & x_2 & \cdots & n & x_{i+1} & \cdots & x_{n-1} & n-1 \end{array}$$

where each of the elements x_i, n and $n-1$ are either barred or unbarred. The top two squares in the last column of the growth diagram look like one of the following:



Here μ and λ differ by a either a vertical domino or a split horizontal domino in the initial column of height 2, respectively. Thus



where $\hat{P}_{n-2}$ is the path tableau of shape ν obtained from the first $n-2$ rows of the growth diagram. The first $n-2$ rows have columns i and n empty, where i is the column containing an X or $\bar{X}$ in the nth row of the square diagram, and these first $n-2$ rows are the growth diagram for

$$\sigma \quad = \quad \begin{array}{cccccccc} 1 & 2 & \cdots & i-1 & i & \cdots & n-2 \\ x_1 & x_2 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_{n-1} \end{array}$$

once empty columns have been removed. In σ, if x_i was barred in π then it will be barred in σ. Note that σ is a colored permutation in S_{n-2}.

By Theorem 2, the insertion tableau for π can be determined by the shadow lines of the square diagram. Since there is no X in the upper right corner, the X in the uppermost row is paired with the X in row $n-1$ of the nth column. Thus, the insertion tableau P begins with a column of height two containing a vertical domino on top of a horizontal domino if the X in row $n-1$ is barred or a split horizontal domino on top of a vertical domino if the X in row $n-1$ is unbarred. When P is evacuated, the shape of the top domino is preserved, leaving an empty split horizontal domino if a horizontal domino is removed and leaving an empty vertical domino (on top of a horizontal domino) if a vertical domino is removed. Thus the initial column of height 2 of $ev(P)$ has a vertical domino containing n on top of a horizontal domino if the X in row $n-1$ is unbarred, which is the same as the placement of the domino containing n in $\hat{P}$. If the X in row $n-1$ is barred, then the initial column of height 2 of $ev(P)$ has a split horizontal domino containing n on top of a vertical domino, which is the same as the placement of the domino containing n in $\hat{P}$.

At the second step of the evacuation process, the domino containing $n-1$ is removed from P, leaving an horizontal domino if the X in row $n-1$ is unbarred and leaving a vertical domino if the X in row $n-1$ is barred. Then

$$ev(P) = \begin{array}{ll} n & \\ n & \\ n\text{-}1 & n\text{-}1 \end{array} \; ev(P_{n-1}) \qquad \text{OR} \qquad ev(P) = \begin{array}{ll} n & \\ n\text{-}1 & \\ n\text{-}1 & n \end{array} \; ev(P_{n-1})$$

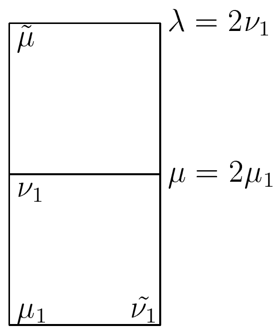
where P_{n-2} is the insertion tableau P without the first column. Comparing $\hat{P}$ and $ev(P)$ we can see that they agree in the first column of height two. As shown in the proof of Theorem 2, $P(\pi)$ has a column of height 2 followed by $P(\sigma)$ where σ is as given above. Since $\sigma \in S_{n-2}$, we can use our inductive hypothesis to obtain

$$ev(P(\pi)) = \hat{P}(\pi).$$

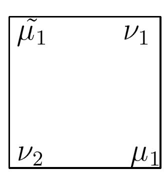
<u>*Case 3:*</u> Suppose the X in column n is in row $a_1 < n-1$. In this case, π is given by:

$$\pi \quad = \quad \begin{array}{ccccccc} 1 & 2 & \cdots & i & \cdots & n-1 & n \\ x_1 & x_2 & \cdots & n & \cdots & x_{n-1} & a_1 \end{array}$$

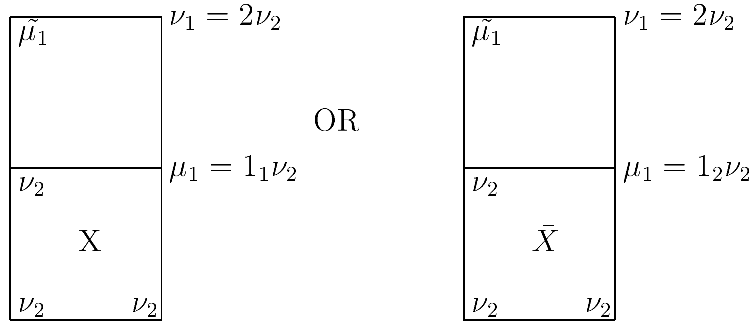
where the elements in the bottom row are each either barred or unbarred. The top two squares in the rightmost column of the growth diagram look like:



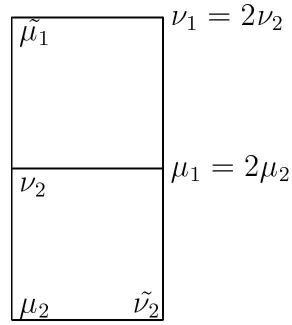
Since $\lambda = 2\nu_1$ and $\mu = 2\mu_1$, then λ and μ differ by the same domino as ν_1 and μ_1. If we remove the upper row and rightmost column, as well as any empty rows and columns, then the partial growth diagram of the new upper right square looks like



As before, if there is an X in the new upper right square, then $\nu_1 = 1_1\mu_1$ if the X is unbarred and $\nu = 1_2\mu$ if the X is barred. If there is an X in the square below this one, then ν_1 and μ_1 differ by a vertical domino in the initial column of height 2 if the X is unbarred and ν_1 and μ_1 differ by a split horizontal domino in the initial column of height 2 if the X is barred:



If there is no X in either square, then the growth diagram looks like



We can continue this procedure until μ_i and ν_i differ by a domino in the first column which implies that λ and μ differ by a domino in the $(i+1)$st column. (Note, a column is considered to be a single domino or a stack or two dominos. For example, a horizontal domino takes up one column in this terminology.)

We now show that the evacuation tableau $ev(P)$ has a domino containing n's in the same place in the tableau as $\hat{P}$. If there is not an X in the nth row or $(n-1)$st row of the nth column of the growth, then by Theorem 2 the first column of P has a domino containing a_1's, with $a_1 < n-1$, on top of a domino containing n's. If the X in column n is unbarred, then the domino containing a_1's is a split horizontal domino on top of a vertical domino containing n's and if the X in column n is barred, then the domino containing a_1's is a vertical domino on top of a horizontal domino containing n's.

After removing the nth row and the nth column and any empty rows and columns from the growth diagram, if there is not an X in one of the top two rows of the rightmost column of the new growth diagram, then the second column of P is a domino containing a_2's, with $a_2 < n-2$, that is split horizontal if the X in the rightmost column is unbarred and vertical if the X in the rightmost column is barred, on top of a domino containing $n-1$'s of the appropriate shape. We can continue in this manner until one of two things happens.

Subcase a: Suppose after i iterations of this process, there is an X, either barred or unbarred, in the uppermost corner of the growth diagram. In this case, the insertion tableau P has i columns of height 2 followed by a single vertical domino if the X is barred and by a single horizontal domino if the X is unbarred. These first $i+1$ columns look like

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & \cdots & a_i & \\ n & n-1 & n-2 & \cdots & n-(i-1) & n-i \end{array}$$

with $a_2 < n-1$, $a_3 < n-2$, …, $a_i < n-(i-1)$ where the column $\begin{array}{c} a_k \\ n-(k-1) \end{array}$ represents a domino containing a_k's on top of a domino containing $n-(k-1)$'s. The shape of the dominos in each of the columns of height two is determined by the X in the a_k row. If the X in the a_k row is barred, then the domino containing a_k's is a vertical domino on top of a horizontal domino containing $n-(k-1)$'s. If the X in the a_k row is unbarred, then the domino containing a_k's is a split horizontal domino on top of a vertical domino containing $n-(k-1)$'s. Thus the top domino in the column determines the shape of the dominos in the column. At the first step of evacuation for P, the domino containing $n-1$ slides one column to the left into the empty domino evacuated by n, $n-2$ slides one column to the left, and so on until $n-i$ slides one column to the left and the evacuation process terminates with an empty single domino in column $i+1$. This single domino is horizontal if the X in the uppermost corner of the growth diagram at this step (i.e. in the $(n-i)$th row) is unbarred and vertical if the X is barred. Thus $ev(P)$ has n's in the single domino in column $i+1$, the same as $\hat{P}$, and after one step of the evacuation procedure the first i columns of the P tableau look like:

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & \cdots & a_i \\ n-1 & n-2 & n-3 & \cdots & n-i \end{array}.$$

where again $\begin{array}{c} a_i \\ n-i \end{array}$ represents a stack of two dominos and the shape of the dominos are again determined by the top domino containing a_i's. The rest of the P tableau remains unchanged by the evacuation procedure.

Subcase b: Suppose after i iterations of this process there is an X, either barred or unbarred, in the second row from the top. In this case, the first $i+1$ columns of the insertion tableau P have height 2. These first $i+1$ columns look like

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & \cdots & a_i & n-(i+1) \\ n & n-1 & n-2 & \cdots & n-(i-1) & n-i \end{array}$$

with $a_1 < n-1$, $a_2 < n-2$, …, $a_i < n-i$, where columns represent stacks of dominos as in Subcase a. In the evacuation process, the dominos containing $n-1$ through $n-i$ all move one column to the left with the shape of the column determined by the top domino. The domino containing $n-(i+1)$ becomes a single horizontal domino if the X reached in row $n-(i+1)$ is unbarred and a vertical domino if the X is barred. This leaves an empty top vertical domino or an empty split horizontal domino, respectively, in column $i+1$. Thus $ev(P)$ has a domino containing n's as the top domino in column $i+1$, as does $\hat{P}$, and of the same shape as in $\hat{P}$. The part of the P tableau to the right of the $(i+1)$st column remains the same.

In both subcases, we can now remove the domino containing n from the $(i+1)$st column of $\hat{P}$ to obtain $\hat{P}_{n-1}$ of shape μ. The path tableau $\hat{P}_{n-1}$ is the path tableau obtained from the first $n-1$ rows of the square diagram, which come from the colored permutation

$$\tau \;=\; \begin{array}{ccccccc} 1 & 2 & \cdots & i-1 & i & \cdots & n-1 \\ x_1 & x_2 & \cdots & x_{i-1} & x_{i+1} & \cdots & a \end{array}.$$

Note that $\tau \in S_{n-1}$. In order to use our inductive hypothesis, it remains to show that after one step of the evacuation of P, we obtain $P(\tau)$. In the proof of Theorem 2, we proved that $P(\pi)$ is equal to a column of height 2 that has a domino containing a's on top of a domino containing n's followed by $P(\sigma)$ where

$$\sigma \;=\; \begin{array}{ccccccc} 1 & 2 & \cdots & i-1 & i & \cdots & n-2 \\ x_1 & x_2 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_{n-1} \end{array}.$$

To obtain $P(\tau)$ we must insert a_1, barred or unbarred, into $P(\sigma)$.

In *Subcase a*, $P(\sigma)$ looks like

$$\begin{array}{cccccc} a_2 & a_3 & \cdots & a_{i-1} & a_i & \\ n-1 & n-2 & \cdots & n-(i-2) & n-(i-1) & n-i \end{array}$$

and a_1 inserted into this tableau gives

$$\begin{array}{ccccc} a_1 & a_2 & \cdots & a_{i-1} & a_i \\ n-1 & n-2 & \cdots & n-(i-1) & n-i \end{array}$$

for the first i columns and does not change the remaining tableau. Again the shape of each column of height 2 is determined by the shape of the domino in the top row. This is exactly what P looks like after one step of the evacuation procedure.

In *Subcase b*, $P(\sigma)$ looks like

$$\begin{array}{ccccc} a_2 & a_3 & \cdots & a_i & a_{n-(i+1)} \\ n-1 & n-2 & \cdots & n-(i-1) & n-i \end{array}$$

and a_1 inserted into this tableau gives

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & \cdots & a_i & \\ n-1 & n-2 & n-3 & \cdots & n-i & n-(i+1) \end{array}$$

for the first $i+1$ columns and does not change the remaining tableau. This is again exactly what P looks like after one step of the evacuation procedure. By induction, $ev(P(\tau)) = \hat{P}(\tau)$ and since $ev(P(\pi))$ and $\hat{P}(\pi)$ agree in the position of the domino containing n, then $ev(P(\pi)) = \hat{P}(\pi)$.

□

Theorem 4. $Q(\pi) = \hat{Q}(\pi)$.

Proof. We will prove this result by induction on the size of $Q(\pi)$. If π is a permutation of a single element, then $\pi = 1$ of $\pi = \bar{1}$. If $\pi = 1$ then there is an unbarred X in the single square in the growth diagram for π and $\hat{Q(\pi)}$ is a horizontal domino with 1's in it. If the X is barred then $\hat{Q(\pi)}$ is a vertical domino with 1's in it. One can easily check that the tableau $Q(\pi)$ for the insertion of this single element agrees with $\hat{Q(\pi)}$.

Now assume that $Q(\sigma) = \hat{Q}(\sigma)$ for σ a colored permutation of length $k < n$ and let π be a colored permutation of length n. Since the growth diagram for π^{-1} is simply the reflection of the growth diagram for π around the diagonal line $y = x$, then $\hat{P}(\pi) = \hat{Q}(\pi^{-1})$ and $\hat{Q}(\pi) = \hat{P}(\pi^{-1})$. Let π_{n-1} be the first $n-1$ elements in the colored permutation π. Then $\hat{Q}(\pi_{n-1})$ is the shape of the $\hat{Q}(\pi)$ tableau at the $(n-1)$st stage.

By reflecting across the diagonal, we have $\hat{Q}(\pi_{n-1}) = \hat{P}_{n-1}(\pi^{-1})$ where $\hat{P}_{n-1}(\pi^{-1})$ is the tableau for the square diagram consisting of the first $n-1$ rows of the square diagram for π^{-1}. Let D represent the domino that $\hat{P}_{n-1}(\pi^{-1})$ and $\hat{P}(\pi^{-1})$ differ by. Then D also represents the domino that $\hat{Q}(\pi_{n-1})$ and $\hat{P}(\pi^{-1})$ differ by. From the growth diagrams we know that the shape of $\hat{Q}(\pi)$ is equal to the shape of $\hat{P}(\pi)$ which is equal to the shape of $\hat{P}(\pi^{-1})$. Thus D represents the domino that $\hat{Q}(\pi)$ and $\hat{Q}(\pi_{n-1})$ differ by. We must show that D is also the domino that $Q(\pi)$ and $Q(\pi_{n-1})$ differ by, which means D must be the domino that $P(\pi)$ and $P(\pi_{n-1})$ differ by, since Q represents a recording tableau for the insertion tableau P.

Since $ev(P(\pi)) = \hat{P}(\pi)$ then these tableau have the same shape and $ev(P(\pi))$ has the same shape as $P(\pi)$ by construction so $P(\pi)$ has the same shape as $\hat{P}(\pi)$. By reflection, the shape of $\hat{P}(\pi)$ is the same as the shape of $\hat{P}(\pi^{-1})$. Suppose a, barred or unbarred, is the last element in the colored permutation π and let σ be the colored permutation in S_{n-1} obtained from π by deleting the last element a and replacing all elements i with $i > a$ by $i-1$. For $i > a$ if i was barred in π then $i-1$ will be barred in σ and for $i \leq a$, if i was barred in π then i will be barred in σ. By the method of insertion, the shape of $P(\pi_{n-1}) = P(\sigma)$ and by Fomin's growth diagram we have that the shape of $\hat{Q}(\pi_{n-1})$ will be the same as the shape of $\hat{Q}(\sigma)$, since $\hat{Q}(\pi_{n-1})$ is the path tableau for the growth diagram of the first $n-1$ columns of the square diagram for π, i.e. for all but the last element a of the square diagram for π. Thus the domino that $\hat{Q}(\pi)$ and $\hat{Q}(\pi_{n-1})$ differ by is the same as the domino that $P(\pi_{n-1})$ and $P(\pi)$ differ by, which is the same as the domino that $Q(\pi)$ and $Q(\pi_{n-1})$ differ by since Q is a recording tableau for P. By induction, $Q(\pi_{n-1}) = \hat{Q}(\pi_{n-1})$ and since $Q(\pi)$ differs from $Q(\pi_{n-1})$ in the same domino that $\hat{Q}(\pi)$ differs from $\hat{Q}(\pi_{n-1})$ by, then $Q(\pi) = \hat{Q}(\pi)$.

□

Theorem 5. *The evacuation procedure is a bijection between standard domino Fibonacci tableaux and Fibonacci path tableaux.*

Proof. The evacuation algorithm is, by definition, an injection from standard domino Fibonacci tableaux to domino Fibonacci path tableaux. The growth diagrams of Fomin shows that $2^n n!$ equals the number of pairs (P, Q) where P and Q are Fibonacci path tableaux of the same shape. The insertion algorithm given in Section 6 shows $2^n n!$ equals the number of pairs $(\hat{P}, \hat{Q})$ where $\hat{P}$ is a standard domino Fibonacci tableau and $\hat{Q}$ is a path Fibonacci tableau. Since $Q = \hat{Q}$ by Theorem 5, then the number of standard domino Fibonacci tableaux must equal the number of Fibonacci path tableaux. Hence, the evacuation algorithm is a bijection. $\square$

8 The Color-to-Spin Property

For a pair (P, Q) in which P is a standard domino Fibonacci tableau and Q is a domino Fibonacci path tableau, we define

$$vert(P, Q) = (\text{the total number of vertical dominos in } P \text{ and } Q).$$

To simplify the *vert* statistic, note than any column of height 2 contains a vertical domino so the number of vertical dominos in P is the number of columns of height 2 in the shape of P plus the number of 1_2's in the shape of P. Since P and Q have the same shape, the number of such columns in Q is the same as in P thus $vert(P, Q) = 2$(the number of columns of height 2 plus the number of 1_2's in the shape of P). Based on the shadow lines, this is the number of shadow lines with 2 X's plus the number of shadow lines with a single $\bar{X}$ on them.

We define

$$split(P, Q) = k - l$$

where k is the number of split horizontal dominos in Q and l is the number of split horizontal dominos in P.

Again we can interpret this statistic in terms of the shadow lines in the square diagram. The number of split horizontal dominos in $P(\pi)$ is the number of columns of height 2 with a split horizontal domino on top, which is the same as the number of shadow lines with 2 X's on them and with the X in the rightmost column unbarred.

Similarly, the number of split horizontal dominos in $Q(\pi)$ is the number of columns of height 2 in Q with a vertical domino on the bottom. This is the same as the number of lines with 2 X's on them with the X in the leftmost column barred for the following reason. In the insertion algorithm, dominos never move from being a bottom domino in a column of height 2 to being a top domino in that column. This means that in the Q, or recording, tableau, the bottom domino in a column of height 2 is created first and maintains its shape throughout the rest of the insertion algorithm. In addition, once a column contains 2 dominos it will have 2 dominos for the remainder of the insertion algorithm. Thus to count the number of split horizontal dominos in Q we need to know the number of columns of height 2, which is the number of shadow lines with 2 X's, whose bottom domino is a vertical domino, which means the leftmost X is barred.

We then define

$$spin(P, Q) = \frac{1}{2} vert(P, Q) + split(P, Q).$$

In the example of (P, Q) from the previous section we have $vert(P, Q) = 8$, $split(P, Q) = 2 - 2 = 0$ and $spin(P, Q) = 4 + 0 = 4$.

For a colored permutation π we define the $color(\pi)$ to be the total number of barred (or colored) elements in π.

Theorem 6. *If π is a colored permutation and (P, Q) is the pair of tableaux obtained through the domino Fibonacci insertion algorithm, then*

$$\mathit{color}(\pi) = \mathit{spin}(P, Q).$$

Proof. If we consider the square diagram of a colored permutation and then look at the shadow lines, we know that every X or $\bar{X}$ lies on some shadow line, so to prove this result we will look at the contribution to color and to spin of each shadow line.

Suppose the shadow line L contains only a single X. If the X is unbarred, then the contribution of this line to color is zero and the contribution of this line to spin is also zero. If the X is barred, then the contribution of this line to color is 1 and the contribution to spin is also 1, since this denotes a vertical domino (a 1_2) in the shape of P. If the shadow line L contains two X's (either barred or unbarred) then there are several cases to consider.

1. Suppose both X's on the line are barred. Then the contribution of this line to color is 2. Since this line contains two X's, the contribution to *vert* is $2(1) = 2$. Since the leftmost X is barred, this designates a split horizontal domino in Q so the contribution of the line to *split* is 1. Thus the contribution of the line to *spin* is $\frac{1}{2}(2) + 1 = 2$.

2. Suppose both X's on the line are unbarred. Then the contribution of this line to color is zero. Since this line contains two X's, the contribution to *vert* is 2. Since the rightmost X is unbarred, this designates a split horizontal domino in P so the contribution of the line to *split* is -1. Thus the contribution of the line to *spin* is $\frac{1}{2}(2) - 1 = 0$.

3. Suppose the leftmost X on the line is barred and the rightmost X on the line is unbarred. Then the contribution of this line to color is 1. Since this line contains 2 X's, the contribution to *vert* is 2. Since the leftmost X is barred, this designates a split horizontal domino in Q so the contribution of the line to *split* is 1. Since the rightmost X is unbarred, this designates a split horizontal domino in P so the contribution of the line to *split* is -1. Thus the total contribution of the line to *spin* is $\frac{1}{2}(2) + 1 - 1 = 1$.

4. Suppose the leftmost X on the line is unbarred and the rightmost X on the line is barred. Then the contribution of this line to color is 1. Since this line contains 2 X's, the contribution to *vert* is 2. Since the leftmost X on the line is unbarred and the rightmost X on the line is barred, there is no contribution to *split*. Thus the contribution of the line to *spin* is $\frac{1}{2}(2) = 1$.

$\square$

For domino Young tableaux Shimozono and White extended their results to define a generalized k-ribbon insertion algorithm [9]. The authors are working on extending the ideas in this paper to a notion of a generalized k-ribbon Fibonacci tableaux. In addition, it is natural to expect that the domino insertion algorithm should extend to semistandard permutations, but such an extension remains elusive. In particular, it is unclear what the correct definition of a semistandard Fibonacci tableaux should be. Such a definition would assist in giving a combinatorial interpretation of the Fibonacci Schur functions which appear in [5].

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