# The Polytope of Degree Partitions

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**Abstract**

The degree partition of a simple graph is its degree sequence rearranged in weakly decreasing order. The polytope of degree partitions (respectively, degree sequences) is the convex hull of degree partitions (respectively, degree sequences) of all simple graphs on the vertex set $[n]$. The polytope of degree sequences has been very well studied. In this paper we study the polytope of degree partitions. We show that adding the inequalities $x_1 \geq x_2 \geq \cdots \geq x_n$ to a linear inequality description of the degree sequence polytope yields a linear inequality description of the degree partition polytope and we show that the extreme points of the degree partition polytope are the $2^{n-1}$ threshold partitions (these are precisely those extreme points of the degree sequence polytope that have weakly decreasing coordinates). We also show that the degree partition polytope has $2^{n-2}(2n-3)$ edges and $(n^2-3n+12)/2$ facets, for $n \geq 4$. Our main tool is an averaging transformation on real sequences defined by repeatedly averaging over the ascending runs.

# 1 Introduction

The degree sequence of a simple graph is a classical and well-studied topic in graph theory. As explained in Chapter 3 of the book *Threshold graphs and related topics* by Mahadev and Peled [**MP**], this subject goes hand in hand with the topic of threshold sequences, i.e., degree sequences of threshold graphs. Threshold sequences satisfy many of the criteria for degree sequences in an extremal way. In this paper we develop a new example of this phenomenon. Our main reference for this paper is Chapter 3 of the book [**MP**]. Another informative recent reference is the paper by Merris and Roby [**MR**].

We consider only simple graphs. Given a simple graph $G = ([n], E)$ on the vertex set $[n] = \{1, 2, \ldots, n\}$, the degree $d_j$ of a vertex $j$ is the number of edges with $j$ as an endpoint and $d_G = (d_1, d_2, \ldots, d_n)$ is the *degree sequence* of $G$. The *degree partition* of $G$ is obtained by rearranging $d_G$ in weakly decreasing order. Let $DS(n)$ denote the set of all degree sequences of simple graphs on the vertex set $[n]$ and let $DP(n)$ denote the set of all degree partitions of $n$-vertex simple graphs (note that some of the entries of a degree partition may be zero. It is usual to have only nonzero terms in a partition, but in this paper it is convenient to have this slight generality).

Define $\mathsf{DS(n)}$, the *polytope of degree sequences*, to be the convex hull (in $\mathbb{R}^n$) of all degree sequences in $DS(n)$ and define $\mathsf{DP(n)}$, the *polytope of degree partitions*, to be the convex hull of all degree partitions in $DP(n)$. The study of $\mathsf{DS(n)}$ was begun by Koren [**K**] who determined its extreme points and showed that the linearized and symmetrized Erdős- Gallai inequalities provide a linear inequality description of $\mathsf{DS(n)}$. Beissinger and Peled [**BP**] determined the (exponential) generating function of the number of extreme points. Peled and Srinivasan [**PS**] determined the edges and facets of $\mathsf{DS(n)}$ and gave another proof of Koren's linear inequality description (we use this proof in the present paper). Finally, Stanley [**S2**] obtained detailed information on $\mathsf{DS(n)}$ including generating functions for all face numbers, volume, number of lattice points, and (the closely related) number of degree sequences (i.e., $\#DS(n)$). In this paper we study the polytope $\mathsf{DP(n)}$ and determine its vertices (and, as a corollary, its volume), edges, and facets.

Threshold graphs were introduced by Chvátal and Hammer [**CH**] and have many different characterizations. For our purposes the most convenient definition is the following: a simple graph $G$ is *threshold* if every induced subgraph of $G$ has a dominating or an isolated vertex. Define $TS(n)$ to be the set of all degree sequences of threshold graphs on the vertex set $[n]$ and define $TP(n)$ to be the set of all degree partitions of $n$-vertex threshold graphs. Elements of $TS(n)$ are called *threshold sequences* and elements of $TP(n)$ are called *threshold partitions*. If $(d_1, \ldots, d_n) \in TP(n)$, then either $d_1 = n - 1$ or $d_n = 0$. Using this fact inductively we easily see that $\#TP(n) = 2^{n-1}$.

We define two further polytopes in $\mathbb{R}^n$. The polytope $\mathsf{K(n)}$ is defined to be the solution set of the following system of linear inequalities:

$$\sum_{i\in S} x_i - \sum_{i\in T} x_i \leq \#S(n-1-\#T), \quad S, T \subseteq [n],\ S \cup T \neq \emptyset,\ S \cap T = \emptyset. \tag{1}$$

We call $\mathsf{K(n)}$ the *Koren polytope* (see [**K**]). Note that taking $S = \{i\}$, $T = \emptyset$ gives

$x_i \leq n-1$ and taking $S = \emptyset,\ T = \{i\}$ gives $x_i \geq 0$, showing that $\mathsf{K}(\mathsf{n})$ is indeed a polytope.

The polytope $\mathsf{F}(\mathsf{n})$ is defined to be the solution set of the following system of linear inequalities:

$$x_1 \geq x_2 \geq \cdots \geq x_n, \tag{2}$$

$$\sum_{i=1}^{k} x_i \ - \sum_{i=n-l+1}^{n} x_i \leq k(n-1-l), \quad 1 \leq k+l \leq n. \tag{3}$$

We call $\mathsf{F}(\mathsf{n})$ the *Fulkerson-Hoffman-McAndrew polytope* (see [**FHM**]). Note that (3) is obtained from (1) by taking $S = \{1, \ldots, k\}$ and $T = \{n-l+1, \ldots, n\}$. Intuitively, $\mathsf{K}(\mathsf{n})$ is obtained by symmetrizing $\mathsf{F}(\mathsf{n})$ and $\mathsf{F}(\mathsf{n})$ is the asymmetric part of $\mathsf{K}(\mathsf{n})$. Also note that $\mathsf{K}(\mathsf{n})$ has exponentially many defining inequalities while $\mathsf{F}(\mathsf{n})$ has only quadratically many defining inequalities.

We now recall the Fulkerson-Hoffman-McAndrew criterion for degree partitions (see [**FHM**] and item 5 in Theorem 3.1.7 in [**MP**]). We give both the partition and sequence versions. It follows from linearizing the well-known nonlinear inequalities of Erdős and Gallai [**EG**].

**Theorem 1.1** *Let $d = (d_1, d_2, \ldots, d_n) \in \mathbb{N}^n$. Then*
*(i) $d \in DP(n)$ if and only if $d \in \mathsf{F}(\mathsf{n})$ and $d_1 + \cdots + d_n$ is even.*
*(ii) $d \in DS(n)$ if and only if $d \in \mathsf{K}(\mathsf{n})$ and $d_1 + \cdots + d_n$ is even.*

Motivated by Theorem 1.1(ii) the following result was proved in [**K, PS**].

**Theorem 1.2** $\mathsf{DS}(\mathsf{n}) = \mathsf{K}(\mathsf{n})$*, with $TS(n)$ as the set of extreme points.*

The main result of this paper is the following generalization and partition analog of Theorem 1.2.

**Theorem 1.3** $\mathsf{DP}(\mathsf{n}) = \mathsf{F}(\mathsf{n})$*, with $TP(n)$ as the set of extreme points.*

We can derive most of Theorem 1.2 as a corollary of Theorem 1.3. Given a real vector $x$, let $[x]$ denote the vector obtained by rearranging the components of $x$ in weakly decreasing order. Then it is easily seen that $x \in \mathsf{K}(\mathsf{n})$ if and only if $[x] \in \mathsf{F}(\mathsf{n})$ and using this we see that Theorem 1.3 implies that $\mathsf{DS}(\mathsf{n}) = \mathsf{K}(\mathsf{n})$ and that every extreme point of $\mathsf{DS}(\mathsf{n})$ is a threshold sequence. To complete the proof of Theorem 1.2 we need to show that every threshold sequence is an extreme point of $\mathsf{DS}(\mathsf{n})$. This can be seen as follows. Every threshold sequence of length $n$ has some entry equal to $n-1$ or 0. Using this fact inductively we see that no threshold sequence can be written as a convex combination of other degree sequences.

The argument in the preceding paragraph is not reversible and there is no such simple proof of Theorem 1.3 from Theorem 1.2. Our proof of Theorem 1.3 has two main ingredients: an averaging operation on real sequences based on descent sets and Theorem 1.2. More precisely, we use not just the statement of Theorem 1.2 but its proof from [**PS**] (for other proofs of Theorem 1.2, see [**K**] and [**BS**]).

Let us consider Theorem 1.3 from a general perspective. Let $P$ be an integral polytope in $\mathbb{R}^n$ that is closed under permutations of its points, i.e., $x \in P$ implies $\pi.x \in P$, for all permutations $\pi$ of $[n]$. For example, $\mathsf{DS(n)}$ is such a polytope. Let $E$ denote the set of extreme points of $P$ and let $E_d \subseteq E$ denote the set of extreme points that have weakly decreasing coordinates. There are two natural ways to define the *asymmetric part* of $P$. In terms of lattice points we define the asymmetric part of $P$ as the polytope

$$P_d = \text{ convex hull of } \{(x_1, x_2, \ldots, x_n) \in P \cap \mathbb{N}^n \mid x_1 \geq x_2 \geq \cdots \geq x_n\}.$$

In terms of linear inequalities we define the asymmetric part of $P$ as the polytope $P_l$ obtained by adding the inequalities $x_1 \geq \cdots \geq x_n$ to the list of inequalities defining $P$. It is easily seen that $P_d \subseteq P_l$ and $E_d \subseteq$ set of extreme points of $P_d$. Equality need not hold in these two inclusions. For instance, consider the polytope $P$ in $\mathbb{R}^2$ defined by: $x_1, x_2 \geq 0,\ x_1 + x_2 \leq 3$. Then it is easily checked that $P_d$ is strictly contained in $P_l$. If we take $P$ to be the polytope in $\mathbb{R}^2$ defined by $x_1, x_2 \geq 0,\ x_1 + x_2 \leq 2$, then we can check that $P_d = P_l$ but $P_d$ has an extreme point $(1, 1)$ that is not contained in $E_d$. Unexpectedly, Theorem 1.3 asserts that, in the case $P = \mathsf{DS(n)}$, we have $P_d = P_l$ and set of extreme points of $P_d = E_d$. Note that Theorem 1.3 implies that the volume of $\mathsf{DS(n)}$ is $n!$ times the volume of $\mathsf{DP(n)}$ (for $n \geq 3$, $\mathsf{DS(n)}$ and $\mathsf{DP(n)}$ are full dimensional).

We now discuss another viewpoint on Theorem 1.3. Let $P$ be a finite poset. For each $p \in P$ introduce a variable $x_p$. The order polytope $\mathsf{O(P)}$ of $P$, defined by Stanley [**S1**], is the solution set of the following system of linear inequalities:

$$x_p \geq x_q, \quad p < q,\ p, q \in P, \tag{4}$$
$$0 \leq x_p \leq 1, \quad p \in P. \tag{5}$$

The constraint matrix of the inequalities (4) is easily seen to be totally unimodular and thus the vertices of $\mathsf{O(P)}$ are integral. It follows that the vertices of $\mathsf{O(P)}$ are the characteristic vectors of order ideals of $P$ (a subset $I \subseteq P$ is an *order ideal* if $q \in I$ and $p \leq q$ imply $p \in I$). Since the linear inequality description of $\mathsf{O(P)}$ is of polynomial size in $\#P$ we can optimize linear functions over $\mathsf{O(P)}$ in polynomial time using linear programming. Picard [**P**] showed that one can optimize linear functions over $\mathsf{O(P)}$ in polynomial time using network flows.

Let $S(n)$ denote the set of all 2-subsets of $[n] = \{1, \ldots, n\}$. We write elements of $S(n)$ as $(i, j)$, where $i < j$. Partially order $S(n)$ as follows: given $X = (a_1, a_2)$ and $Y = (b_1, b_2)$ in $S(n)$ define $X \leq Y$ if $a_i \leq b_i,\ i = 1, 2$. Let $\mathsf{S(n)}$ denote the order polytope of $S(n)$ and define $\mathsf{C(n)}$ to be the hypercube in $\binom{n}{2}$-space. The defining inequalities of $\mathsf{C(n)}$ are (here we write the variable corresponding to a 2-element subset $(i, j)$ as $x_{i,j}$)

$$0 \leq x_{i,j} \leq 1, \quad (i, j) \in S(n),$$

and the defining inequalities of $\mathsf{S(n)}$ are

$$x_{i,j} \geq x_{k,l}\,,\ (i, j) < (k, l), \quad (i, j), (k, l) \in S(n),$$
$$0 \leq x_{i,j} \leq 1, \quad (i, j) \in S(n).$$

Now let $M(n)$ denote the $n \times \binom{n}{2}$ incidence matrix of singletons vs. doubletons in $[n]$, i.e., the rows of $M(n)$ are indexed by $[n]$ and the columns of $M(n)$ (indexed by $S(n)$) are the characteristic vectors of elements of $S(n)$. We think of $M(n)$ as the linear transformation $\mathbb{R}^{\binom{n}{2}} \to \mathbb{R}^n$, $y \mapsto M(n)y$.

It is easily seen that the image of $\mathsf{C(n)}$ under the transformation $M(n)$ is $\mathsf{DS(n)}$. Theorem 1.2 gives the defining inequalities for this image along with the extreme points.

Now let us consider the image of $\mathsf{S(n)}$ under $M(n)$. It is well known (see **[CH, MP]**) that the order ideals in $S(n)$ are precisely the edge sets of threshold graphs on the vertex set $[n]$ whose degree sequences $(d_1, \ldots, d_n)$ satisfy $d_1 \geq \cdots \geq d_n$. It follows that $M(n)(\mathsf{S(n)}) = \mathsf{TP(n)}$, where $\mathsf{TP(n)}$ is the convex hull of $TP(n)$. As we have already seen above, no threshold sequence can be written as a convex combination of other degree sequences. Thus the set of extreme points of $\mathsf{TP(n)}$ is precisely $TP(n)$. At this point we have the inclusions (the second of these follows from Theorem 1.1)

$$\mathsf{TP(n)} \subseteq \mathsf{DP(n)} \subseteq \mathsf{F(n)}.$$

In Section 4 we prove that $\mathsf{TP(n)} = \mathsf{F(n)}$, thereby proving Theorem 1.3.

This paper is organized as follows. In Section 2 we recall two characterizations of threshold graphs. In Section 3 we give a simple polynomial time dynamic programming algorithm for optimizing linear functions over $\mathsf{S(n)}$. We do not use this algorithm in the rest of the paper. Its main purpose is to point out that, in contrast to general order polytopes, optimizing linear functions over $\mathsf{S(n)}$ does not require linear programming or network flows. Since $\mathsf{TP(n)}$ is a linear image of $\mathsf{S(n)}$ this also gives a polynomial time algorithm for optimizing linear functions over $\mathsf{TP(n)}$. In Section 4 we first introduce an averaging operation on real sequences based on descent sets and then use this operation to give another algorithm for optimizing linear functions over $\mathsf{TP(n)}$. We then show that this algorithm also optimizes linear functions over $\mathsf{F(n)}$, thus showing that $\mathsf{TP(n)} = \mathsf{DP(n)} = \mathsf{F(n)}$. In Section 5 we determine the facets of $\mathsf{DP(n)}$ and give an adjacency criterion for the extreme points of $\mathsf{DP(n)}$. As a consequence, we obtain the following.

**Theorem 1.4** *For $n \geq 4$, $\mathsf{DP(n)}$ has $2^{n-1}$ vertices, $2^{n-2}(2n-3)$ edges, and $(n^2-3n+12)/2$ facets.*

It would be interesting to determine all the face numbers of $\mathsf{DP(n)}$. In particular, in analogy with the face numbers of the hypercube, we can ask whether the number of dimension $k$ faces of $\mathsf{DP(n)}$, for $k = 0, 1, \ldots, n-1$, is of the form $P_k(n)2^{n-1-k}$, where $P_k(n)$ is a polynomial in $n$.

## 2 Threshold graphs

In this short section we recall two characterizations of threshold graphs. The proofs are straightforward and can be found in **[CH, MP]**. Let $i \leq j$. A graph $T = (\{i, \ldots, j\}, E)$ on the vertex set $\{i, \ldots, j\}$ is said to be a *proper threshold graph* if $T$ is threshold and $d_i \geq d_{i+1} \geq \cdots \geq d_j$, where $d_\ell$ is the degree of vertex $\ell$.

**Theorem 2.1** *Let $T = ([n], E)$ be a simple graph on the vertex set $[n]$. The following are equivalent:*
*(i) $T$ is a proper threshold graph.*
*(ii) $E$ is an order ideal in $S(n)$.*
*(iii) There exist real numbers $b_1 \geq b_2 \geq \cdots \geq b_n$ such that $(i,j) \in E$ if and only if $b_i + b_j \geq 0$.*$\square$

Consider the set $TP(n)$ of degree partitions of $n$-vertex threshold graphs. Partially order $TP(n)$ by componentwise $\leq$, i.e., $(d_1, \ldots, d_n) \leq (e_1, \ldots, e_n)$ iff $d_i \leq e_i$ for all $i$. Let $\mathcal{O}(S(n))$ denote the poset (actually, a lattice) of all order ideals of $S(n)$ under containment.

**Theorem 2.2** *(i) $TP(n)$ is a lattice with join and meet given by componentwise maximum and minimum, i.e., for $d = (d_1, \ldots, d_n),\ e = (e_1, \ldots, e_n) \in TP(n)$*

$$d \vee e = (max(d_1, e_1), \ldots, max(d_n, e_n)), \quad d \wedge e = (min(d_1, e_1), \ldots, min(d_n, e_n)).$$

*(ii) The map $\mathcal{D} : \mathcal{O}(S(n)) \to TP(n)$ given by*

$$\mathcal{D}(E) = \text{ degree sequence of } ([n], E)$$

*is a lattice isomorphism.* $\square$

# 3 Optimizing linear functions over $\mathsf{S(n)}$

In this section we give a simple dynamic programming algorithm for optimizing linear functions over $\mathsf{S(n)}$.

Given real weights $c = (c_{i,j} : (i,j) \in S(n))$ consider the linear program

$$\begin{aligned} &\text{maximize} \sum_{(i,j)\in S(n)} c_{i,j} x_{i,j} \\ &\text{subject to } (x_{i,j} : (i,j) \in S(n)) \in \mathsf{S(n)}. \end{aligned} \tag{6}$$

We noted in the introduction that the extreme points of $\mathsf{S(n)}$ are characteristic vectors of order ideals in $S(n)$ and thus we can solve (6) by solving the following combinatorial optimization problem (where $c(I) = \sum_{(i,j)\in I} c_{i,j}$ denotes the *weight* of the order ideal $I$)

$$\begin{aligned} &\text{maximize } c(I) \\ &\text{subject to } I \in \mathcal{O}(S(n)). \end{aligned} \tag{7}$$

**Lemma 3.1** *Given real weights $c = (c_{i,j} : (i,j) \in S(n))$, the set of maximum weight order ideals is closed under union and intersection. Thus, among the maximum weight order ideals, there is a unique maximal and a unique minimal element (under containment).*

*Proof* Let $I$ and $J$ be maximum weight order ideals. Then $c(I \cup J) \leq c(I)$, $c(I \cap J) \leq c(J)$ and $c(I) + c(J) = c(I \cup J) + c(I \cap J)$. The result follows. $\square$

**Lemma 3.2** *Let $I, J \subseteq S(n)$ be order ideals such that $\chi(I)$ and $\chi(J)$ (the characteristic vectors of $I$ and $J$) are adjacent vertices of $\mathsf{S(n)}$. Then $I \subseteq J$ or $J \subseteq I$.*

*Proof* There is a cost vector $c$ such that $I$ and $J$ are the only maximum weight ideals w.r.t $c$. The result now follows from Lemma 3.1. $\Box$

We now give a dynamic programming algorithm to find maximum weight ideals in $S(n)$. Let $c = (c_{i,j} : (i,j) \in S(n))$ be a cost vector. By Theorem 2.1(ii) order ideals in $S(n)$ are precisely edge sets of proper threshold graphs on $[n]$ and thus finding a maximum weight order ideal is equivalent to finding a proper threshold graph $T = ([n], E)$ with $c(E)$ maximum. In the algorithm below, for $i \leq j$, $(\{i, \ldots, j\}, E_{i,j})$ will be the unique edge maximal proper threshold graph on the vertices $\{i, \ldots, j\}$ with maximum weight.

**Algorithm 1**
*Input*: $c = (c_{i,j} : (i,j) \in S(n))$.
*Output*: The unique edge maximal proper threshold graph on $[n]$ with maximum weight.
*Method*:
1. **for** $i$ from 1 to $n$ **do** $E_{i,i} \leftarrow \emptyset$
2. **for** $i$ from $n-1$ downto 1 **do**
3. **for** $j$ from $i+1$ to $n$ **do**
4. **if** $(c_{i,i+1} + c_{i,i+2} + \cdots + c_{i,j} + c(E_{i+1,j})) \geq c(E_{i,j-1})$
5. **then** $E_{i,j} \leftarrow \{(i,i+1), (i,i+2), \ldots, (i,j)\} \cup E_{i+1,j}$
6. **else** $E_{i,j} \leftarrow E_{i,j-1}$
7. **Output** $([n], E_{1,n})$

**Lemma 3.3** *Algorithm 1 is correct, i.e., for all $i < j$, $(\{i, \ldots, j\}, E_{i,j})$ is the unique edge maximal proper threshold graph on the vertices $\{i, \ldots, j\}$ with maximum weight w.r.t. $c$.*

*Proof* By induction on $j - i$. The statement is clear for $j = i$. For $i < j$ consider the unique maximum weight edge maximal proper threshold graph $T$ on the vertices $\{i, \ldots, j\}$ with edge set, say, $E$. Then either $i$ is dominating in $T$ or $j$ is isolated in $T$. If $i$ is dominating then, by induction, we have $E = \{(i,i+1), \ldots, (i,j)\} \cup E_{i+1,j}$. If $j$ is isolated then, by induction, we get $E = E_{i,j-1}$. It is easy to see that $i$ is dominating in $T$ if and only if $(c_{i,i+1} + c_{i,i+2} + \cdots + c_{i,j} + c(E_{i+1,j})) \geq c(E_{i,j-1})$. That completes the proof. $\Box$

Given real numbers $c_i, i \in [n]$ consider the following linear program

$$\begin{aligned} &\text{maximize } \sum_{i \in [n]} c_i x_i \\ &\text{subject to } (x_i : i \in [n]) \in \mathsf{TP(n)}. \end{aligned} \tag{8}$$

We noted in the introduction that the extreme points of $\mathsf{TP(n)}$ are the threshold partitions in $TP(n)$ and thus we can solve (8) by solving the following combinatorial optimization problem

$$\begin{aligned} &\text{maximize } \sum_{i \in [n]} c_i d_i \\ &\text{subject to } (d_1, \ldots, d_n) \in TP(n). \end{aligned} \tag{9}$$

Consider problem (9). Define weights $c = (c_{i,j} : (i,j) \in S(n))$ by $c_{i,j} = c_i + c_j$. Recall the poset isomorphism $\mathcal{D} : \mathcal{O}(S(n)) \to TP(n)$ and observe that, for $d = (d_1, d_2, \ldots, d_n) \in TP(n)$, we have

$$\sum_{i \in [n]} c_i d_i \;=\; \sum_{(i,j) \in \mathcal{D}^{-1}(d)} c_{i,j} \;=\; c(\mathcal{D}^{-1}(d)).$$

Thus solving (9) is a special case of solving (7). From Lemmas 3.1, 3.2 and the poset isomorphism $\mathcal{D}$ we now have

**Lemma 3.4** *(i) Given real weights $(c_i : i \in [n])$, the set of optimal threshold sequences in (9) is closed under $\vee$ and $\wedge$. Thus, among the optimal threshold sequences, there is a unique maximal and a unique minimal element.*
*(ii) Let $d, e \in TP(n)$ be adjacent vertices of* $\mathsf{TP(n)}$. *Then $d$ and $e$ are comparable in the partial order on $TP(n)$.* $\square$

In Section 5 we shall characterize comparable pairs $d, e \in TP(n)$ that are adjacent vertices of $\mathsf{TP(n)}$.

# 4 Repeated averaging over ascending runs

In this section we prove Theorem 1.3. We begin by defining an averaging operation on real sequences.

Let $c = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$. We define its *descent set*, denoted $Des(c)$, by

$$Des(c) = \{i \in [n-1] \mid c_i > c_{i+1}\}.$$

For instance, if $c = (\mathbf{1}, \mathbf{3}, \mathbf{2}, \mathbf{7}, \mathbf{2}, \mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{5})$ then $Des(c) = \{2, 4, 6\}$. Write the descent set of $c$ as $\{i_1, i_2, \ldots, i_k\}$, where $i_1 < i_2 < \cdots < i_k$. The subsequences

$$c_1, c_2, \ldots, c_{i_1} \;;\; c_{i_1+1} \cdots c_{i_2} \;;\; \ldots \;;\; c_{i_k+1} \cdots c_n$$

are called the *ascending runs* of $c$. In the example above the ascending runs are

$$\mathbf{1}, \mathbf{3}; \mathbf{2}, \mathbf{7}; \mathbf{2}, \mathbf{3}; \mathbf{1}, \mathbf{1}, \mathbf{5}.$$

Given a real vector $c = (c_1, \ldots, c_n)$ define $\mathcal{A}(c) \in \mathbb{R}^n$ as follows: replace each $c_i$ by the average of the elements of the (unique) ascending run of $c$ in which $c_i$ appears. For the example from the preceding paragraph we have

$$\mathcal{A}(c) = \left(\mathbf{2}, \mathbf{2}, \frac{\mathbf{9}}{\mathbf{2}}, \frac{\mathbf{9}}{\mathbf{2}}, \frac{\mathbf{5}}{\mathbf{2}}, \frac{\mathbf{5}}{\mathbf{2}}, \frac{\mathbf{7}}{\mathbf{3}}, \frac{\mathbf{7}}{\mathbf{3}}, \frac{\mathbf{7}}{\mathbf{3}}\right),$$

and

$$\mathcal{A}(\mathcal{A}(c)) = \left(\frac{\mathbf{13}}{\mathbf{4}}, \frac{\mathbf{13}}{\mathbf{4}}, \frac{\mathbf{13}}{\mathbf{4}}, \frac{\mathbf{13}}{\mathbf{4}}, \frac{\mathbf{5}}{\mathbf{2}}, \frac{\mathbf{5}}{\mathbf{2}}, \frac{\mathbf{7}}{\mathbf{3}}, \frac{\mathbf{7}}{\mathbf{3}}, \frac{\mathbf{7}}{\mathbf{3}}\right).$$

Set $\mathbb{R}^n_{\geq} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \cdots \geq x_n\}$.

**Lemma 4.1** *(i) $\mathcal{A}(c) = c$ if and only if $c \in \mathbb{R}^n_\geq$.*
*(ii) Given $c \in \mathbb{R}^n$, there exists $0 \leq t \leq n-1$ such that $\mathcal{A}^t(c) \in \mathbb{R}^n_\geq$.*

*Proof* (i) This is clear.
(ii) Clearly, $Des(\mathcal{A}(c)) \subseteq Des(c)$. If $Des(\mathcal{A}(c)) = Des(c)$ then $\mathcal{A}(c) \in \mathbb{R}^n_\geq$. Thus each application of the operation $\mathcal{A}$ either strictly decreases the descent set or else the process terminates. The result follows. □

Define $\mathcal{P} : \mathbb{R}^n \to \mathbb{R}^n_\geq$ by $\mathcal{P}(c) = \mathcal{A}^{n-1}(c)$. Alladi Subramanyam has pointed out to us that the function $\mathcal{P}$ arises in the simply ordered case of isotonic regression studied in order restricted statistical inference (see Chapter 1 of [**RWD**]), where the following geometric interpretation is given: for $c \in \mathbb{R}^n$, $\mathcal{P}(c)$ is the unique closest point (under Euclidean distance) to $c$ in the closed, convex set $\mathbb{R}^n_\geq$.

The next two lemmas use the function $\mathcal{P}$ to reduce the problem of maximizing linear functions over $\mathsf{TP}(\mathsf{n})$ to that of maximizing linear functions over $\mathsf{DS}(\mathsf{n})$, where a simple greedy method works.

**Lemma 4.2** *Consider the combinatorial optimization problem (9) with cost vector $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$.*
*(i) Suppose that $c_i \leq c_{i+1}$ for some $i \leq n-1$. Let $d^* = (d_1^*, d_2^*, \ldots, d_n^*)$ be the unique maximal optimal solution to (9). Then $d_i^* = d_{i+1}^*$.*
*(ii) Suppose that $c_i \leq c_{i+1}$ for some $i \leq n-1$. Let $d^* = (d_1^*, d_2^*, \ldots, d_n^*)$ be the unique minimal optimal solution to (9). Then $d_i^* = d_{i+1}^*$.*
*(iii) Suppose that $c_i < c_{i+1}$ for some $i \leq n-1$. Let $d^* = (d_1^*, d_2^*, \ldots, d_n^*)$ be any optimal solution to (9). Then $d_i^* = d_{i+1}^*$.*

*Proof* We prove part (i). The proofs for parts (ii) and (iii) are similar.

The proof is by induction on $n$, the case $n = 2$ being clear. Let $n \geq 3$ and consider the following three cases:
(a) $2 \leq i < i+1 \leq n-1$: Either $d_1^* = n-1$ or $d_n^* = 0$. In the first case $(d_2^* - 1, \ldots, d_n^* - 1)$ is the unique maximal optimal solution to (9) with cost vector $(c_2, \ldots, c_n)$ and in the second case $(d_1^*, \ldots, d_{n-1}^*)$ is the unique maximal optimal solution to (9) with cost vector $(c_1, \ldots, c_{n-1})$. By induction we now see that $d_i^* = d_{i+1}^*$.
(b) $i = 1, i+1 = 2$: Let $T = ([n], E)$ be the proper threshold graph with degree sequence $d^*$, i.e., $E = \mathcal{D}^{-1}(d^*)$ and assume that $d_1^* > d_2^*$. Since $E$ is an order ideal of $S(n)$ we see that, for some $2 \leq j < l$, the vertices adjacent to 1 are $\{2, 3, \ldots, l\}$ and the vertices adjacent to 2 are $\{1, 2, \ldots, j\} - \{2\}$. Let $E' = \{(1, k) \mid j < k \leq l\}$ and $E'' = \{(2, k) \mid j < k \leq l\}$. Note that $T' = ([n], E - E')$ is a proper threshold graph and thus, since $d^*$ is an optimal solution to (9), it follows that $c(E') \geq 0$ (for a subset $X \subseteq S(n)$ we set $c(X) = \sum_{(i,j) \in X} c_i + c_j$). Since $c(E'') - c(E') = (l - j)(c_2 - c_1) \geq 0$ we have $c(E'') \geq 0$ and thus, since $T'' = ([n], E \cup E'')$ is a proper threshold graph, the degree sequence of $T''$ is also an optimal solution to (9), contradicting the maximality of $d^*$. So $d_1^* = d_2^*$.
(c) $i = n-1, i+1 = n$: Similar to case (b).□

**Lemma 4.3** *Let $c \in \mathbb{R}^n$. Consider two instances of the combinatorial optimization problem (9), one with cost vector $c$ and another with cost vector $\mathcal{P}(c)$. Then*

*(i) The unique maximal optimal solutions to these two instances are equal.*
*(ii) The unique minimal optimal solutions to these two instances are equal.*

*Proof* We prove part (i). The proof for part (ii) is similar. We show that the unique maximal optimal solutions to (9) with cost vectors $c$ and $\mathcal{A}(c)$ are the same. This will prove part (i).

Let $d^* = (d_1^*, \ldots, d_n^*)$ be the unique maximal optimal solution to (9) with cost vector $c$ and let $e^* = (e_1^*, \ldots, e_n^*)$ be the unique maximal optimal solution to (9) with cost vector $\mathcal{A}(c)$.

Let $c = (c_1, \ldots, c_n)$ and $\mathcal{A}(c) = (b_1, \ldots, b_n)$. Write $Des(c) = \{i_1, i_2, \ldots, i_k\}$, where $i_1 < i_2 < \cdots < i_k$. Put $i_0 = 0, i_{k+1} = n$ and, for $\ell = 1, 2, \ldots, k+1$, set

$$B_\ell = \{i_{\ell-1} + 1, i_{\ell-1} + 2, \ldots, i_\ell\}, \tag{10}$$

i.e., $B_\ell$ is the set of indices of the $\ell$th ascending run of $c$.

By Lemma 4.2 and the definition of the map $\mathcal{A}$ we have

$$d_i^* = d_j^* \text{ and } e_i^* = e_j^* \text{ whenever } i, j \in B_\ell, \text{ for some } \ell.$$

We now have, using the definition of the map $\mathcal{A}$,

$$\begin{aligned} c_1 d_1^* + c_2 d_2^* + \cdots + c_n d_n^* &= \sum_{\ell=1}^{k+1} \left\{ \sum_{s \in B_\ell} c_s \right\} d_{i_\ell}^* \\ &= \sum_{\ell=1}^{k+1} \left\{ \sum_{s \in B_\ell} b_s \right\} d_{i_\ell}^* \\ &= b_1 d_1^* + b_2 d_2^* + \cdots + b_n d_n^*. \end{aligned}$$

Similarly we can show

$$c_1 e_1^* + c_2 e_2^* + \cdots + c_n e_n^* = b_1 e_1^* + b_2 e_2^* + \cdots + b_n e_n^*.$$

Now we use the fact that $d^*$ is optimal for the cost vector $c$ and $e^*$ is optimal for the cost vector $\mathcal{A}(c)$. We have

$$\sum_{i=1}^n c_i d_i^* \geq \sum_{i=1}^n c_i e_i^* = \sum_{i=1}^n b_i e_i^* \geq \sum_{i=1}^n b_i d_i^* = \sum_{i=1}^n c_i d_i^*.$$

It follows that $\sum_{i=1}^n c_i d_i^* = \sum_{i=1}^n c_i e_i^*$ and $\sum_{i=1}^n b_i d_i^* = \sum_{i=1}^n b_i e_i^*$.

Since $d^*$ is the unique maximal solution to (9) with cost vector $c$ we have $e^* \leq d^*$ and since $e^*$ is the unique maximal solution to (9) with cost vector $\mathcal{A}(c)$ we have $d^* \leq e^*$. Thus $d^* = e^*$. $\square$

We can now give our second algorithm to solve the optimization problem (9).

**Algorithm 2**
*Input*: $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$.

*Output*: The unique maximal $d^* = (d_1^*, \dots, d_n^*) \in TP(n)$ maximizing $\sum_{i=1}^n c_i d_i$ over all $(d_1, \dots, d_n) \in TP(n)$.
*Method*:
1. $(b_1, \dots, b_n) \leftarrow \mathcal{P}(c)$
2. $E \leftarrow \emptyset$
3. **for all** $((i,j) \in S(n)$ with $b_i + b_j \geq 0)$ **do** $E \leftarrow E \cup \{(i,j)\}$
4. $(d_1^*, \dots, d_n^*) \leftarrow$ degree sequence of $([n], E)$

**Lemma 4.4** *Algorithm 2 is correct.*

*Proof* Consider the problem of maximizing $b_1 d_1 + \cdots + b_n d_n$ over all degree sequences $(d_1, \dots, d_n) \in DS(n)$. An edge $(i,j)$ contributes $b_i + b_j$ to the objective function and hence it follows that among all simple graphs on $[n]$ whose degree sequences maximize $\sum_{i=1}^n b_i d_i$, the graph $([n], E)$ computed in Step 3 of Algorithm 2 is the unique maximal one (under containment of edges). Since $b_1 \geq b_2 \geq \cdots \geq b_n$, it follows from Theorem 2.1(iii) that $d^*$ is a threshold partition. Using Lemma 4.3 we now see that $d^*$ is the unique maximal optimal solution to (9) with cost vector $c$. $\square$
**Remarks** (i) To get the unique minimal optimal solution in Algorithm 2 we change line 3 to the following:
3. **for all** $((i,j) \in S(n)$ with $b_i + b_j > 0)$ **do** $E \leftarrow E \cup \{(i,j)\}$
(ii) As observed in the introduction it is easy to see that every element of $TS(n)$ (respectively, $TP(n)$) is an extreme point of $\mathsf{DS(n)}$ (respectively, $\mathsf{DP(n)}$). Algorithm 2 shows that the converse statement for $\mathsf{DS(n)}$, namely, that every extreme point of $\mathsf{DS(n)}$ is in $TS(n)$ also has a short direct proof not involving the polytope $\mathsf{K(n)}$. In contrast, our proof that every extreme point of $\mathsf{DP(n)}$ is in $TP(n)$ is indirect and uses the polytope $\mathsf{F(n)}$.

We can now give the proof of our main theorem.
*Proof* (of Theorem 1.3) Let $c = (c_1, \dots, c_n) \in \mathbb{R}^n$ and let $d^* = (d_1^*, \dots, d_n^*)$ be the output of Algorithm 2. We show that $d^*$ maximizes $\sum_{i=1}^n c_i x_i$ over $\mathsf{F(n)}$. This will prove that $\mathsf{TP(n)} = \mathsf{F(n)}$ and since, as already observed in the introduction, the set of extreme points of $\mathsf{TP(n)}$ is $TP(n)$, this also proves Theorem 1.3.

Write the constraints of $\mathsf{F(n)}$ in the form

$$-x_i + x_{i+1} \leq 0, \quad 1 \leq i \leq n-1. \tag{11}$$

$$\sum_{i=1}^{k} x_i \; - \sum_{i=n-l+1}^{n} x_i \leq k(n-1-l), \quad 1 \leq k+l \leq n. \tag{12}$$

We shall show that the row vector $c$ can be written as a nonnegative rational combination of the row vectors of lhs coefficients of those constraints from (11) and (12) that are satisfied with equality by $d^*$. By the (weak) duality theorem of linear programming, this will prove the result.

Write $Des(c) = \{i_1, i_2, \dots, i_k\}$, where $i_1 < i_2 < \cdots < i_k$. For $\ell = 1, \dots, k+1$, define $B_\ell$ to be the set of indices of the $\ell$th ascending run of $c$, as in (10). Let $\mathcal{P}(c) = (b_1, \dots, b_n)$. Since the ascending runs of $\mathcal{A}^j(c)$ are contained in the ascending runs of $\mathcal{A}^{j+1}(c)$ it follows from the definition of $\mathcal{P}(c)$ that $b_i = b_{i+1}$ whenever $i, i+1 \in B_\ell$, for some $\ell$. It now follows from Algorithm 2 that the inequality $-x_i + x_{i+1} \leq 0$ is satisfied with equality by $d^*$ whenever $i, i+1 \in B_\ell$, for some $\ell$.

For $1 \leq i \leq n-1$ define $v_i = (0, \ldots, 0, -1, 1, 0, \ldots, 0)$, with $-1$ in the $i$ th spot and 1 in the $(i+1)$ st spot, i.e., $v_i$ is the row vector of lhs coefficients of the $i$ th inequality from (11).

For $\ell = 1, \ldots, k+1$, let $m_\ell = \min B_\ell = i_{\ell-1}+1$, $M_\ell = \max B_\ell = i_\ell$ and let $a_\ell = \frac{\sum_{i \in B_\ell} c_i}{\# B_\ell}$ be the average of the elements of the $\ell$ th ascending run of $c$. We assert that

$$c = \mathcal{A}(c) + \sum_{\ell=1}^{k+1} \sum_{i=m_\ell}^{M_\ell - 1} \{(i+1-m_\ell)a_\ell - (c_{m_\ell} + \cdots + c_i)\} \, v_i. \tag{13}$$

Before proving the assertion we note that the coefficient of $v_i$ in (13) is nonnegative since $a_\ell$ is the average of the $\ell$ th ascending run and the entries in this run are weakly increasing. Also note that for every $v_i$ that appears in (13) the corresponding inequality in (11) is satisfied with equality by $d^*$.

Let $1 \leq \ell \leq k+1$. We now show that, for $i \in B_\ell$, the $i$ th coordinate on the rhs of (13) is $c_i$. This will prove (13). The $m_\ell$ th coordinate on the rhs of (13) is $a_\ell - (a_\ell - c_{m_\ell}) = c_{m_\ell}$ and the $M_\ell$ th coordinate on the rhs of (13) is $a_\ell + ((M_\ell - m_\ell)a_\ell - (c_{m_\ell} + \cdots + c_{M_\ell - 1})) = c_{M_\ell}$. For $m_\ell < i < M_\ell$, the $i$ th coordinate on the rhs of (13) is $a_\ell + ((i - m_\ell)a_\ell - (c_{m_\ell} + \cdots + c_{i-1})) - ((i+1-m_\ell)a_\ell - (c_{m_\ell} + \cdots + c_i)) = c_i$.

Repeating the above procedure for $\mathcal{A}(c), \mathcal{A}^2(c), \ldots$ we eventually arrive at an expression of the form

$$c = \mathcal{P}(c) + \sum \alpha_i v_i,$$

where the sum is over all $1 \leq i \leq n-1$ such that $d_i^* = d_{i+1}^*$, and $\alpha_i$ is a nonnegative rational for all $i$.

We observed in the proof of Lemma 4.4 that $d^*$ is an optimal degree sequence for the cost vector $\mathcal{P}(c)$. Therefore, by Theorem 1.2 and the (strong) duality theorem of linear programming, $\mathcal{P}(c)$ can be written as a nonnegative rational combination of the row vectors of lhs coefficients of those inequalities from (1) that are satisfied with equality by $d^*$. In fact, the proof of Theorem 4.2 from [**PS**] (also see the proof of Theorem 3.3.14 from [**MP**]) shows how to write $\mathcal{P}(c)$ as a nonnegative rational combination of the row vectors of lhs coefficients of those inequalities from (12) that are satisfied with equality by $d^*$.

That completes the proof of Theorem 1.3. □

# 5 Facets and edges of $\mathsf{DP(n)}$

In this section we determine the facets and edges of the polytope $\mathsf{DP(n)}$. Most of the steps needed to determine the facets of $\mathsf{DP(n)}$ are similar to the case of $\mathsf{DS(n)}$ and therefore we just quote the corresponding results from [**MP**].

In order to identify the facets of $\mathsf{DP(n)}$ we need to know its dimension and the dimension of a related polytope. For $m, n \geq 1$, let $K(m, n)$ denote the complete bipartite graph with bipartition $\{1, \ldots, m\}$ and $\{m+1, \ldots, m+n\}$. For a spanning subgraph of $K(m, n)$ we define its *degree bipartition* as the sequence obtained by rearranging the first $m$ and last $n$ components of its degree sequence $(d_1, \ldots, d_m, d_{m+1}, \ldots, d_{m+n})$ into weakly decreasing order. Define $\mathsf{DS(m,n)}$ (respectively, $\mathsf{DP(m,n)}$) to be the convex hull of degree sequences (respectively, degree bipartitions) of spanning subgraphs of $K(m, n)$.

**Lemma 5.1** *(i) dim* $\mathsf{DP(1)} = 0$*, dim* $\mathsf{DP(2)} = 1$*.*
*(ii) dim* $\mathsf{DP(m,n)} = m + n - 1$*, for* $m, n \geq 1$*.*
*(iii) dim* $\mathsf{DP(n)} = n$*, for* $n \geq 3$*.*

*Proof* (i) Clear.
(ii) and (iii) This follows from Lemma 3.3.16 in [**MP**] which proves dim $\mathsf{DS(m,n)} = m + n - 1$, for $m, n \geq 1$ and dim $\mathsf{DS(n)} = n$, for $n \geq 3$. A direct proof can also be given. □

**Theorem 5.2** *For* $n \geq 4$ *the facets of* $\mathsf{DP(n)}$ *are given by*
*(i)* $x_i \geq x_{i+1}, \;\; i = 1, \ldots, n-1.$
*(ii)* $\displaystyle\sum_{i=1}^{k} x_i \; - \sum_{i=n-l+1}^{n} x_i \leq k(n-1-l),$
*where* $k = 1, l = 0$ *or* $k = 0, l = 1$ *or* $k, l \neq 0, \;\; k + l = 2, 3, \ldots, n-3, n.$
*It follows that, for* $n \geq 4$*,* $\mathsf{DP(n)}$ *has* $\dfrac{n(n-3)}{2} + 6$ *facets.*

*Proof* By Theorem 1.3 and the full dimensionality of $\mathsf{DP(n)}$ it follows that every facet of $\mathsf{DP(n)}$ is of the form $x_i \geq x_{i+1}$, for some $i = 1, \ldots, n-1$ or of the form (3), for some $1 \leq k + l \leq n$.

We first show that $x_i \geq x_{i+1}, \; i = 1, \ldots, n-1$ determines a facet for $n \geq 3$. Since $(0, \ldots, 0)$ satisfies $x_1 = \cdots = x_n$ it is enough to produce $n-1$ linearly independent elements of $TP(n)$ satisfying $x_i = x_{i+1}$, for $i = 1, \ldots, n-1$. This we do by induction on $n$. For $n = 3$, $\{(1,1,0), (2,2,2)\}$ and $\{(2,1,1), (2,2,2)\}$ are sets of 2 linearly independent elements in $TP(3)$ satisfying $x_1 = x_2$ and $x_2 = x_3$ respectively.

Let $n \geq 4$. We consider two cases.

(a) $1 \leq i \leq n-2$: By induction, there is a set $S$ of $n-2$ linearly independent elements of $TP(n-1)$ satisfying $x_i = x_{i+1}$. Let $S' \subseteq TP(n)$ denote the set obtained from $S$ by adding a zero at the end of every element of $S$. Then it is easily checked that $S' \cup \{(n-1, \ldots, n-1)\}$ is a set of $n-1$ linearly independent elements of $TP(n)$ satisfying $x_i = x_{i+1}$.

(b) $i = n-1$: By induction, there is a set $S$ of $n-2$ linearly independent elements of $TP(n-1)$ satisfying $x_{n-2} = x_{n-1}$. Let $S''$ denote the set obtained from $S$ by adding a zero at the beginning of every element of $S$ and let $\Delta = (n-1, 1, \ldots, 1) \in TP(n)$. Then it is easily checked that $\{\Delta + u \; : \; u \in S''\} \cup \{\Delta\}$ is a set of $n-1$ linearly independent elements of $TP(n)$ satisfying $x_{n-1} = x_n$.

Now we have to show that if $n \geq 4$ and (3) is a facet of $\mathsf{DP(n)}$ then $k, l$ satisfy the conditions listed under (ii) of the statement. This can be done using Lemma 5.1 in the same way as Theorem 3.3.17 (determining the facets of $\mathsf{DS(n)}$) is deduced from Lemma 3.3.16 in [**MP**] and we omit the proof.

For $n \geq 4$, the number of pairs $(k, l)$ of positive integers with $k + l = p$ $(2 \leq p \leq n)$ is $p - 1$ and thus the number of facets of $\mathsf{DP(n)}$ is given by

$$n - 1 + 2 + \left( \sum_{p=2}^{n} (p-1) \right) - (n-3) - (n-2) = \frac{n(n-3)}{2} + 6. \quad \square$$

By an *interval* of $[n]$ we mean a nonempty subset $I \subseteq [n]$ of the form $I = \{i, i+1, \ldots, j\}$, for some $i \leq j$. An interval $I$ is *nontrivial* if $\# I \geq 2$. Given a nontrivial interval $I \subseteq [n]$, define the vector $\Omega_I = (a_1, \ldots, a_n)$ by $a_i = 0$ if $i \notin I$, and $a_i = \# I - 1$ if $i \in I$.

Consider a disjoint pair of intervals of $[n]$. Write this disjoint pair as $(I, J)$, where $\max I < \min J$. Define the vector $\Omega_{I,J} = (a_1, \ldots, a_n)$ by $a_i = 0$ if $i \notin I \cup J$, $a_i = \# J$ if $i \in I$, and $a_i = \# I$, if $i \in J$.

**Theorem 5.3** *Let $n \geq 3$ and let $d, e \in TP(n)$. Then $d$ and $e$ are adjacent extreme points of $\mathsf{DP(n)}$ if and only if $d$ and $e$ are comparable (in the partial order on threshold partitions) and*

$$|d - e| = \Omega_I \text{ or } \Omega_{I,J} \ ,$$

*for some nontrivial interval $I$ or some disjoint pair of intervals $(I, J)$.*

*Proof* Let $d = (d_1, \ldots, d_n)$ and $e = (e_1, \ldots, e_n)$.

(if) Assume $e \leq d$. The proof is by induction on $n$. The case $n = 3$ is easily verified (there are four threshold partitions of length 3 and any two of them are adjacent). Let $n \geq 4$ and consider the following three cases.

(a) $d_n = e_n = 0$: By induction, $(d_1, \ldots, d_{n-1})$ and $(e_1, \ldots, e_{n-1})$ are adjacent extreme points of $\mathsf{DP(n-1)}$ ($n$ cannot be a member of $I$ or $J$). The facet of $\mathsf{DP(n)}$ given by $x_n \geq 0$ is isomorphic to $\mathsf{DP(n-1)}$ and since $d$ and $e$ lie on this facet it follows that they are adjacent.

(b) $d_1 = e_1 = n-1$: By induction $(d_2 - 1, \ldots, d_n - 1)$ and $(e_2 - 1, \ldots, e_n - 1)$ are adjacent extreme points of $\mathsf{DP(n-1)}$. The facet of $\mathsf{DP(n)}$ given by $x_1 \leq n - 1$ is isomorphic to $\mathsf{DP(n-1)}$ and since $d$ and $e$ lie on this facet it follows that they are adjacent.

(c) $d_1 = n-1$ and $e_n = 0$: In this case $d - e$ will have nonzero first and last components. We consider two subcases.

(i) $d - e = \Omega_{[n]}$: We must have $e = (0, \ldots, 0)$ and $d = (n-1, \ldots, n-1)$. Since $d$ and $e$ both satisfy the $n-1$ linearly independent defining inequalities $x_i \geq x_{i+1}$, $i = 1, \ldots, n-1$ with equality they are adjacent.

(ii) $d - e = \Omega_{I,J}$ for some disjoint intervals $(I, J)$ with $1 \in I$ and $n \in J$: Let $T_d$ and $T_e$ be the proper threshold graphs on the vertex set $[n]$ with degree sequences $d$ and $e$ respectively. Write $d - e = (r, \ldots, r, 0, \ldots, 0, s, \ldots, s)$ where $r = \# J$, $s = \# I$, $r$ occurs $s$ times and $s$ occurs $r$ times. Now $e_1 = d_1 - r = n - 1 - r$ and hence $e_{n-r+1} = \cdots = e_n = 0$. Similarly we can show that $d_1 = \cdots = d_s = n - 1$.

We now have

$$\begin{aligned} d &= (n-1,\ldots,n-1,\ldots,s,\ldots,s), \\ e &= (n-1-r,\ldots,n-1-r,\ldots,0,\ldots,0), \\ d_i &= e_i, \quad i=s+1,\ldots,n-r, \end{aligned}$$

where $n-1$ occurs $s$ times and $s$ occurs $r$ times (in $d$) and $n-1-r$ occurs $s$ times and 0 occurs $r$ times (in $e$).

It follows that

- the vertices $\{1,\ldots,s\}$ induce a clique in both $T_d$ and $T_e$,
- the vertices $\{n-r+1,\ldots,n\}$ induce a stable set in both $T_d$ and $T_e$,
- every vertex of $\{s+1,\ldots,n-r\}$ is connected to every vertex in $\{1,\ldots,s\}$ in both $T_d$ and $T_e$,
- no vertex in $\{s+1,\ldots,n-r\}$ is connected to any vertex in $\{n-r+1,\ldots,n\}$ in both $T_d$ and $T_e$.
- the vertices of $\{1,\ldots,s\}$ are connected to all the vertices of $\{n-r+1,\ldots,n\}$ in $T_d$ and to none of them in $T_e$.

Thus the proper threshold subgraphs of $T_d$ and $T_e$ induced on the vertices $\{s+1,\ldots,n-r\}$ have identical degree sequences and hence, by Theorem 2.2, are identical, say $T$. By Theorem 1.3 there exist real coefficients $(c_{s+1},\ldots,c_{n-r})$ such that $T$ is the unique proper threshold graph on $\{s+1,\ldots,n-r\}$ of maximum weight (i.e., whose degree sequence maximizes $\sum_{i=s+1}^{n-r} c_i x_i$).

Let $M = \max\{|c_i| \ : \ s+1 \le i \le n-r\}$. Choose positive numbers $0 < c_1 < c_2 < \cdots < c_s$ and negative numbers $c_{n-r+1} < \cdots < c_n < 0$ with $c_1 > M$, $|c_n| > M$, and $r(c_1+\cdots+c_s) = -s(c_{n-r+1}+\cdots+c_n)$. Put $c=(c_1,\ldots,c_n)$ and consider problem (9). It is easy to check, using Lemma 4.2(iii) and the uniqueness of $T$, that an optimal threshold partition with last component 0 must be $e$ and an optimal threshold partition with first component $n-1$ must be $d$. Since $\sum_{i=1}^n c_i d_i = \sum_{i=1}^n c_i e_i$ it follows that $d$ and $e$ are the only optimal threshold partitions and thus are adjacent.

(only if) By Theorem 3.4, $e$ and $d$ are comparable, say $e \le d$. There is a real cost vector $c \in \mathbb{R}^n$ such that $d$ and $e$ are the only two optimal solutions to problem (9). Then $e$ must be the unique minimal solution and $d$ must be the unique maximal solution for problem (9) with cost vector $c$ and, by Lemma 4.3, also for cost vector $\mathcal{P}(c)$. Thus Algorithm 2 will produce $d$ as output and Algorithm 2, with step 3 modified as in remark (i) following the algorithm, will produce $e$ as output.

Let $\mathcal{P}(c) = (b_1,b_2,\ldots,b_n)$ and write $Des(\mathcal{P}(c)) = \{i_1,i_2,\ldots,i_k\}$, where $i_1 < i_2 < \cdots < i_k$. For $\ell = 1,\ldots,k+1$, define $B_\ell$ to be the set of indices of the $\ell$th ascending run of $\mathcal{P}(c)$, as in (10). We have

(i) $b_i = b_j$ whenever $i, j \in B_\ell$, for some $\ell$. For $\ell = 1, \ldots, k+1$, define $a_\ell = b_i$, for (any) $i \in B_\ell$. Note that $a_1 > a_2 > \cdots > a_{k+1}$.

(ii) Set $\Sigma_0 = \{\ell \,:\, 1 \le \ell \le k+1,\ a_\ell = 0\}$. Note that $\#\Sigma_0 \le 1$.

(iii) Set $\Sigma_c = \{(i,j) \,:\, 1 \le i < j \le k+1,\ a_i + a_j = 0\}$. Note that the ordered pairs in $\Sigma_c$ are disjoint and incomparable (in the partial order on $S(k+1)$).

Let $E_d$ and $E_e$ be the edge sets of the unique proper threshold graphs on $[n]$ with degree sequences $d$ and $e$ respectively. Then from Algorithm 2 (and the subsequent remark (i)) we see that

$$E_d - E_e = \left( \biguplus_{i \in \Sigma_0} \binom{B_i}{2} \right) \biguplus \left( \biguplus_{(i,j) \in \Sigma_c} B_i \times B_j \right),$$

where $\biguplus$ denotes disjoint union and $\binom{X}{2}$ (for $X$ a set of integers) denotes the set of all ordered pairs $(i,j)$ with $i < j$ and $i, j \in X$.

Now we observe that, for $i \in \Sigma_0$, $E_e \cup \binom{B_i}{2}$ and $E_d - \binom{B_i}{2}$ are both order ideals in $S(n)$ and, for $(i,j) \in \Sigma_c$, $E_e \cup (B_i \times B_j)$ and $E_d - (B_i \times B_j)$ are also both order ideals in $S(n)$. Since $d$ and $e$ are optimal it follows that $c(\binom{B_i}{2}) = 0$, for $i \in \Sigma_0$ and $c(B_i \times B_j) = 0$, for $(i,j) \in \Sigma_c$ (here, for a subset $E$ of edges, $c(E) = \sum_{(i,j) \in E} (c_i + c_j)$). Since $d$ and $e$ are the only optimal solutions it now follows that

$$\delta(\Sigma_0) + \#\Sigma_c = 1,$$

where $\delta(\Sigma_0) = 1$ if $\#\Sigma_0 = 1$ and the unique element $i$ of $\Sigma_0$ satisfies $\#B_i \ge 2$ and $\delta(\Sigma_0) = 0$ otherwise. The result follows. □

In order to determine the higher dimensional faces of $\mathsf{TP(n)}$ we need to know more about the interaction between the lattice structure of $TP(n)$ and the faces of $\mathsf{TP(n)}$. In particular, we would like to know the answer to the following question: given a face of $\mathsf{TP(n)}$ the set of elements of $TP(n)$ lying on this face is closed under $\wedge$ and $\vee$. What are the join irreducible elements on this face?.

**Theorem 5.4** *For $n \ge 3$, the number of edges of $\mathsf{DP(n)}$ is $2^{n-2}(2n-3)$.*

*Proof* Let $TD(n)$ denote the set of all threshold partitions $d = (d_1, \ldots, d_n) \in TP(n)$ satisfying $d_1 = n-1$ (or, equivalently, $d_n \ne 0$). Given $d \in TD(n)$ as above, let $m(d)$ denote the number of dominating vertices in $d$, i.e., $m(d)$ is the largest index $j$ with $d_j = n-1$. We assert that

$$\sum_{d \in TD(n)} m(d) = 2^{n-1}. \tag{14}$$

The proof is by induction on $n$, the cases $n = 1, 2$ being checked easily. Let $n \ge 3$ and let $d = (d_1, \ldots, d_n) \in TD(n)$. Write

$$(d_1, \ldots, d_n) = (n-1, 1, \ldots, 1) + (0, e_1, \ldots, e_{n-1}),$$

where $e = (e_1, \ldots, e_{n-1}) \in TP(n-1)$.

If $e_{n-1} = 0$ then $m(d) = 1$. The number of $e \in TP(n-1)$ with $e_{n-1} = 0$ is $2^{n-3}$. If $e_{n-1} \neq 0$ then $m(d) = m(e) + 1$. The number of $e \in TP(n-1)$ with $e_{n-1} \neq 0$ is also $2^{n-3}$.

It follows by induction that

$$\sum_{d \in TD(n)} m(d) = 2^{n-3} + 2^{n-3} + \sum_{e \in TD(n-1)} m(e) = 2^{n-3} + 2^{n-3} + 2^{n-2} = 2^{n-1}.$$

Let $E_n$ denote the number of edges of $\mathsf{DP(n)}$. We prove the following recurrence

$$E_n = 2E_{n-1} + 2^{n-1}, \ \ n \geq 4,$$

with $E_3 = 6$. The result follows easily from this recurrence by induction.

The polytope $\mathsf{DP(3)}$ has 4 vertices any two of which are adjacent, so $E_3 = 6$. Let $n \geq 4$ and let $d = (d_1, \ldots, d_n)$ and $e = (e_1, \ldots, e_n)$ be adjacent extreme points of $\mathsf{DP(n)}$. We say that $d$ and $e$ are *straight neighbors* if both $d$ and $e$ are dominating (i.e., $d_1 = e_1 = n-1$) or both $d$ and $e$ are isolated (i.e., $d_n = e_n = 0$) and we say that $d$ and $e$ are *cross neighbors* if one of them is dominating and the other isolated. Straight neighbors lie on one of the facets $x_1 \leq n-1$ or $x_n \geq 0$ and thus, by induction, the number of straight neighbors is $2E_{n-1}$. We shall now show that the number of cross neighbors is $2^{n-1}$. This will prove the recurrence.

Let $e = (e_1, \ldots, e_n) \in TP(n)$ with $e_n = 0$. We want to count the number of cross neighbors of $e$. The following two cases arise:

(i) $e = (0, \ldots, 0)$: If $d$ is a cross neighbor of $e$ then $d - e$ will have nonzero first and last components and thus, by Theorem 5.3, $d = \Omega_{[n]}$ or $\Omega_{I,J}$, with $1 \in I$ and $n \in J$. It is easily checked that the only possibility for $(I, J)$ is $I = \{1\}$ and $J = \{2, \ldots, n\}$. Thus $e$ has two cross neighbors.

(ii) $e \neq (0, \ldots, 0)$: Let $e_1 = j$, where $1 \leq j \leq n-2$. Then $e_{j+2} = \cdots = e_n = 0$. Put $e' = (e_1, \ldots, e_{j+1}) \in TD(j+1)$. We assert that $d \in TP(n)$ is a cross neighbor of $e$ if and only if

$$d - e = \Omega_{I, \{j+2, \ldots, n\}},$$

where $I = \{1, \ldots, k\}$ for some $k \leq m(e')$. To see this consider the following two cases.

(a): $d - e = \Omega_I$. Since $d_n > 0 = e_n$, we have $\max I = n$ and since $d_1 = n - 1 > e_1$, we have $\min I = 1$. Hence $\Omega_I = (n-1, \ldots, n-1)$ and $T_d$ is obtained from $T_e$ by adding a clique on $[n]$, but this makes $T_d$ non-simple, so case (a) is impossible.

(b): $d - e = \Omega_{I,J}$.

(if): $T_d$ is obtained from $T_e$ by adding a complete bipartite graph between an initial segment of the dominating vertices of $e'$ and all the isolated vertices of $e$. It is easily seen that $T_d$ is a proper threshold graph.

(only if): as in case (a) we have $\max J = n$ and $\min I = 1$. We have $n - 1 = d_1 = e_1 + \#J = j + \#J$, so $\#J = n - 1 - j$ and $J = \{j+2, \ldots, n\}$. Moreover,

$$\begin{aligned} \#I &= d_{j+2} \\ &= \text{number of times } d_1 \text{ occurs in } (d_1, \ldots, d_{j+1}) \\ &\leq \text{number of times } e_1 \text{ occurs in } (e_1, \ldots, e_{j+1}). \end{aligned}$$

Hence $I = \{1, \ldots, k\}$ for some $k \leq m(e')$.

It follows that $e$ has $m(e')$ cross neighbors.

Using (i), (ii) above and the formula (14) we see that the total number of cross neighbors is equal to

$$2 + \sum_{j=2}^{n-1} 2^{j-1} = 2 + 2(2^{n-2} - 1) = 2^{n-1}. \quad \square$$

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