Tilings of the sphere with right triangles II: The $(1, 3, 2)$, $(0, 2, n)$ subfamily

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Abstract

Sommerville [8] and Davies [2] classified the spherical triangles that can tile the sphere in an edge-to-edge fashion. Relaxing this condition yields other triangles, which tile the sphere but have some tiles intersecting in partial edges. This paper shows that no right triangles in a certain subfamily can tile the sphere, although multilayered tilings are possible.

Keywords: spherical right triangle, monohedral tiling, non-normal, non-edge-to-edge, asymptotically right

1 Introduction

A tiling in which all tiles are congruent is said to be a *monohedral* (or sometimes *homohedral*) tiling; and if two tiles that intersect always do so in a single point or an entire edge, the tiling is called *edge-to-edge* (or sometimes *normal*). In 1923, D.M.Y. Somerville [8] classified the edge-to-edge monohedral tilings of the sphere with triangles, subject to certain restrictions. H.L. Davies obtained a complete classification of edge-to-edge monohedral tilings in 1967 [2], though many details were omitted; these were provided in 2002

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by Ueno and Agaoka [9]. There are, of course, reasons why the edge-to-edge tilings are of special interest; however, non-edge-to-edge tilings do exist. Some of these use tiles that also tile in an edge-to-edge fashion; others use tiles that cannot tile edge-to-edge.

In [3] a complete classification of isosceles spherical triangles that tile the sphere was given. Of course, every isosceles tile yields a right-angled tile by bisection, but there are other right-angled triangles that tile as well. This paper forms a part of a sequence of articles (also including [4, 5, 6, 7]) that classifies those triangles.

For any right triangle, let β be the larger non-right angle and γ the smaller. It is clear that a tile cannot tile the sphere unless it permits some vertex configurations with at least as many β as γ angles. We call such a configuration (or, rather, the triple containing the numbers of the various sorts of angles appearing there) a "β source"; it turns out to be useful to classify triangles in terms of these, and to consider triangles with a common β source together.

This paper deals with one family of right-angled triangles, whose members are shown not to tile the sphere. We present, however, two interesting multiple covers. The family consists of all those triangles for which $90° + 3\beta + 2\gamma = 2\beta + n\gamma = 360°$, where β and γ are the two non-right angles of the triangle, with $\beta > \gamma$. It will be shown below that there are infinitely many such triangles, indexed by n. It may be seen that $\lim_{n\to\infty} \beta_n = 90°$; we call such a family *asymptotically right-angled*.

While most triangles in this family can be shown not to tile by a simple counting argument, two of them have some extra vertex configurations not shared by the others and require many cases to be considered. For that reason, the present paper has been separated from [5], of which it would otherwise be a natural part. The reader is referred to that paper for basic definitions and general results.

2 Basic Results

In this paper, we consider the family of right triangles T_n which have $(1, 3, 2)$ as a β source and $(0, 2, n)$ as a γ source. T_n has

$$\beta_n = \frac{270°n - 720°}{3n-4}, \gamma_n = \frac{540°}{3n-4} \tag{1}$$

We will see that none of these tile the sphere, although T_6 and T_8 yield interesting multiple covers.

We can use the values of these angles to derive the following lemma, which rules out certain relations among edge lengths.

Lemma 1 *We do not have $B = 2C$, $H = 2C$, or $2B = H + C$ for any T_n.*

Proof: Note that γ_n is a strictly decreasing function of n, while β_n is strictly increasing. It therefore follows that B/C is strictly increasing. Numerical calculation shows that when $n < 8$, $B/C < 2$, while for $n = 8$, $B/C = 2.30121\ldots > 2$. The proof that $H \neq 2C$ follows the same pattern. Finally, after verifying that $2B \neq H + C$ for $n < 8$, we observe that for $n \geq 8$ we have $H + C < B + 2C < 2B$. ■

We now find the vertex vectors (a, b, c) of the triangles T_n.

Proposition 1 *The vertex vectors of T_n are precisely $(0, 2, n)$, $(1, 3, 2)$, $(4, 0, 0)$; and, if n is even, $(1, 0, \frac{3n-4}{2})$ and $(2, 1, \frac{n}{2})$; and also, if n is even and ≤ 8, $(0, 5, \frac{8-n}{2})$.*

Proof: This follows the pattern of Proposition 4 of [5] ■

When n is odd, there is no second split, so that (in light of [2] and [4]) the triangle does not tile. Setting $\beta > \gamma$ in (1) we get $n > 14/3$; so we need to consider only T_n for $n = 6, 8, 10, \ldots$.

For $n = 2$ we get a negative value for β. For $n = 4$ we have $\beta = 45°$, $\gamma = 67.5°$ with $\beta < \gamma$. With the angles in the proper order, this is classified as a member of the $(0, 4, 2)$ "quarterlune" family. It has γ source $(0, 0, 8)$, tiles the sphere in an edge-to-edge fashion, and is listed by Davies [2], though not by Sommerville [8].

For $n > 8$ the vertex vector $(0, 5, \frac{8-n}{2})$ does not exist and there is no β source except for $(1, 3, 2)$. This provides only a slight surplus of β angles over γ angles, and it is easy to show that it cannot serve as the sole β source in a tiling, by showing that wherever it appears it is associated with enough nearby γ angles to require a global surplus of γ angles over β angles.

Proposition 2 *No triangle T_n tiles the sphere using the $(1, 3, 2)$ vertex as its only β source.*

Proof: Suppose, on the contrary, that such a tiling exists. At any $(1, 3, 2)$ vertex O the angles have, between them, three medium edges, four short edges, and five hypotenuses. Either there are two or more mismatched pairs, or there is a medium edge mismatched with a hypotenuse.

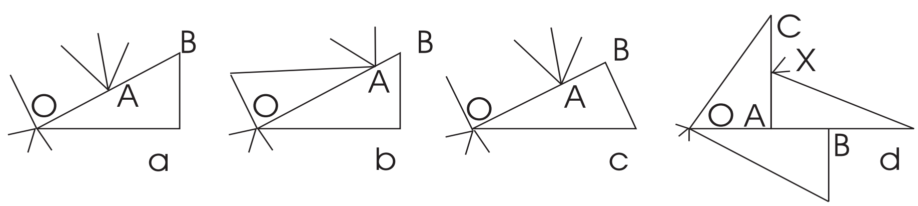

Figure 1: Splits associated with a $(1, 3, 2)$ vertex

We say that a $(0, 2, n)/2$ split vertex X is *associated with* a $(1, 3, 2)$ vertex O if it is connected to O by a single short or medium edge (Fig. $1a - c$), or if it is connected by a short edge to a $(4, 0, 0)/2$ split which is itself connected to O by another short edge (Fig. $1d$). It may be easily verified, using Lemma 1, that no split X is associated with more than 2 $(1, 3, 2)$ vertices.

Suppose $\overline{OA}$ and $\overline{OB}$ are a mismatched pair of edges, belonging to triangles $\triangle OAC$ and $\triangle OBD$. Assume (without loss of generality) that $\overline{OA}$ is shorter than $\overline{OB}$. If the angles $\angle OAC$ is not a right angle, it is a part of a $(0,2,n)/2$ split, associated with O. If both are right angles, another right angle is needed to fill the gap $\angle CAB$, and the β angle of that triangle will be a part of a split vertex X, either on the extended edge $\overleftrightarrow{OA}$ or on an extended edge that terminates at that extended edge, and X will again be associated with O and with no other $(1,3,2)$ vertex.

As each split associated with a $(1,3,2)$ vertex yields at least 3 γ angles and one β angle, we conclude that even one mismatched edge leads to a net surplus of γ angles in the tiling, which is impossible. ■

Corollary 1 *No triangle $T_n, n > 8$, tiles the sphere.*

The next lemma, ruling out certain edge length dependencies, will be used frequently.

Lemma 2 (Overhang Lemma) *Let $\triangle XYZ$ be a triangle in a (hypothetical) monohedral tiling of the sphere with some T_n. If $\overleftrightarrow{XY}$ overhangs at Y, then $\overleftrightarrow{XZ}$ does not overhang at Z.*

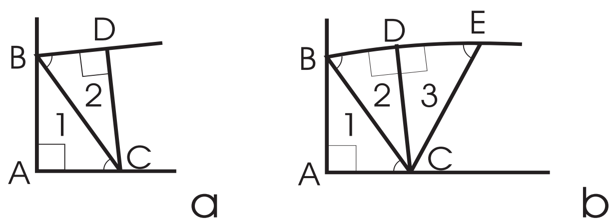

Figure 2: An included right angle

Proof: Consider first the case in which the included angle is the right angle A, and suppose there are overhangs at B and C as shown. All other angles at C must be γ angles, so triangle 2 is forced as shown (Fig. 2a). The remaining angles in the gap at B are all γ angles, so that $\overline{BD}$ is mismatched with either a medium edge or hypotenuse. The edge $\overline{CD}$ must be covered as shown in (Fig. 2b) by Triangle 3; and by Lemma 1 the extended edge $\overleftrightarrow{BD}$ does not end at E, creating a further overhang. The edge $\overline{CE}$ cannot be covered by a hypotenuse in either orientation without putting two β angles at a split, which is impossible.

Suppose now that the included angle between the edges is β and the edges overhang at A and C:

The split at A requires another right angle. By Lemma 1, $\overline{AC}$ must be covered by another medium edge, forcing Triangle 2; and by the same lemma, there is an overhang as shown at D. Triangle 3 is then forced, to cover the hypotenuse $\overline{CD}$ while avoiding a second β at D. The remaining gap at C must be filled by one or more γ angles, so the extended edge $\overleftrightarrow{CE}$ overhangs triangle 3 at E. Triangle 4 and the overhang at F follow, and $\overline{DF}$ cannot then be covered in any way.

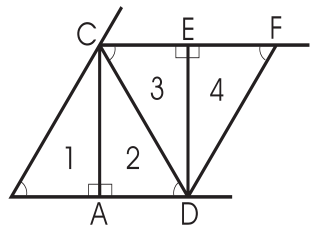

Figure 3: An included β angle

Finally, if the included angle is a γ angle, and there are overhangs at A and B, the edge $\overline{AB}$ would have to be covered by another short edge, with a split vertex at either end; but this requires either a split vertex with two β angles or one with a β angle and a right angle, neither of which exists for these triangles. ∎

3 The triangle T_6

This triangle has angles $(90°, 64\frac{2}{7}°, 38\frac{4}{7}°)$, and vertex vectors $(0,2,6)$, $(0,5,1)$, $(1,0,7)$, $(1,3,2)$, $(2,1,3)$, and $(4,0,0)$. Its area is 1/56 that of the sphere. In this section we show that this triangle does not tile the sphere, although it admits a multiple cover.

Because of the second β source $(0,5,1)$ which has a large surplus of β angles over γ angles, a counting argument of the sort used in the previous section will not work; we have to resort to "brute force" tile chasing.

Theorem 1 *The* $(90°, 64\frac{2}{7}°, 38\frac{4}{7}°)$ *triangle does not tile the sphere.*

Proof: If it did so, it would (by Prop. 2) do so using at least one $(0,5,1)$ vertex. Let O be such a vertex, with triangle 1 contributing the only γ angle (Figure 4a).

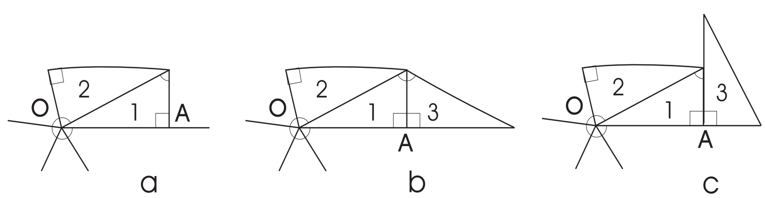

Figure 4: The neighborhood of a $(0,5,1)$ vertex

Since all other angles are β angles, the medium edge of triangle 1 cannot be matched, forcing an overhang at A. The hypotenuse of triangle 1 must then, by Lemma 2, be matched, with a β angle at O, forcing triangle 2 as shown in Figure 4a. There are two possible orientations for triangle 3, which contributes the remaining right angle at A (Figure 4b, c).

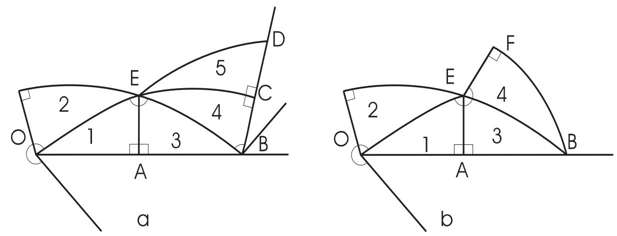

Figure 5: The neighborhood of a $(0, 5, 1)$ vertex

Proposition 3 *If the configuration of Figure* 4b *appears in a tiling of the sphere with triangle* T_6*, this configuration must extend to that of Figure* 5b*.*

Proof: By Lemma 1, there is an overhang at B and the hypotenuse of triangle 3 must be matched. If triangle 4, which contributes this hypotenuse, were positioned as in Fig. 5a with a β angle at B, then, when the split at B is filled, it will create an overhang at C, forcing triangle 5. By Lemma 1, the extended edge $\overleftrightarrow{BC}$ overhangs triangle 5 at D; the edge $\overline{DE}$ must be matched, but neither vertex D nor vertex E can take an additional β angle. We conclude that triangle 4 must be positioned as in Figure 5b. ■

It follows that E could only be a $(1, 3, 2)$ vertex or a $(0, 5, 1)$ vertex. The next proposition eliminates one of these possibilities.

Proposition 4 *If the configuration of Figure* 5b *appears in a tiling of the sphere with triangle* T_6*, then* E *is a* $(0, 5, 1)$ *vertex.*

Proof: Suppose instead that E is a $(1, 3, 2)$ vertex. If the right angle were next to triangle 2 (Figure 6a), the short edge $\overline{EF}$ could not match either edge adjacent to a γ angle; so to obtain the matching required by Lemma 2 we must have triangles 5 and 6 as shown.

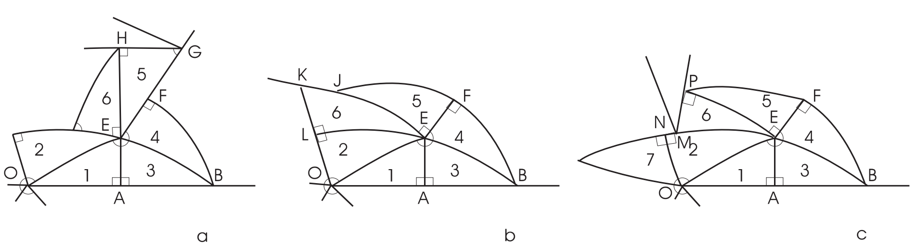

Figure 6: The configuration if E is a $(1, 3, 2)$ vertex

Overhangs at G and H are then forced as shown; and the overhang at H requires a split with a β and a right angle, which does not exist. Therefore the right angle at E must be adjacent to triangle 4 (Figure 6b, c). Lemma 2 requires one edge adjacent to that angle to be matched, which forces triangle 5 as shown. We now consider the orientation of triangle 6, contributing the γ angle at E.

If the medium edge of triangle 6 matches that of triangle 2 (Figure 6*b*), it creates a split at J, and an overhang at K. By Lemma 1, the extended edge $\overline{OK}$ is covered by two more short edges. One of these must be a side of a β angle at O, forcing a $(4, 0, 0)$ vertex at L; but then there are two β angles at K, which is impossible.

The medium edge of triangle 6 thus matches that of triangle 5 (Figure 6*c*), creating an overhang at M. Triangle 7 is then forced at O. When the gap at N is filled with γ angles, it creates a split vertex at P with a right angle and a γ angle; but this cannot exist. We conclude that E is a $(0, 5, 1)$ vertex. ■

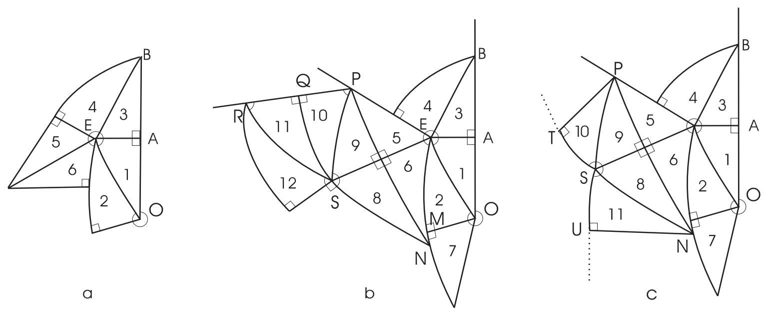

Figure 7: The neighborhood of the $(0, 5, 1)$ vertex E

Proposition 5 *If the configuration of Figure* 5*b appears in a tiling of the sphere with triangle* T_6*, this configuration must extend to that of Figure 7a.*

The $(0, 5, 1)$ vertex at E requires 2 more β angles (Figure 7 - note that these diagrams have been reoriented!) Let triangle 5 be the one adjacent to triangle 4 at E. If it is placed with its hypotenuse against the short leg of triangle 4 (Figure 7*b*, *c*), then triangle 6 is forced as shown by Lemmas 1 and 2. The overhang at M forces triangle 7 as shown, creating an overhang at N. By Lemma 1, there must be an overhang at P. The extended edge $\overline{NP}$ must be matched by two medium edges, forcing triangles 8 and 9. There cannot be an overhang at S, so the hypotenuses of triangles 8 and 9 must each be matched.

Suppose (Figure 7*b*) that the hypotenuse of triangle 9 is covered by triangle 10 contributing an angle β at P. When the split at P is filled it creates an overhang at Q. Triangle 11 is forced on edge $\overline{QS}$, with the right angle at Q, forcing S to be a $(1, 3, 2)$ vertex. By Lemma 1, $\overrightarrow{PQ}$ overhangs triangle 11 at R, forcing triangle 12 as shown. But now we cannot match $\overline{SN}$ while providing the missing right angle at S. A similar contradiction arises if we try to cover the hypotenuse of triangle 8 with the β angle of the new triangle at N.

We must thus have triangles 10 and 11 as shown in Figure 7c. The vertex at S must be $(0, 5, 1)$, and one of $\overline{ST}$, $\overline{SU}$ must be covered with an edge adjacent to the γ angle, which must overhang as shown. The corresponding medium edge, $\overline{PT}$ or $\overline{NU}$, cannot be covered, as the split at N or P cannot accommodate either a fourth γ or a right angle. We conclude that triangle 5 is as shown in Figure 7*a*, with triangle 6 forced by Lemma 2. ■

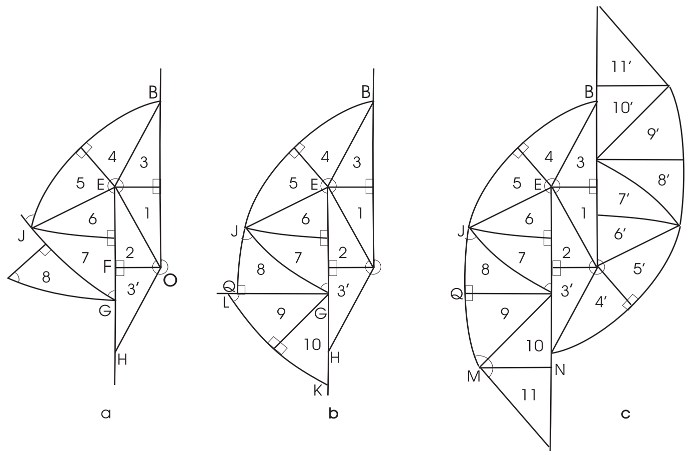

Figure 8: A configuration appears twice.

Proposition 6 *If the configuration of Figure* 4*b appears in a tiling of the sphere with the triangle* T_6*, centered on the* $(0, 5, 1)$ *vertex* O*, then the tiling contains another copy of the same configuration centered on the vertex* E *of Figure 7a.*

Proof: We have seen that triangles 4-6 are forced. Triangle 7 is also forced by Lemma 2 (Fig. 8a). By Lemma 1 there is an overhang at G which forces triangle $3'$ to be as shown. Triangles $2, 1$ and $3'$ have the same configuration as triangles $1, 2$ and 3. Thus triangles $4 - 7$ have counterparts $4' - 7'$ (Fig. 8c). ■

Proposition 7 *If the configuration of Figure* 4*b appears in a tiling of the sphere with the triangle* T_6*, it must extend to the configuration of Figure 8c.*

Proof: We first show that the hypotenuse of triangle 7 must be matched. Otherwise, there would be an overhang at J (Fig. 8a). The edge $\overline{BJ}$ must be filled with two medium edges; but this is impossible since the split at J requires a β. Triangle 8 is thus as shown in Fig. 8b. (Note that the extended edge $\overrightarrow{BJ}$ must therefore extend beyond J).

The split at G requires two more γ angles, provided by triangles 9 and 10. By Lemma 2, these must be positioned as in Fig. 8b or c; but the configuration of Fig. 8b has overhangs at K and L. $\overline{KL}$ is thus a complete extended edge, which by Lemma 1 must be covered by two more short edges; but this cannot be done in the absence of a split with two β angles. We must then have triangles 9 and 10 as in Fig. 8c; there is an overhang at N, forcing triangle 11. Triangles $8'$-$11'$ are forced similarly. ■

$\overline{BM}$ is thus a complete extended edge, and must be matched by two medium edges and two short edges.

Proposition 8 *If the configuration of Figure* 8*c appears in a tiling of the sphere with the triangle* T_6*, the medium edge of triangle 4 cannot be matched.*

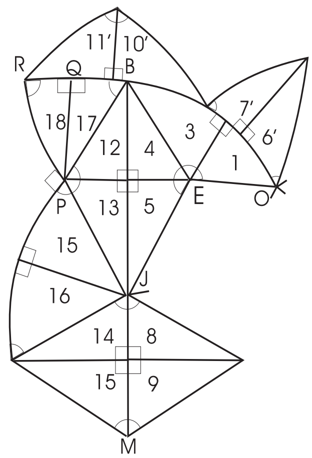

Figure 9: The medium edge of triangle 4 cannot be matched

Proof: Suppose that it is (Figure 9); then triangle 12 must be as shown. Triangle 13 is then forced by Lemma 2, and triangles 14 and 15 are then forced . The edge $\overline{PJ}$ must be covered, without a third β at P, forcing triangle 15; and Lemma 2 then forces triangle 16. The edge $\overline{BP}$ must be covered by triangle 17 as shown. Referring back to Fig. 8*c*, we see that the length of $\overline{QR}$ is exactly C; so triangle 18 must cover $\overline{QP}$ as shown. P must therefore be a $(1, 3, 2)$ vertex; but Lemma 2 shows that it is impossible to fill the right-angled gap. ■

Proposition 9 *The neighborhood of every* $(0, 5, 1)$ *vertex in a tiling of the sphere by the triangle* T_6 *must have the configuration of Figure 4c.*

Proof: Suppose not; in light of Proposition 8, we must have triangle 12 as in Figure 10. Triangles 13, 14, and 15 are forced as before. The edge $\overline{PS}$ must be matched, forcing triangle 16; the remaining γ gap at P is filled by triangle 17, whose orientation is dictated by Lemma 2.

The segment $\overline{OR}$ has length $2C + 2B$, while the segment $\overline{OB}$ has length $2B$ (refer back to Figure 9.) The segment $\overline{BR}$ therefore has length $2C$. This is equal to neither H nor B; so that side of triangle 18, filling the γ gap at B, cannot be matched. Lemma 2 thus requires triangle 18 to be positioned as shown, with its hypotenuse matching that of triangle 12. Triangle 19 is forced as shown; this leaves a right-angled gap at S. However, the sides of this gap are both longer than either leg of the triangle, so Lemma 2 shows that the tiling cannot be completed. ■

Proposition 10 *The neighhborhood of any* $(0, 5, 1)$ *vertex in a tiling of the sphere by the triangle* T_6 *must have the configuration of Figure 11b.*

Proof: By the previous proposition, triangle 3 must be as shown in Figure 11.

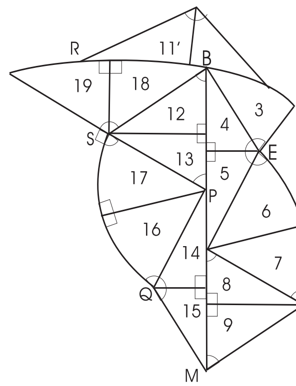

Figure 10: A contradiction is reached.

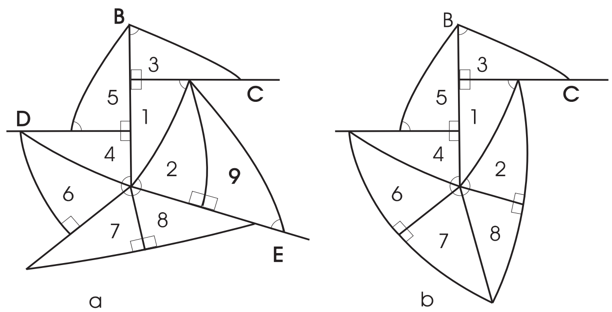

Figure 11: The configuration near a $(0, 5, 1)$ vertex

As there is an overhang at C, there cannot be one at B, forcing triangles 4 and 5 as shown. There must be an overhang as shown at D, forcing triangle 6. Triangles 7 and 8 provide the remaining two β angles at O; by Lemma 2, they must either be as in Figure 11a or b. In the first case triangle 9 is forced; there is an overhang at E, requiring the hypotenuse of triangle 9 to be matched, but the splits at both ends already have a β angle; thus this configuration is impossible. We conclude that the short edges of triangles 6 and 7 are matched, as in Figure 11b. ■

Proposition 11 *If the configuration of Figure 11b arises in a tiling of the sphere by the triangle T_6, the vertex B is also a $(0, 5, 1)$ vertex.*

Proof: If $\overrightarrow{CB}$ extended beyond B, it would be impossible to cover the hypotenuse of triangle 5; so the hypotenuse of triangle 3 must be matched (triangle 9 in Figure 12$a-c$.) If triangle 9 contributes a β angle to the split at C (Figure 12a), the next angle must be a γ and the overhang at E results. As already observed, we cannot have a split at B, so triangle 10 is forced, with an overhang at F by Lemma 1. But it is now impossible to cover $\overline{BG}$, as neither end can accept another β angle. We conclude that triangle 9 must be oriented as in 12b, c and must contribute a second β at B, which is thus not split. Triangle 12 is then forced on $\overline{BG}$.

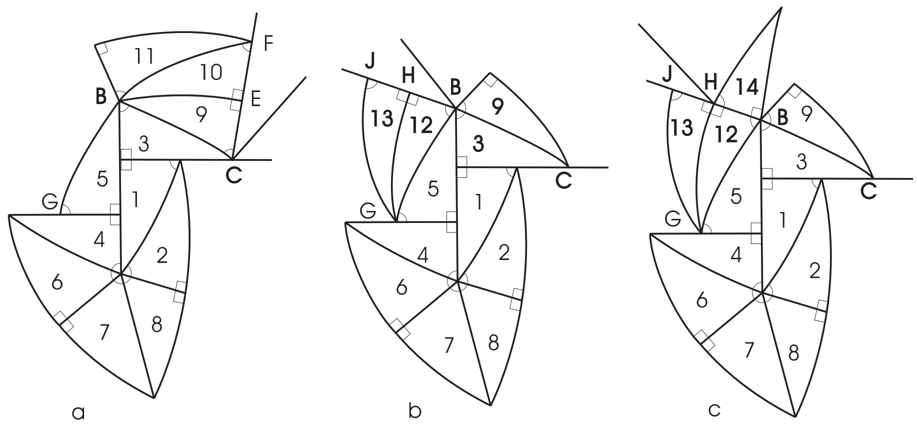

Figure 12: The configuration near a $(0, 5, 1)$ vertex, continued

If there is an overhang at H(Fig. 12b), then triangle 13 is forced as shown. By Lemma 1 there is also an overhang at J; and the hypotenuse of triangle 13 cannot be matched, as neither end can accept another β. We conclude that the short edge of triangle 12 must be matched. But if triangle 14, matching the short edge of triange 12, contributes a right angle at B, a similar impossible configuration arises (Fig. 12c). We conclude that triangle 14 contributes a β angle at B, which is therefore a $(0, 5, 1)$ vertex, with the remaining triangle as in Fig. 13a. ■

As before, all our deductions henceforth apply to the vertex B as well as to O; so forced triangles will appear in pairs. Beginning with Figure 13, we will renumber the diagrams to reflect this.

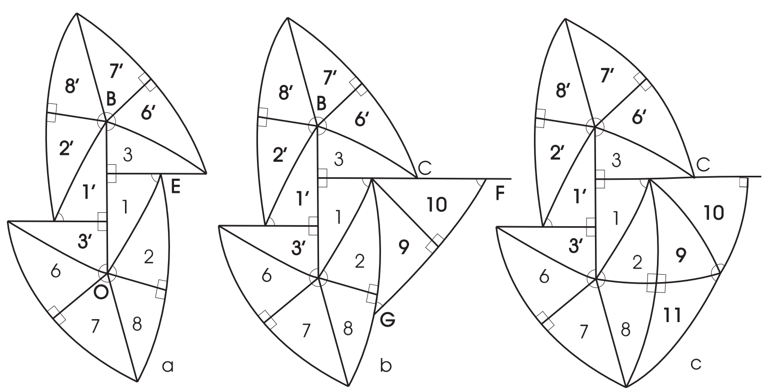

Figure 13: A pair of $(0, 5, 1)$ vertices

Proposition 12 *In any tiling of the sphere by the triangle T_6, the neighborhood of any* $(0, 5, 1)$ *vertex has the configuration of Figure* 15b.

Proof: The split at E requires two more γ angles; we will show that the medium edge of triangle 2 is matched, as in Figure 13c. Suppose not; we would have triangle 9 as in

13b; and by Lemma 2 triangle 10 would forced. The exposed segment $\overline{CF}$ has length $H+C-B$, which, as we may easily verify, is not a sum of edge lengths; thus there would be an overhang at F. But the extended edge $\overline{FG}$ (with length $2C$) cannot be covered, as neither end can accept another β angle. We thus have triangle 9 as in Figure 13c. Triangles 10 and 11 are forced by Lemma 2.

The gap at C requires a β and another γ. By Lemma 2, the triangles (12 and 13) that provide these angles must have their hypotenuses matched. We shall show that the medium edge of triangle $6'$ is also matched.

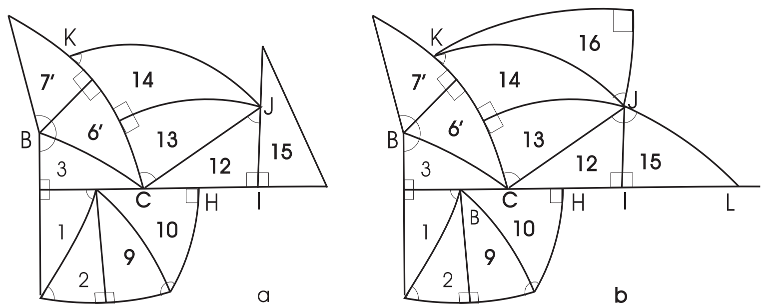

Figure 14: An impossible configuration at vertex C.

Suppose that this is not the case; we have triangles 12 and 13 as in Fig. 14. Triangle 14 is forced by Lemma 2. The segment $\overline{HI}$ has length $B-C$ and cannot be covered exactly, so there is an overhang at I witha right-angled gap. If the gap were filled as in Fig. 14a, triangle 15 would create an overhang at J, and the edge $\overline{JK}$ could not be covered; so triangle 15 must be positioned as in Fig. 14b.

The edge $\overline{JK}$ can be covered only as shown, so that J must be a $(1,3,2)$ vertex. However, the segment $\overline{IL}$ has length $2B-C$, which is not a sum of edge lengths, so there is an overhang at L. The edge $\overline{JL}$ is a hypotenuse, and cannot be covered with a right angle at J, a contradiction.

The medium edge of triangle $6'$ is therefore matched, as in Fig. 15a. Triangle 14 is now forced by Lemma 2; triangles $9'-14'$ are obtained by applying the same arguments after interchanging the $(0,5,1)$ vertices B and O. ■

We are now almost finished; the reader will observe the apparent emergence of a periodic pattern.

Proposition 13 *If the configuration of Figure 15b arises in a tiling of the sphere by the triangle T_6, the vertex N is also a $(0,5,1)$ vertex.*

Proof: The extended edge $\overline{MN}$ has length $B+C$ and must be covered by a short edge and a medium edge. We will rule out some of the ways in which this might be done.

Let triangle 15 be the tile covering $\overline{MN}$ near N. If this triangle had a β angle at N (Fig. 16a), N would be a $(0,5,1)$ vertex with four β angles as shown. This would force another copy of the entire configuration of Fig.15b, oriented so as to require four γ angles at B, which is impossible.

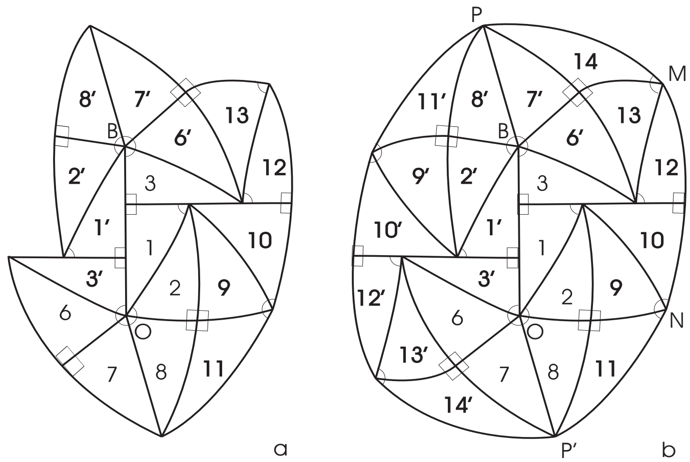

Figure 15: A 24-tile configuration forced at any $(0,5,1)$ vertex

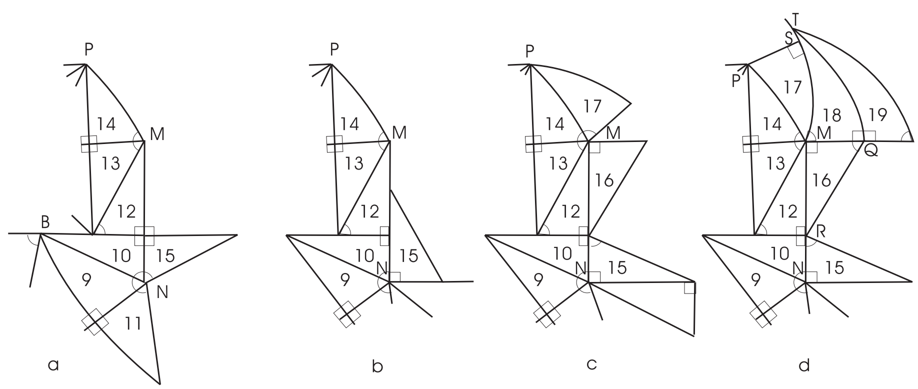

Figure 16: Covering the segment $\overline{MN}$

Supppose that triangle 15 has a right angle at N; then N is a $(1,3,2)$ vertex, and the two remaining triangles have γ angles and no short edge there. Thus, by Lemma 2, triangle 15 cannot have the orientation shown in Figure 16b; it must be as in Figure 16c,d, forcing triangle 16 at M as shown.

There cannot be an overhang at vertex P, which already has four γ angles; so the hypotenuse of triangle 14 must be matched. The tile (triangle 17) doing so cannot be positioned as in Fig. 16c, because the remaining γ angle at M could not have either adjacent edge matched, contrary to Lemma 2. If instead it is positioned as in Fig. 16d, triangle 18 must be as shown, creating an overhang at S and hence another at T. As there cannot be an overhang at Q, triangle 19 is forced. But now it is impossible to cover the edge $\overline{QR}$, as neither of the splits at Q,R can accept another β angle. We conclude that triangle 15 can only have a γ angle at vertex N (Figure 17).

To cover the hypotenuse of triangle 15 while satisfying Lemma 2, triangle 17 must be placed as shown; triangle 18 is then forced. N is thus a $(0,5,1)$ vertex. ■

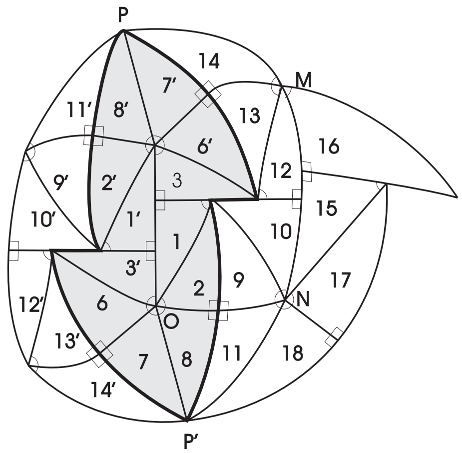

Figure 17: N has the same configuration as O.

The rest of the (hypothetical) tiling is therefore forced. However, we note that every angle at the vertex P and P' has measure γ. As γ does not divide 360°, the tiling does not close up at P and P'; we conclude that T_6 does not tile the sphere. ■

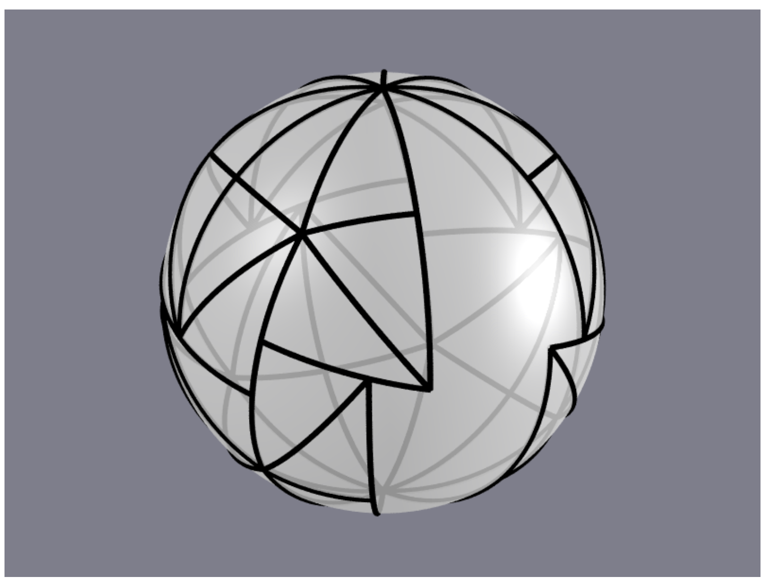

Figure 18: The tiling fails to close up.

The configuration at the poles is illustrated in Fig. 18. However, $28\gamma = 1080°$; so if we continue the tiling around the poles beyond the point at which it begins to overlap, we obtain a threefold cover of the sphere with 168 tiles (Fig. 19).

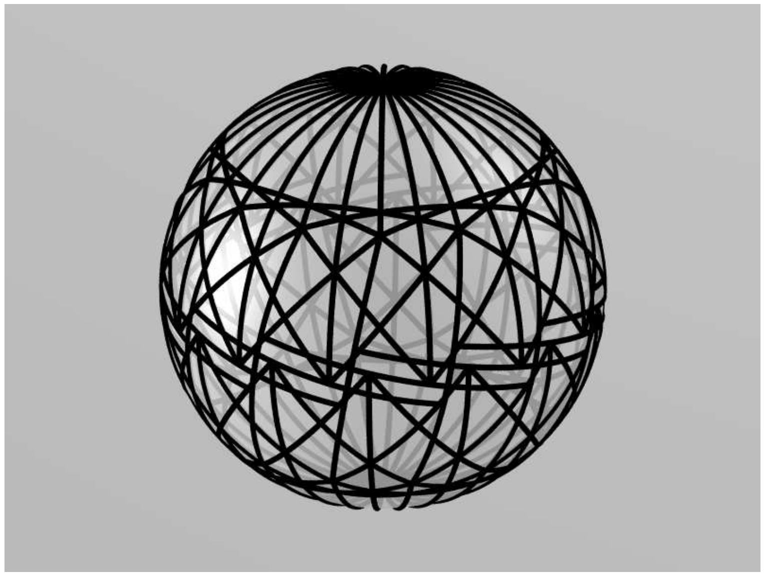

Figure 19: Three-layered tiling of the sphere with the $(90°, 64\frac{2}{7}°, 38\frac{4}{7}°)$ triangle.

4 The triangle T_8

This triangle has angles $(90°, 72°, 27°)$ and vertex vectors $(0, 2, 8)$, $(0, 5, 0)$, $(1, 0, 10)$, $(1, 3, 2)$, $(2, 1, 4)$, and $(4, 0, 0)$. Its area is 1/80 that of the sphere.

Theorem 2 *The triangle T_8 does not tile the sphere.*

Proof: As before, we examine the properties of a hypothetical tiling and eventually reach a contradiction.

Proposition 14 *In any tiling of the sphere with the triangle T_8, the configuration of Figure 20c must occur.*

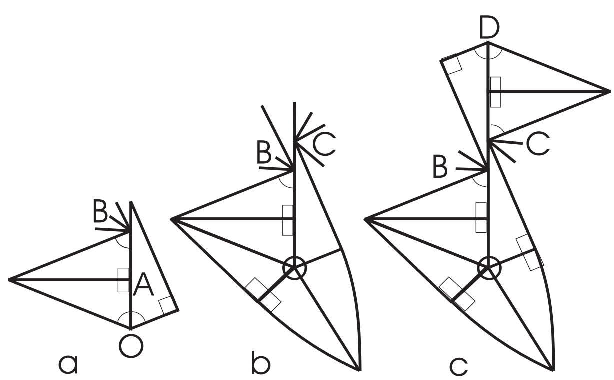

Figure 20: Configurations of T_8 at a $(0, 5, 0)$ vertex.

Proof: As before, such a hypothetical tiling must use the β source $(0,5,0)$ at least once. The triangles meeting at such a vertex have, between them, five short edges and five hypotenuses. As many short edges as hypotenuses must be unmatched; parity permits one, three, or five unmatched pairs. Each unmatched pair must (by Lemma 2) have the configuration of Fig. 20a, with a $(0,2,8)/2$ split as shown that cannot be related in the same way to any other $(0,5,0)$ vertex. Nor can the split be related to a $(1,3,2)$ vertex in any of the ways illustrated in Figure 1. For Figure $1a, b$ a shared split would require $3C = H$ or $3C = B$ respectively; and the other two cases have a right angle where Figure 20 requires a β angle.

If there were three (or five) of these, the $(0,5,0)$ and the associated splits between them would have 12 (or 20) γ angles associated with 8 (or 10)β angles, and the configuration would have a net shortage of β angles. We conclude that there exists a $(5,0,0)$ vertex O with exactly one unmatched pair, as in Fig. 20b.

The overhang and split at C are forced. If neither of these split vertices is shared with another β source, then O, B, and C have between them seven β angles and eight γ angles, for a net γ surplus. It follows that some instance of this configuration must have another β source at D sharing these associated split vertices. The vertex D cannot be split; so matching edge lengths forces the configuration of Fig. 20c. ■

We cannot immediately rule out the possibility that D is a $(1,3,2)$ vertex or a $(0,5,0)$ vertex with three mismatched pairs of edges. However, the next proposition will establish that it is in fact identical to O.

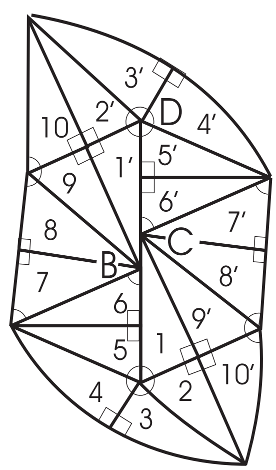

Figure 21: The configuration at D.

Proposition 15 *Any instance of the configuration of Figure 20c must extend to that of Figure 21.*

Proof: We consider the other triangles at B. If triangle 9 (covering the medium edge of triangle $1'$) were as shown in Fig. 22a, triangle $2'$ would be forced (because there can

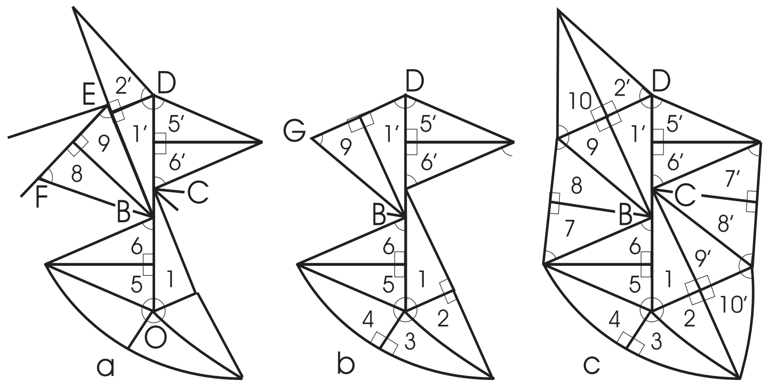

Figure 22: Configuration at B.

be no overhang at D.) The split at E then requires γ angles; the extended edge $\overline{EF}$ forces triangle 8, and there is no way to cover the edge $\overline{BF}$. We conclude that triangle 9 must be as in Fig. 22b.

If the extended edge $\overrightarrow{DG}$ overhangs at G, covering the edge $\overline{BG}$ would require a split at B or G with two β angles, so triangles 10 and $2'$ must be as shown in Fig. 22c. Again, $\overrightarrow{BG}$ cannot overhang at G, so triangle 8 is forced, with triangle 7 following from Lemma 2. Similar arguments yield triangles $7'$-$10'$.

We now consider what might cover the hypotenuse of triangle $5'$ (see Figure 23a). Certainly $\overrightarrow{DH}$ cannot extend beyond H, as this would make $\overline{HJ}$ a complete extended edge of length $2C$; but there is not room for a right angle or β angle at H. Thus $\overline{DH}$ is covered by another hypotenuse. If it is oriented as in Fig. 23b, triangles $12'$ and $13'$ are forced. Triangle $14'$ must be as shown, by Lemma 2.

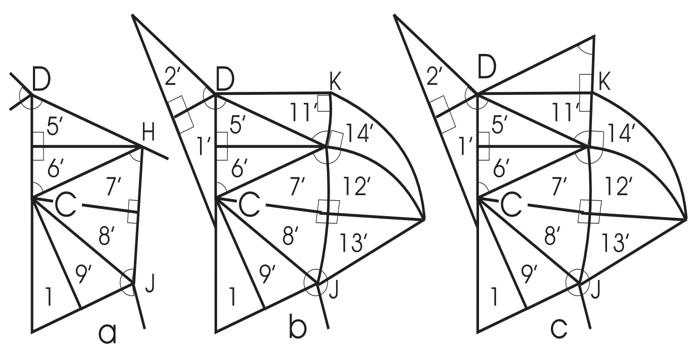

Figure 23: Some configurations at D.

A further right angle and γ are needed at D. If the γ angle were next to triangle $11'$, it would have to have its medium edge against $\overline{DK}$ (Figure 23c), leaving a right angled gap at D that cannot be filled (Lemma 2). The triangle must instead be positioned as triangle 15 in Figure 24.

Lemma 2 forces triangle 16 to be as shown, with triangle 17 overhanging at L. Triangle 18 is then forced, with another overhang at M. The hypotenuse of triangle 18 must be

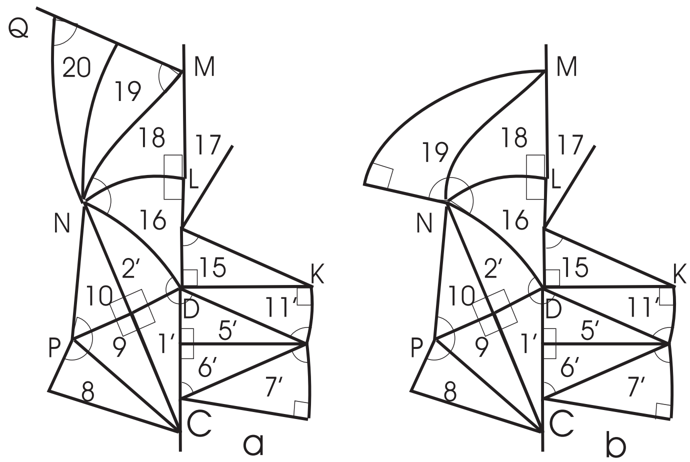

Figure 24: The two alternatives for triangle 19.

covered; Figure 24 shows the two alternatives.

If triangle 19 is as in Figure 24a, triangle 20 must be as shown, with an overhang at Q. But then the hypotenuse $\overline{NQ}$ cannot be covered, as neither end can accept another β angle. Conversely, if triangle 19 is positioned as in Figure 24*b*, we have a right angle gap at N which does not permit the hypotenuse edge $\overline{NP}$ to be covered. We conclude that the hypotenuse of triangle $5'$ is in fact covered as in Figure 21, so that D is a $(0, 5, 0)$ vertex. The position of triangle $3'$ follows immediately by Lemma 2. ■

Proposition 16 *Triangle 11 cannot appear as in Figure 25a.*

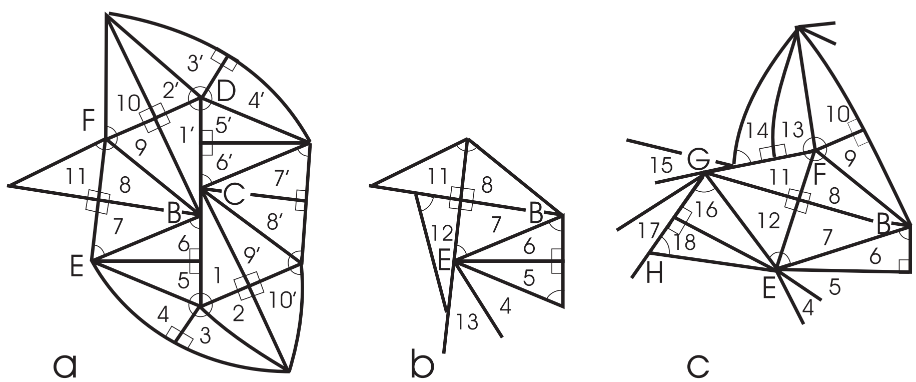

Figure 25: An impossible configuration

Proof: We first note that the right angled gap cannot be filled by a triangle positioned as triangle 12 in Figure 25b, as this would create an overhang and a γ gap at E. Filling this gap (triangle 13 in Figure 25b) would place triangle 12 in a configuration forbidden by Lemma 2. We thus have triangle 12 as in Figure 25c.

There must be a fifth β angle at F. This cannot be provided by a triangle positioned as triangle 13 in Figure 25c; otherwise Lemma 2 gives us triangle 14, creating a split which must be filled with γ angles, giving us triangle 15 and an overhang at G. The hypotenuse of triangle 12 must be matched, and as there are already 2 β and three γ angles at E, triangle 16 must be as shown. Triangles 17 and 18 follow; but then the hypotenuse $\overline{HE}$ of triangle 18 cannot be covered, as neither end can accept another β angle.

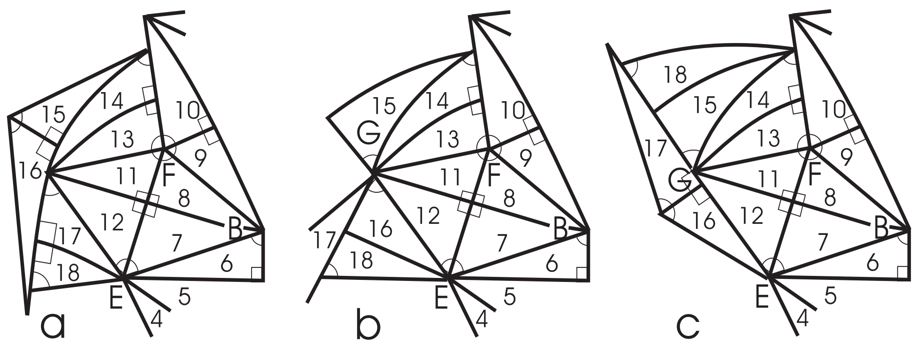

Figure 26: Various attempts to extend Figure 26a

We thus have triangle 13 as in Figure 26. Triangle 14 follows from Lemma 2, creating a split that must be completed with γ angles. If triangle 15 is as shown in Figure 26a, then triangles 16-18 are forced, and the hypotenuse of triangle 18 cannot be covered; otherwise, it is as in Figure 26b, c. As before, there must be a γ angle at E; either orientation for this triangle leads to an impossible configuration (Figure 26b, c). We conclude that the configuration of Figure 25a cannot occur. ■

Corollary 2 *The extended edge $\overrightarrow{FE}$ continues beyond E in Figure 27a.*

Proof: Suppose, on the contrary, that $\overrightarrow{FE}$ was covered by two short edges; then by the previous proposition one of those triangles would have a right angle at F. There would then be either three right angles and a β angle or two right angles and two β angles at G, but neither of these is possible. ■

Proposition 17 *The configuration of Figure 27a extends to that of 27c.*

Proof: As observed above, there is an overhang at E; hence there is a γ gap that must be filled. If triangle 11, filling it, were oriented as in Figure 27b, we would also have triangle 12 as shown there; but Proposition 16 and Lemma 2 would make it impossible to cover the extended edge $\overline{FH}$.

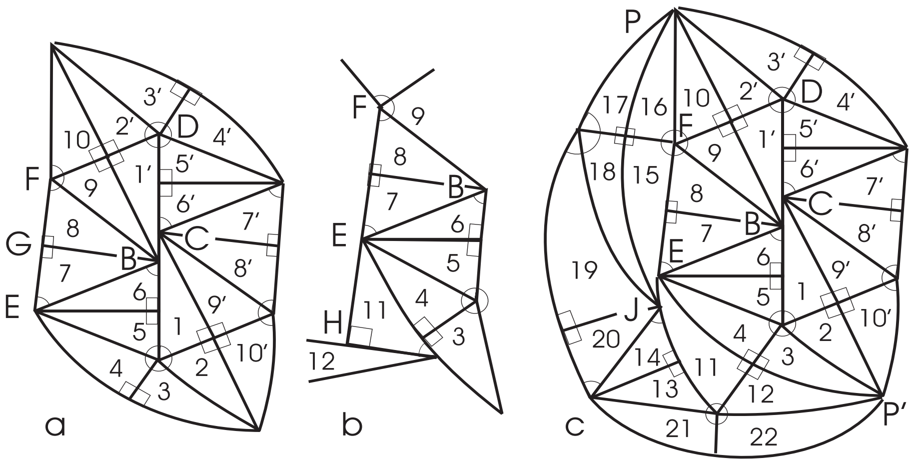

Figure 27: Propagating the configuration of Figure 21

Triangle 11 is thus positioned as in Figure 27*c*. Triangle 12 follows immediately by Lemma 2. There cannot be a split at G, so by Proposition 16 triangles 13, 14, and 15 must be as shown to cover the extended edge $\overline{FJ}$. Triangle 16 must be positioned as shown to provide the fifth β angle at F consistent with Lemma 2. But now triangles $7-11$ and $13-16$ form the configuration of Figure 20*c*, and triangles $17-22$ are forced by Proposition 15. ■

We can now finish the proof of the theorem. Repeatedly applying Proposition 17, we obtain configurations with $5, 7, 9 \ldots$ γ angles at the poles P, P', eventually reaching a contradiction as there is no vertex configuration with 11 or more γ angles. ■

However, 40 γ angles will cover the region around a point exactly three times, so there is a triple cover of the sphere with 240 T_8 tiles. It is worth noting that all the right triangles in this tiling are arranged in pairs; the multiple cover (and the failed tiling) are thus derived from the multiple cover by the $(72°, 72°, 54°)$ isosceles triangle mentioned in [3].

5 Conclusion

We have seen that no spherical triangle in the T_n family (except for the improper member T_4) can tile the sphere. For $n > 8$ this is fairly straightforward, but for $n = 6$ and $n = 8$, the powerful counting arguments that establish the result in other cases fail, and it is necessary to rule out a lot of false leads. Some of these come very close to complete tilings of the sphere.

This result forms a part of the classification of all spherical triangles that tile the sphere, which will be completed in a sequence of papers now in preparation.

The authors would like to thank the anonymous referee for a very thorough reading and many helpful suggestions.

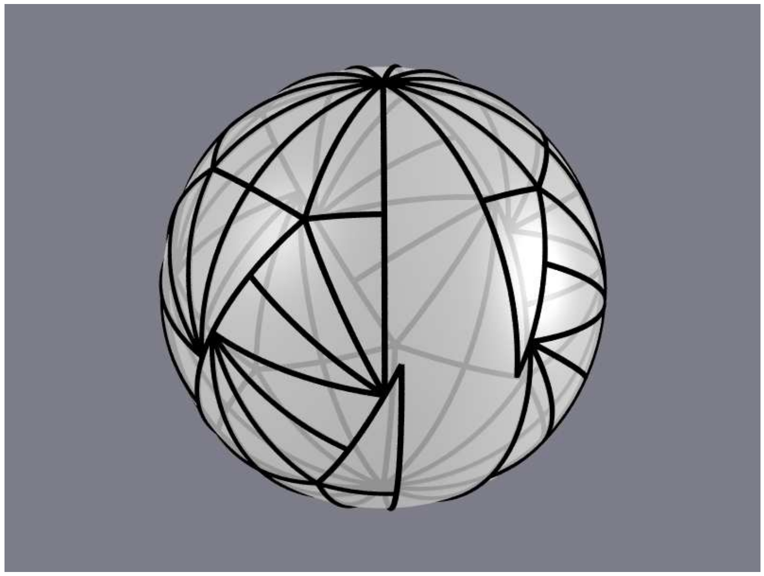

Figure 28: A partial tiling that fails to close up.

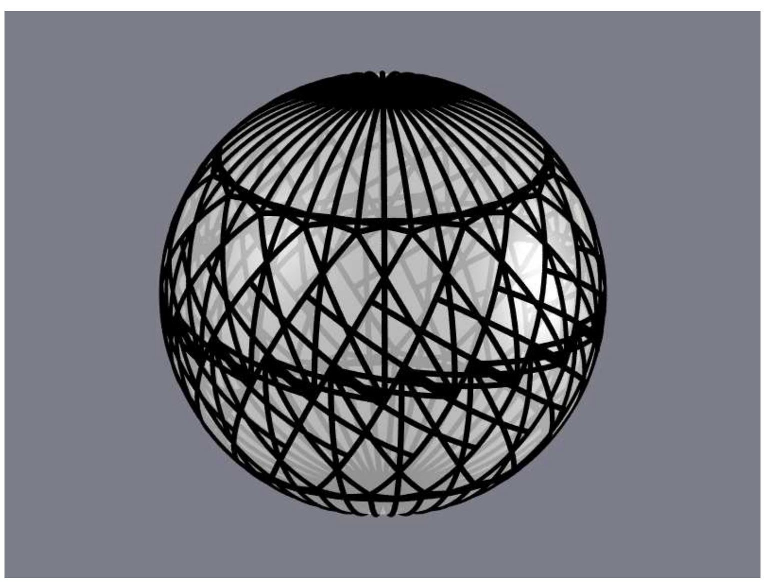

Figure 29: Three-layered tiling of the sphere with the $(90°, 72°, 27°)$ triangle.

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