The Generating Function of Ternary Trees and Continued Fractions

Ira M. Gessel* and Guoce Xin†

Department of Mathematics
Brandeis University
Waltham MA 02454-9110
gessel@brandeis.edu

Department of Mathematics
Brandeis University
Waltham MA 02454-9110
guoce.xin@gmail.com

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Abstract

Michael Somos conjectured a relation between Hankel determinants whose entries $\frac{1}{2n+1}\binom{3n}{n}$ count ternary trees and the number of certain plane partitions and alternating sign matrices. Tamm evaluated these determinants by showing that the generating function for these entries has a continued fraction that is a special case of Gauss's continued fraction for a quotient of hypergeometric series. We give a systematic application of the continued fraction method to a number of similar Hankel determinants. We also describe a simple method for transforming determinants using the generating function for their entries. In this way we transform Somos's Hankel determinants to known determinants, and we obtain, up to a power of 3, a Hankel determinant for the number of alternating sign matrices. We obtain a combinatorial proof, in terms of nonintersecting paths, of determinant identities involving the number of ternary trees and more general determinant identities involving the number of r-ary trees.

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1 Introduction

Let $a_n = \frac{1}{2n+1}\binom{3n}{n} = \frac{1}{3n+1}\binom{3n+1}{n}$ be the number of ternary trees with n vertices and define the Hankel determinants

$$U_n = \det\left(a_{i+j}\right)_{0\le i,j\le n-1} \tag{1}$$
$$V_n = \det\left(a_{i+j+1}\right)_{0\le i,j\le n-1} \tag{2}$$
$$W_n = \det\left(a_{(i+j+1)/2}\right)_{0\le i,j\le n-1}, \tag{3}$$

where we take a_k to be 0 if k is not an integer. (We also interpret determinants of 0×0 matrices as 1.) The first few values of these determinants are

n	1	2	3	4	5	6	7
U_n	1	2	11	170	7429	920460	323801820
V_n	1	3	26	646	45885	9304650	5382618660
W_n	1	1	2	6	33	286	4420

This paper began as an attempt to prove the conjectures of Michael Somos [27] that

(a) U_n is the number of of cyclically symmetric transpose complement plane partitions whose Ferrers diagrams fit in an $n \times n \times n$ box,

(b) V_n is the number of $(2n+1) \times (2n+1)$ alternating sign matrices that are invariant under vertical reflection, and

(c) W_n is the number of $(2n+1) \times (2n+1)$ alternating sign matrices that are invariant under both vertical and horizontal reflection.

Mills, Robbins, and Rumsey [22] (see also [5, Eq. (6.15), p. 199]) showed that the number of objects of type (a) is

$$\prod_{i=1}^{n-1} \frac{(3i+1)(6i)!\,(2i)!}{(4i+1)!\,(4i)!}. \tag{4}$$

Mills [25] conjectured the formula

$$\prod_{i=1}^{n} \frac{\binom{6i-2}{2i}}{2\binom{4i-1}{2i}} \tag{5}$$

for objects of type (b) and this conjecture was proved by Kuperberg [19]. A formula for objects of type (c) was conjectured by Robbins [26] and proved by Okada [23]. A determinant formula for these objects was proved by Kuperberg [19].

It turns out that it is much easier to evaluate Somos's determinants than to relate them directly to (a)–(c). It is easy to see that $W_{2n} = U_n V_n$ and $W_{2n+1} = U_{n+1} V_n$, so it is only necessary show that U_n is equal to (4) and V_n is equal to (5) to prove Somos's conjectures.

This was done by Tamm [28], who was unaware of Somos's conjectures. Thus Somos's conjectures are already proved; nevertheless, our study of these conjecture led to some additional determinant evaluations and transformations that are the subject of this paper.

Tamm's proof used the fact that Hankel determinants can be evaluated using continued fractions; the continued fraction that gives these Hankel determinants is a special case of Gauss's continued fraction for a quotient of hypergeometric series. The determinant V_n was also evaluated, using a different method, by Eğecioğlu, Redmond, and Ryavec [6, Theorem 4], who also noted the connection with alternating sign matrices and gave several additional Hankel determinants for V_n:

$$V_n = \det\left(b_{i+j}\right)_{0\le i,j\le n-1} = \det\left(r_{i+j}\right)_{0\le i,j\le n-1} = \det\left(s_{i+j}(u)\right)_{0\le i,j\le n-1}, \tag{6}$$

where $b_n = \frac{1}{n+1}\binom{3n+1}{n}$, $r_n = \binom{3n+2}{n}$, and

$$s_n(u) = \sum_{k=0}^{n} \frac{k+1}{n+1}\binom{3n-k+1}{n-k}u^k,$$

where u is arbitrary. As noted in [6, Theorem 4], $s_n(0) = b_n$, $s_n(1) = a_{n+1}$, and $s_n(3) = r_n$.

In Section 2, we describe Tamm's continued fraction method for evaluating these determinants. In Section 3, we give a systematic application of the continued fraction method to several similar Hankel determinants. In Theorem 3.1 we give five pairs of generating functions similar to that for a_n whose continued fractions are instances of Gauss's theorem. Three of them have known combinatorial meanings for their coefficients, including the number of two-stack-sortable permutations (see West [29]).

In Section 4 we discuss a simple method, using generating functions, for transforming determinants and use it to show that

$$U_n = \det\left(\binom{i+j}{2i-j}\right)_{0\le i,j\le n-1} \tag{7}$$

and

$$V_n = \det\left(\binom{i+j+1}{2i-j}\right)_{0\le i,j\le n-1}. \tag{8}$$

We also prove Eğecioğlu, Redmond, and Ryavec's identity (6) and the related identity

$$\det\left(s_{i+j-1}(u)\right)_{0\le i,j\le n-1} = U_n/u, \quad n>0, \tag{9}$$

where $s_{-1}(u) = u^{-1}$. When $u=1$, (9) reduces to (1) and when $u=3$, (9) reduces to

$$\det\left(r_{i+j-1}\right)_{0\le i,j\le n-1} = U_n/3, \quad n>0. \tag{10}$$

Note that $r_{n-1} = \frac{1}{3}\binom{3n}{n}$, so (10) is equivalent to $\det\left(\binom{3n}{n}\right)_{0\le i,j\le n-1} = 3^{n-1}U_n$ for $n>0$.

In Section 5 we consider the Hankel determinants of the coefficients of

$$\frac{1-(1-9x)^{1/3}}{3x}.$$

We first evaluate them using continued fractions, and then show that the method of Section 4 transforms them into powers of 3 times the determinant

$$\det\left(\binom{i+j}{i-1}+\delta_{ij}\right)_{0\le i,j\le n-1},$$

which counts descending plane partitions and alternating sign matrices. Similarly, the Hankel determinant corresponding to

$$\frac{1-(1-9x)^{2/3}}{3x}$$

is transformed to a power of 3 times the determinant

$$\det\left(\binom{i+j}{i}+\delta_{ij}\right)_{0\le i,j\le n-1},$$

which counts cyclically symmetric plane partitions.

Determinants of binomial coefficients can often be interpreted as counting configurations of non-intersecting paths (see, for example, Gessel and Viennot [11] and Bressoud [5]) and both sides of (7) (8) have such interpretations. In Section 6, we describe the nonintersecting lattice path interpretation for (7). We give a new class of interpretations of a_n in terms of certain paths called K-paths in Theorem 6.3. From this new interpretation of a_n, (7) follows easily. The proof of Theorem 6.3 relies on a "sliding lemma", which says that the number of certain K-paths does not change after sliding their starting and ending points.

In Section 7, we study another class of paths called T-paths, which are related to trinomial coefficients, and KT-paths, which are analogous to K-paths. We find another class of interpretations of a_n in terms of KT-paths, using which we find a new determinant identity involving U_n (Theorem 7.3). Unfortunately, we do not have a nonintersecting path interpretation for this determinant. There is a natural bijection from K-paths to KT-paths, and the sliding lemma for KT-paths is easier to prove than that for K-paths.

In Section 8, we study $KT^{(r)}$-paths, which reduce to KT paths when $r=2$. The results of Section 7 generalize, and we obtain determinant identities involving Hankel determinants for the number of $(r+1)$-ary trees (see (72) and (73)).

In Section 9, we give algebraic proofs of the results of Section 8 using partial fractions.

2 Hankel Determinants and Gauss's Continued Fraction

Let $A(x)=\sum_{n\ge 0}A_n x^n$ be a formal power series. We define the Hankel determinants $H_n^{(k)}(A)$ of $A(x)$ by

$$H_n^{(k)}(A)=\det\left(A_{i+j+k}\right)_{0\le i,j\le n-1}.$$

We shall write $H_n(A)$ for $H_n^{(0)}(A)$ and $H_n^1(A)$ for $H_n^{(1)}(A)$. We also define $\hat{H}_n(A)$ to be $H_n(A(x^2))$. It is not difficult to show that $\hat{H}_{2n}(A) = H_n(A)H_n^1(A)$ and $\hat{H}_{2n+1}(A) = H_{n+1}(A)H_n^1(A)$.

Let $g(x)$ be the generating function for ternary trees:

$$g(x) = \sum_{n\geq 0} a_n x^n = \sum_{n\geq 0} \frac{1}{2n+1}\binom{3n}{n} x^n, \tag{11}$$

which is uniquely determined by the functional equation

$$g(x) = 1 + xg(x)^3. \tag{12}$$

Then $U_n = H_n(g(x))$, $V_n = H_n^1(g(x))$, and $W_n = \hat{H}_n(g(x))$.

In general, it is difficult to say much about $H_n(A(x))$. However, if $A(x)$ can be expressed as a continued fraction, then there is a very nice formula. This is the case for $g(x)$: Tamm [28] observed that $g(x)$ has a nice continued fraction expression, which is a special case of Gauss's continued fraction. We introduce some notation to explain Tamm's approach.

We use the notation $S(x;\lambda_1,\lambda_2,\lambda_3,\ldots)$ to denote the continued fraction

$$S(x;\lambda_1,\lambda_2,\lambda_3,\ldots) = \cfrac{1}{1-\cfrac{\lambda_1 x}{1-\cfrac{\lambda_2 x}{1-\cfrac{\lambda_3 x}{\ddots}}}} \tag{13}$$

The following theorem is equivalent to [14, Theorem 7.2]. Additional information about continued fractions and Hankel determinants can be found in Krattenthaler [17, Section 5.4].

Lemma 2.1. *Let $A(x) = S(x;\lambda_1,\lambda_2,\lambda_3,\ldots)$ and let $\mu_i = \lambda_1\lambda_2\cdots\lambda_i$. Then for $n\geq 1$,*

$$H_n(A) = (\lambda_1\lambda_2)^{n-1}(\lambda_3\lambda_4)^{n-2}\cdots(\lambda_{2n-3}\lambda_{2n-2}) = \mu_2\mu_4\cdots\mu_{2n-2} \tag{14}$$

$$H_n^1(A) = \lambda_1^n(\lambda_2\lambda_3)^{n-1}\cdots(\lambda_{2n-2}\lambda_{2n-1}) = \mu_1\mu_3\cdots\mu_{2n-1} \tag{15}$$

$$\hat{H}_n(A) = \lambda_1^{n-1}\lambda_2^{n-2}\cdots\lambda_{n-2}^2\lambda_{n-1} = \mu_1\mu_2\cdots\mu_{n-1}. \tag{16}$$

We define the hypergeometric series by

$${}_2F_1\left(a,b;c \mid x\right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!\,(c)_n} x^n,$$

where $(u)_n = u(u+1)\cdots(u+n-1)$.

Gauss proved the following theorem [14, Theorem 6.1], which gives a continued fraction for a quotient of two hypergeometric series:

Lemma 2.2. *If c is not a negative integer then we have the continued fraction*

$$ {}_2F_1\left(a, b+1; c+1 \mid x\right) \big/ {}_2F_1\left(a, b; c \mid x\right) = S(x; \lambda_1, \lambda_2, \ldots), \tag{17} $$

where

$$ \begin{aligned} \lambda_{2n-1} &= \frac{(a+n-1)(c-b+n-1)}{(c+2n-2)(c+2n-1)}, \qquad && n = 1, 2, \ldots, \\ \lambda_{2n} &= \frac{(b+n)(c-a+n)}{(c+2n-1)(c+2n)}, && n = 1, 2, \ldots. \end{aligned} \tag{18} $$

Combining Lemmas 2.1 and 2.2 gives a formula for evaluating certain Hankel determinants.

Lemma 2.3. *Let*

$$ A(x) = {}_2F_1\left(a, b+1; c+1 \mid \rho x\right) \big/ {}_2F_1\left(a, b; c \mid \rho x\right). $$

Then

$$ H_n(A) = \prod_{i=0}^{n-1} \frac{(a)_i (b+1)_i (c-b)_i (c-a+1)_i}{(c)_{2i} (c+1)_{2i}} \rho^{2i} \tag{19} $$

$$ H_n^1(A) = \prod_{i=1}^{n} \frac{(a)_i (b+1)_{i-1} (c-b)_i (c-a+1)_{i-1}}{(c)_{2i-1} (c+1)_{2i-1}} \rho^{2i-1} \tag{20} $$

$$ = \prod_{i=1}^{n} \frac{(c-1)c}{b(c-a)\rho} \frac{(a)_i (b)_i (c-b)_i (c-a)_i}{(c)_{2i} (c-1)_{2i}} \rho^{2i} \tag{21} $$

Proof. By Lemma 2.2, $A(x)$ has the continued fraction expansion $A(x) = S(x; \lambda_1, \lambda_2, \cdots)$ where

$$ \begin{aligned} \lambda_{2n-1} &= \frac{(a+n-1)(c-b+n-1)}{(c+2n-2)(c+2n-1)} \rho, \\ \lambda_{2n} &= \frac{(b+n)(c-a+n)}{(c+2n-1)(c+2n)} \rho. \end{aligned} $$

Then

$$ \lambda_1 \lambda_3 \cdots \lambda_{2i-1} = \frac{(a)_i (c-b)_i}{(c)_{2i}} \rho^i $$

and

$$ \lambda_2 \lambda_4 \cdots \lambda_{2i} = \frac{(b+1)_i (c-a+1)_i}{(c+1)_{2i}} \rho^i. $$

So with the notation of Lemma 2.1,

$$ \mu_{2i} = \lambda_1 \lambda_2 \cdots \lambda_{2i} = \frac{(a)_i (c-b)_i (b+1)_i (c-a+1)_i}{(c)_{2i} (c+1)_{2i}} \rho^{2i} $$

and

$$\mu_{2i-1} = \lambda_1 \lambda_2 \cdots \lambda_{2i-1} = \frac{(a)_i (c-b)_i (b+1)_{i-1} (c-a+1)_{i-1}}{(c)_{2i} (c+1)_{2i-2}} \rho^{2i-1}.$$

Then (19) follows immediately from (14), and (20) follows from (15) with the help of the identity $(c)_{2i}(c+1)_{2i-2} = (c)_{2i-1}(c+1)_{2i-1}$, and (21) follows easily from 20. □

There is also a simple formula for $H_n^{(2)}(A)$, although we will not need it.

Lemma 2.4. *Let* $Q(a,b,c \mid x) = {}_2F_1\left(a, b+1; c+1 \mid x\right) / {}_2F_1\left(a, b; c \mid x\right)$. *Then*

$$Q(b,a,c \mid x) = \frac{c(a-b)}{a(c-b)} + \frac{b(c-a)}{a(c-b)} Q(a,b,c \mid x).$$

Proof. The formula is an immediate consequence of the contiguous relation

$$c(a-b) {}_2F_1\left(a, b; c \mid x\right) + b(c-a) {}_2F_1\left(a, b+1; c+1 \mid x\right) + a(b-c) {}_2F_1\left(a+1, b; c+1 \mid x\right) = 0,$$

which is easily proved by equating coefficients of powers of x. □

Equivalently, Lemma 2.4 asserts that $ca + b(c-a)Q(a,b,c \mid x)$ is symmetric in a and b.

Proposition 2.5. *With* $A(x)$ *as in Lemma 2.3, we have*

$$H_n^{(2)}(A) = \left(\frac{a(c-b)}{c(a-b)} \frac{(a+1)_n (c-b+1)_n}{(b+1)_n (c-a+1)_n} - \frac{b(c-a)}{c(a-b)} \right) H_{n+1}(A).$$

Proof. First note that if $u(x) = \alpha + \beta v(x)$, where α and β are constants, then

$$H_{n+1}(u) = \beta^{n+1} H_{n+1}(v) + \alpha \beta^n H_n^{(2)}(v),$$

so

$$H_n^{(2)}(v) = \frac{1}{\alpha \beta^n} H_{n+1}(u) - \frac{\beta}{\alpha} H_{n+1}(v). \tag{22}$$

Now take $u = Q(b,a,c \mid x)$ and $v = Q(a,b,c \mid x)$, so that $u = \alpha + \beta v$ by Lemma 2.4, where $\alpha = c(a-b)/a(c-b)$ and $\beta = b(c-a)/a(c-b)$. Then by Lemma 2.3, we have

$$\begin{aligned} \frac{H_{n+1}(u)}{H_{n+1}(v)} &= \prod_{i=1}^{n} \frac{(a+1)_i}{(a)_i} \frac{(b)_i}{(b+1)_i} \frac{(c-b+1)_i}{(c-b)_i} \frac{(c-a)_i}{(c-a+1)_i} \\ &= \prod_{i=1}^{n} \frac{b(c-a)}{a(c-b)} \frac{(a+i)(c-b+i)}{(b+i)(c-a+i)} = \left[\frac{b(c-a)}{a(c-b)} \right]^n \frac{(a+1)_n (c-b+1)_n}{(b+1)_n (c-a+1)_n}, \end{aligned} \tag{23}$$

and by (22) we have

$$\frac{H_n^{(2)}(v)}{H_{n+1}(v)} = \frac{a(c-b)}{c(a-b)} \left[\frac{a(c-b)}{b(c-a)} \right]^n \frac{H_{n+1}(u)}{H_{n+1}(v)} - \frac{b(c-a)}{c(a-b)}. \tag{24}$$

The result follows from (23) and (24). □

Tamm [28] evaluated the determinants U_n and V_n by first showing that

$$\sum_{n=0}^{\infty} a_n x^n = {}_2F_1\left(\frac{2}{3}, \frac{4}{3}; \frac{3}{2} \;\middle|\; \frac{27}{4}x\right) \Big/ {}_2F_1\left(\frac{2}{3}, \frac{1}{3}; \frac{1}{2} \;\middle|\; \frac{27}{4}x\right). \tag{25}$$

Given (25), it follows from Lemma 2.3 that

$$U_n = \prod_{i=1}^{n-1} \frac{(\frac{2}{3})_i(\frac{1}{6})_i(\frac{4}{3})_i(\frac{5}{6})_i}{(\frac{1}{2})_{2i}(\frac{3}{2})_{2i}} \left(\frac{27}{4}\right)^{2i}$$

and

$$V_n = \prod_{i=0}^{n} \frac{2}{3} \frac{(\frac{2}{3})_i(\frac{1}{6})_i(\frac{1}{3})_i(-\frac{1}{6})_i}{(\frac{1}{2})_{2i}(-\frac{1}{2})_{2i}} \left(\frac{27}{4}\right)^{2i}$$

So (4) and (5) will follow from

$$\frac{(\frac{2}{3})_i(\frac{1}{6})_i(\frac{4}{3})_i(\frac{5}{6})_i}{(\frac{1}{2})_{2i}(\frac{3}{2})_{2i}} \left(\frac{27}{4}\right)^{2i} = \frac{(3i+1)(6i)!\,(2i)!}{(4i+1)!\,(4i)!} \tag{26}$$

and

$$\frac{2}{3} \frac{(\frac{2}{3})_i(\frac{1}{6})_i(\frac{1}{3})_i(-\frac{1}{6})_i}{(\frac{1}{2})_{2i}(-\frac{1}{2})_{2i}} \left(\frac{27}{4}\right)^{2i} = \frac{\binom{6i-2}{2i}}{2\binom{4i-1}{2i}} \tag{27}$$

for $i \geq 1$. These identities are most easily verified by using the fact that if $A_1 = B_1$ and $A_{i+1}/A_i = B_{i+1}/B_i$ for $i \geq 1$, then $A_i = B_i$ for all $i \geq 1$. It is interesting to note that although (26) holds for $i = 0$, (27) does not.

3 Hypergeometric series evaluations

Let $f = g - 1 = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{1}{2n+1}\binom{3n}{n} x^n$. In this section we study cases of Gauss's continued fraction (17) that can be expressed in terms of f. We found empirically that there are ten cases of (17) that can be expressed as polynomials in f. We believe there are no others, but we do not have a proof of this. Since $a \neq b$ in all of these cases, by Lemma 2.4 they must come in pairs which are the same, except for their constant terms, up to a constant factor. It turns out that one element of each of these pairs factors as $(1+f)(1+rf)$, where r is 0, 1, $\frac{1}{2}$, $-\frac{1}{2}$, or $\frac{2}{5}$, while the other does not factor nicely. We have no explanation for this phenomenon.

Note that (28a) is the same as (25).

Theorem 3.1. *We have the following cases of Gauss's continued fraction:*

$$1+f = {}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{3}{2}\;\middle|\;\frac{27}{4}x\right)\Big/\,{}_2F_1\left(\frac{2}{3},\frac{1}{3};\frac{1}{2}\;\middle|\;\frac{27}{4}x\right) \tag{28a}$$

$$(1+f)^2 = {}_2F_1\left(\frac{4}{3},\frac{5}{3};\frac{5}{2}\;\middle|\;\frac{27}{4}x\right)\Big/\,{}_2F_1\left(\frac{4}{3},\frac{2}{3};\frac{3}{2}\;\middle|\;\frac{27}{4}x\right) \tag{28b}$$

$$(1+f)(1+\tfrac{1}{2}f) = {}_2F_1\left(\frac{5}{3},\frac{7}{3};\frac{7}{2}\;\middle|\;\frac{27}{4}x\right)\Big/\,{}_2F_1\left(\frac{5}{3},\frac{4}{3};\frac{5}{2}\;\middle|\;\frac{27}{4}x\right) \tag{28c}$$

$$(1+f)(1-\tfrac{1}{2}f) = {}_2F_1\left(\frac{5}{3},\frac{7}{3};\frac{5}{2}\;\middle|\;\frac{27}{4}x\right)\Big/\,{}_2F_1\left(\frac{5}{3},\frac{4}{3};\frac{3}{2}\;\middle|\;\frac{27}{4}x\right) \tag{28d}$$

$$(1+f)(1+\tfrac{2}{5}f) = {}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{5}{2}\;\middle|\;\frac{27}{4}x\right)\Big/\,{}_2F_1\left(\frac{2}{3},\frac{1}{3};\frac{3}{2}\;\middle|\;\frac{27}{4}x\right) \tag{28e}$$

Their companions are

$$1-\tfrac{1}{2}f = {}_2F_1\left(\frac{1}{3},\frac{5}{3};\frac{3}{2}\;\middle|\;\frac{27}{4}x\right)\Big/\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{1}{2}\;\middle|\;\frac{27}{4}x\right) \tag{29a}$$

$$1+\tfrac{1}{5}f+\tfrac{1}{10}f^2 = {}_2F_1\left(\frac{2}{3},\frac{7}{3};\frac{5}{2}\;\middle|\;\frac{27}{4}x\right)\Big/\,{}_2F_1\left(\frac{2}{3},\frac{4}{3};\frac{3}{2}\;\middle|\;\frac{27}{4}x\right) \tag{29b}$$

$$1+\tfrac{6}{7}f+\tfrac{2}{7}f^2 = {}_2F_1\left(\frac{4}{3},\frac{8}{3};\frac{7}{2}\;\middle|\;\frac{27}{4}x\right)\Big/\,{}_2F_1\left(\frac{4}{3},\frac{5}{3};\frac{5}{2}\;\middle|\;\frac{27}{4}x\right) \tag{29c}$$

$$1-\tfrac{2}{5}f+\tfrac{2}{5}f^2 = {}_2F_1\left(\frac{4}{3},\frac{8}{3};\frac{5}{2}\;\middle|\;\frac{27}{4}x\right)\Big/\,{}_2F_1\left(\frac{4}{3},\frac{5}{3};\frac{3}{2}\;\middle|\;\frac{27}{4}x\right) \tag{29d}$$

$$1+\tfrac{1}{2}f+\tfrac{1}{7}f^2 = {}_2F_1\left(\frac{1}{3},\frac{5}{3};\frac{5}{2}\;\middle|\;\frac{27}{4}x\right)\Big/\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{3}{2}\;\middle|\;\frac{27}{4}x\right) \tag{29e}$$

In order to prove Theorem 3.1, we need formulas for some rational functions of f that are easily proved by Lagrange inversion.

Lemma 3.2. *Let* $f=\sum_{n=1}^{\infty}\frac{1}{2n+1}\binom{3n}{n}x^n$*. Then* f *satisfies the functional equation* $f=x(1+f)^3$ *and*

$$f^k = \sum_{n=k}^{\infty}\frac{k}{n}\binom{3n}{n-k}x^n \tag{30}$$

$$(1+f)^k = \sum_{n=0}^{\infty}\frac{k}{3n+k}\binom{3n+k}{n}x^n \tag{31}$$

$$\frac{(1+f)^{k+1}}{1-2f} = \sum_{n=0}^{\infty}\binom{3n+k}{n}x^n. \tag{32}$$

In particular,

$$1 + f = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{3}{2} \;\middle|\; \frac{27}{4}x\right) \tag{33}$$

$$\frac{1+f}{1-2f} = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2} \;\middle|\; \frac{27}{4}x\right) \tag{34}$$

$$\frac{(1+f)^2}{1-2f} = {}_2F_1\left(\frac{4}{3}, \frac{2}{3}; \frac{3}{2} \;\middle|\; \frac{27}{4}x\right). \tag{35}$$

Proof. We use the following form of the Lagrange inversion formula (see [9, Theorem 2.1] or [13, Theorem 1.2.4]): If $G(t)$ is a formal power series, then there is a unique formal power series $h = h(x)$ satisfying $h = xG(h)$, and

$$[x^n]\, h^k = \frac{k}{n}[t^{n-k}]\, G(t)^n, \text{ for } n, k > 0, \tag{36}$$

$$[x^n]\, \frac{h^k}{1 - xG'(h)} = [t^{n-k}]\, G(t)^n, \text{ for } n, k \geq 0. \tag{37}$$

Let us define f to be the unique formal power series satisfying $f = x(1+f)^3$. With $G(t) = (1+t)^3$, (36) gives (30), and the case $k = 1$ gives that the coefficient of x^n in f for $n \geq 1$ is $\frac{1}{n}\binom{3n}{n-1} = \frac{1}{2n+1}\binom{3n}{n}$.

Replacing with f with $x(1+f)^3$ and k with j in (30), and dividing both sides by x^j, gives

$$(1+f)^{3j} = \sum_{n=0}^{\infty} \frac{j}{n+j}\binom{3n+3j}{n} x^n.$$

Since the coefficient of x^n on each side is a polynomial in j, we may set $j = k/3$ to obtain (31).

From (37) we have

$$\frac{f^j}{1 - 3x(1+f)^2} = \sum_{n=j}^{\infty} \binom{3n}{n-j} x^n.$$

Replacing f by $x(1+f)^3$ in the numerator, and replacing $x(1+f)^2$ by $f/(1+f)$ in the denominator, gives

$$\begin{aligned} \frac{x^j(1+f)^{3j+1}}{1-2f} &= \sum_{n=j}^{\infty} \binom{3n}{n-j} x^n \\ &= \sum_{n=0}^{\infty} \binom{3n+3j}{n} x^{n+j}, \end{aligned}$$

so

$$\frac{(1+f)^{3j+1}}{1-2f} = \sum_{n=0}^{\infty} \binom{3n+3j}{n} x^n.$$

As before, we may set $j = k/3$ to obtain (32). ∎

Proof of Theorem 3.1. Formulas (28a)–(28e) follow from the evaluations of their numerators and denominators: (33), (34), (35), and

$$ {}_2F_1\left(\frac{2}{3}, \frac{4}{3}; \frac{5}{2} \,\middle|\, \frac{27}{4}x\right) = (1+f)^2(1+\tfrac{2}{5}f) \tag{38} $$

$$ {}_2F_1\left(\frac{4}{3}, \frac{5}{3}; \frac{5}{2} \,\middle|\, \frac{27}{4}x\right) = \frac{(1+f)^4}{1-2f} \tag{39} $$

$$ {}_2F_1\left(\frac{5}{3}, \frac{7}{3}; \frac{7}{2} \,\middle|\, \frac{27}{4}x\right) = \frac{(1+f)^5(1+\frac{1}{2}f)}{1-2f} \tag{40} $$

$$ {}_2F_1\left(\frac{4}{3}, \frac{5}{3}; \frac{3}{2} \,\middle|\, \frac{27}{4}x\right) = \frac{(1+f)^4}{(1-2f)^3} \tag{41} $$

$$ {}_2F_1\left(\frac{5}{3}, \frac{7}{3}; \frac{5}{2} \,\middle|\, \frac{27}{4}x\right) = \frac{(1+f)^5(1-\frac{1}{2}f)}{(1-2f)^3}. \tag{42} $$

Our original derivations of these formulas were through the ${}_2F_1$ contiguous relations [1, p. 558], but once we have found them, we can verify (38)–(40) by by taking appropriate linear combinations of (30) and (32). Formulas (41) and (42) can be proved by applying the formula

$$ {}_2F_1\left(a+1, b+1; c+1 \mid x\right) = \frac{c}{ab}\frac{d}{dx}{}_2F_1\left(a, b; c \mid x\right) $$

to (34) and (35) and using the fact that $df/dx = (1+f)^4/(1-2f)$.

Formulas (29a)–(29e) can be proved similarly; alternatively, they can be derived from (28a)–(28e) by using Lemma 2.4. ☐

Now we apply Lemma 2.3 to the formulas of Theorem 3.1. First we normalize the coefficient sequences that occur in (28a)–(28e) to make them integers, using (30) to find formulas for the coefficients. We define the sequence a_n, b_n, c_n, d_n, and e_n by

$$ 1+f = \sum_{n=0}^{\infty} a_n x^n \qquad a_n = \frac{(3n)!}{n!\,(2n+1)!} = \frac{1}{2n+1}\binom{3n}{n} $$

$$ (1+f)^2 = \sum_{n=0}^{\infty} b_n x^n \qquad b_n = \frac{(3n+1)!}{(n+1)!\,(2n+1)!} = \frac{1}{n+1}\binom{3n+1}{n} $$

$$ (1+f)(2+f) = \sum_{n=0}^{\infty} c_n x^n \qquad c_n = 2\frac{(3n)!}{(n+1)!\,(2n)!} = \frac{2}{n+1}\binom{3n}{n} = a_n + b_n $$

$$ (1+f)(2-f) = \sum_{n=0}^{\infty} d_n x^n \qquad d_n = 2\frac{(3n)!}{(n+1)!\,(2n+1)!} = 3a_n - b_n $$

$$ (1+f)(5+2f) = \sum_{n=0}^{\infty} e_n x^n \qquad e_n = (9n+5)\frac{(3n)!}{(n+1)!\,(2n+1)!} = 3a_n + 2b_n $$

Here is a table of the first few values of these numbers

n	0	1	2	3	4	5	6	7
a_n	1	1	3	12	55	273	1428	7752
b_n	1	2	7	30	143	728	3876	21318
c_n	2	3	10	42	198	1001	5304	29070
d_n	2	1	2	6	22	91	408	1938
e_n	5	7	23	96	451	2275	12036	65892

The sequences a_n and b_n are well-known, and have simple combinatorial interpretations in terms of lattice paths: a_n is the number of paths, with steps $(1,0)$ and $(0,1)$, from $(0,0)$ to $(2n,n)$ that never rise above (but may touch) the line $x=2y$ and b_n is the number of paths from $(0,0)$ to $(2n,n)$ that never rise above (but may touch) the line $x=2y-1$ (see, e.g., Gessel [10]). Moreover, for $n>0$, d_n is the number of two-stack-sortable permutations of $\{1,2,\ldots,n\}$. (See, e.g., West [29] and Zeilberger [30].) The sequences c_n and e_n are apparently not well-known.

Let us write $H_n(a)$ for $H_n\bigl(\sum_{n=0}^{\infty} a_n x^n\bigr)$, and similarly for other letters replacing a. Then applying Lemma 2.3 and Theorem 3.1 gives

$$H_n(a)=\prod_{i=0}^{n-1}\frac{(\frac{2}{3})_i(\frac{1}{6})_i(\frac{4}{3})_i(\frac{5}{6})_i}{(\frac{1}{2})_{2i}(\frac{3}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}$$

$$H_n^1(a)=\prod_{i=1}^{n}\frac{2}{3}\frac{(\frac{2}{3})_i(\frac{1}{6})_i(\frac{1}{3})_i(-\frac{1}{6})_i}{(\frac{1}{2})_{2i}(-\frac{1}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}$$

$$H_n(b)=\prod_{i=0}^{n-1}\frac{(\frac{4}{3})_i(\frac{5}{6})_i(\frac{5}{3})_i(\frac{7}{6})_i}{(\frac{3}{2})_{2i}(\frac{5}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}$$

$$H_n^1(b)=\prod_{i=1}^{n}\frac{(\frac{4}{3})_i(\frac{5}{6})_i(\frac{2}{3})_i(\frac{1}{6})_i}{(\frac{3}{2})_{2i}(\frac{1}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}$$

$$H_n(c)=\prod_{i=0}^{n-1}2\frac{(\frac{5}{3})_i(\frac{7}{6})_i(\frac{7}{3})_i(\frac{11}{6})_i}{(\frac{5}{2})_{2i}(\frac{7}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}$$

$$H_n^1(c)=\prod_{i=1}^{n}\frac{(\frac{5}{3})_i(\frac{7}{6})_i(\frac{4}{3})_i(\frac{5}{6})_i}{(\frac{5}{2})_{2i}(\frac{3}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}$$

$$H_n(d)=\prod_{i=0}^{n-1}2\frac{(\frac{5}{3})_i(\frac{1}{6})_i(\frac{7}{3})_i(\frac{5}{6})_i}{(\frac{5}{2})_{2i}(\frac{3}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}$$

$$H_n^1(d)=(-1)^n\prod_{i=1}^{n}\frac{(\frac{5}{3})_i(\frac{1}{6})_i(\frac{4}{3})_i(-\frac{1}{6})_i}{(\frac{3}{2})_{2i}(\frac{1}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}$$

$$H_n(e) = \prod_{i=0}^{n-1} 5\frac{(\frac{2}{3})_i(\frac{7}{6})_i(\frac{4}{3})_i(\frac{11}{6})_i}{(\frac{3}{2})_{2i}(\frac{5}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}$$
$$H_n^1(e) = \prod_{i=1}^{n} 2\frac{(\frac{2}{3})_i(\frac{7}{6})_i(\frac{1}{3})_i(\frac{5}{6})_i}{(\frac{3}{2})_{2i}(\frac{1}{2})_{2i}}\left(\frac{27}{4}\right)^{2i}$$

Here is a table of the values of these Hankel determinants:

n	1	2	3	4	5	6	7
$H_n(a)$	1	2	11	170	7429	920460	323801820
$H_n^1(a)$	1	3	26	646	45885	9304650	5382618660
$H_n(b)$	1	3	26	646	45885	9304650	5382618660
$H_n^1(b)$	2	11	170	7429	920460	323801820	323674802088
$H_n(c)$	2	11	170	7429	920460	323801820	323674802088
$H_n^1(c)$	3	26	646	45885	9304650	5382618660	8878734657276
$H_n(d)$	2	3	10	85	1932	120060	20648232
$H_n^1(d)$	1	2	10	133	4830	485460	136112196
$H_n(e)$	5	66	2431	252586	74327145	62062015500	147198472495020
$H_n^1(e)$	7	143	8398	1411510	677688675	928501718850	3628173844041420

It is apparent from the table that

$$U_n = H_n(a) = H_{n-1}^1(b) = H_{n-1}(c), \tag{43}$$

and that $V_n = H_n^1(a) = H_n(b) = H_{n-1}^1(c)$, and these are easily verified from the formulas. The combinatorial interpretations of U_n and V_n have already been discussed. The numbers $H_n(e)$ were shown by Kuperberg [19, Theorem 5] to count certain alternating sign matrices. In Kuperberg's notation, $H_n(e) = A_{\mathrm{UU}}^{(2)}(4n; 1, 1, 1)$.

There are also Hankel determinant evaluations corresponding to (29a)–(29e), normalized to make the entries integers. These evaluations can be found in Krattenthaler [17, Theorem 30].

4 Determinants and Two-Variable Generating Functions

In this section we describe a method for transforming determinants whose entries are given as coefficients of generating functions. (A related approach was used in [8] to evaluate Hankel determinants of Bell numbers.) Using this technique, we are able to convert the determinants for U_n and V_n in (1) and (2) into the known determinant evaluations given in (7) and (8). (Conversely, the evaluations of these Hankel determinants give new proofs of (7) and (8).) These two determinants are special cases of a determinant evaluation of

Mills, Robbins, and Rumsey [22] (see [16, Theorem 37] for related determinants):

$$\det\left(\binom{i+j+r}{2i-j}\right)_{0\le i,j\le n-1}$$
$$= (-1)^{\chi(n\equiv 3 \bmod 4)} 2^{\binom{n-1}{2}} \prod_{i=1}^{n-1} \frac{(r+i+1)_{\lfloor (i+1)/2\rfloor}(-r-3n+i+\frac{3}{2})_{\lfloor i/2\rfloor}}{(i)_i}, \quad (44)$$

where $\chi(S)=1$ if S is true and $\chi(S)=0$ otherwise. There exist short direct proofs of (44) (see [3, 15, 24]), but no really simple proof.

Suppose that we have a two-variable generating function

$$D(x,y)=\sum_{i,j=0}^{\infty} d_{i,j}x^i y^j.$$

Let $[D(x,y)]_n$ be the determinant of the $n\times n$ matrix

$$(d_{i,j})_{0\le i,j\le n-1}.$$

The following rules can be used to transform the determinant $[D(x,y)]_n$ to a determinant with the same value:

Constant Rules. Let c be a non-zero constant. Then

$$[cD(x,y)]_n = c^n[D(x,y)]_n,$$

and

$$[D(cx,y)]_n = c^{\binom{n}{2}}[D(x,y)]_n.$$

Product Rule. If $u(x)$ is any formal power series with $u(0)=1$, then

$$[u(x)D(x,y)]_n = [D(x,y)]_n.$$

Composition Rule. If $v(x)$ is any formal power series with $v(0)=0$ and $v'(0)=1$, then

$$[D(v(x),y)]_n = [D(x,y)]_n.$$

The product and composition rules hold because the transformed determinants are obtained from the original determinants by elementary row operations. Equivalently, the new matrix is obtained by multiplying the old matrix on the left by a matrix with determinant 1. Note that all of these transformations can be applied to y as well as to x.

The Hankel determinants $H_n(A)$ and $H_n^1(A)$ of a formal power series $A(x)$ are given by

$$H_n(A)=\left[\frac{xA(x)-yA(y)}{x-y}\right]_n, \quad (45)$$

$$H_n^1(A)=\left[\frac{A(x)-A(y)}{x-y}\right]_n. \quad (46)$$

Proof of (7) *and* (8). The generating function for the Hankel determinant $H_n(g)$ is

$$\frac{xg(x) - yg(y)}{x - y}. \tag{47}$$

Since $f/(1+f)^3 = x$, f is the compositional inverse of $x/(1+x)^3$, and thus $f\big(x/(1+x)^3\big) = x$. Since $g = 1 + f$, we have $g\big(x/(1+x)^3\big) = 1 + x$.

Now let us substitute $x \to x/(1+x)^3$, $y \to y/(1+y)^3$ in (47). After simplifying, we obtain

$$\frac{(1-xy)(1+x)(1+y)}{1 - xy^2 - 3xy - x^2y}.$$

Then dividing by $(1+x)(1+y)$, we get

$$\frac{1-xy}{1 - xy^2 - 3xy - x^2y}.$$

Next, we show that

$$\frac{1-xy}{1 - xy^2 - 3xy - x^2y} = \sum_{i,j} \binom{i+j}{2i-j} x^i y^j. \tag{48}$$

Multiplying both sides of (48) by $1 - xy^2 - 3xy - x^2y$ and equating coefficients of $x^m y^n$ shows that (48) is equivalent to the recurrence

$$\binom{m+n}{2m-n} - \binom{m+n-3}{2m-n} - 3\binom{m+n-2}{2m-n-1} - \binom{m+n-3}{2m-n-3} = \begin{cases} 1, & \text{if } m = n = 0 \\ -1, & \text{if } m = n = 1 \\ 0, & \text{otherwise} \end{cases}$$

where we interpret the binomial coefficient $\binom{a}{b}$ as 0 if either a or b is negative, and the verification of the recurrence is straightforward. (We will give another proof of (48) in Example 9.2.) This completes the proof of (7).

For equation (8), we need to consider the generating function

$$(g(x) - g(y))/(x - y).$$

Making the same substitution as before gives

$$\frac{(1+x)^3(1+y)^3}{1 - xy^2 - 3xy - x^2y}.$$

Dividing by $(1+x)^2(1+y)^3$ gives

$$\frac{1+x}{1 - xy^2 - 3xy - x^2y}, \tag{49}$$

which can be shown, by the same method as before, to equal

$$\sum_{i,j} \binom{i+j+1}{2i-j} x^i y^j. \qquad \square$$

Using the same approach, we can prove a result of Eğecioğlu, Redmond, and Ryavec [6] that gives another Hankel determinant for V_n. (It should be noted that our V_n is their V_{n-1}.) In Section 4 of [6] they define numbers μ_n and gave several characterizations for them. In later sections they transform the Hankel determinant for these numbers several times, as described on page 5 of their paper, ultimately reducing it to the Hankel determinant for the numbers $\binom{3n+2}{n}$. We will give a direct reduction of the generating function for these Hankel determinants to (49).

In their Theorem 2, Eğecioğlu, Redmond, and Ryavec give several characterizations of the numbers μ_n. We will use a characterization given not in the statement of this theorem, but in the proof, on page 16: the generating function

$$M(x) = \sum_{n=0}^{\infty} \mu_n x^{n+1} \tag{50}$$

satisfies

$$M(x) = x + 3xM(x)^2 + xM(x)^3. \tag{51}$$

Theorem 4.1. *Let μ_n be defined by* (50) *and* (51). *Then* $\det\left(\mu_{i+j}\right)_{0\le i,j\le n-1} = V_n$.

Proof. By (45),

$$\det\left(\mu_{i+j}\right)_{0\le i,j\le n-1} = \left[\frac{M(x)-M(y)}{x-y}\right]_n.$$

By (51), $M(x)$ is the compositional inverse of $x/(1+3x^2+x^3)$, so making the substitution $x \to x/(1+3x^2+x^3)$, $y \to y/(1+3y^2+y^3)$ in $(M(x)-M(y))/(x-y)$ and simplifying gives

$$\frac{(1+3x^2+x^3)(1+3y^2+y^3)}{1-xy^2-3xy-x^2y}.$$

Applying the product rule, we reduce this generating function to (49), for which the corresponding determinant, as we have seen, is V_n. ∎

We note that if (51) is replaced with $M(x) = x + \alpha x M(x) + 3xM(x)^2 + xM(x)^3$, where α is arbitrary, then the Hankel determinants are unchanged.

To transform in this way the more general determinant on the left side of (44), we would start with the generating function

$$\sum_{i,j}\binom{i+j+r}{2i-j}x^i y^j = \frac{\sum_{n=0}^{\infty}\left[\binom{r+n}{2n}+\binom{r+n-2}{2n-1}y\right]x^n}{1-xy^2-3xy-x^2y}. \tag{52}$$

The generating function in r of (52) is derived in (77). The sums in the numerator can be evaluated explicitly by making an appropriate substitution in the identities

$$\sum_{n=0}^{\infty}\binom{r+n}{2n}(-4\sin^2\theta)^n=\frac{\cos(2r+1)\theta}{\cos\theta}$$

$$\sum_{n=0}^{\infty}\binom{r+n-2}{2n-1}(-4\sin^2\theta)^n=-2\tan\theta\,\sin 2(r-1)\theta.$$

However we have not been able to use these formulas to prove (44).

Another application of this method gives a family of generating functions that have the same Hankel determinants.

Theorem 4.2. *Let $A(x)$ be a formal power series with $A(0)=1$ and let c be a constant. Then we have*

$$H_n\left(\frac{A(x)}{1-cxA(x)}\right)=H_n(A) \tag{53}$$

for all n, and

$$H_n\left(\frac{1}{1-cxA(x)}\right)=c^{n-1}H^1_{n-1}(A) \tag{54}$$

for $n\geq 1$.

Proof. We use the method of generating functions to evaluate these determinants. By (45),

$$\begin{aligned} H_n\left(\frac{A(x)}{1-cxA(x)}\right)&=\left[\frac{\dfrac{xA(x)}{1-cxA(x)}-\dfrac{yA(y)}{1-cyA(y)}}{x-y}\right]_n\\ &=\left[\frac{1}{(1-cxA(x))(1-cyA(y))}\frac{xA(x)-yA(y)}{x-y}\right]_n.\end{aligned}$$

Since$(1-cxA(x))^{-1}$ is a formal power series with constant term 1, we get

$$H_n\left(\frac{A(x)}{1-cxA(x)}\right)=\left[\frac{xA(x)-yA(y)}{x-y}\right]_n=H_n(A).$$

A similar computation shows that

$$\begin{aligned} H_n\left(\frac{1}{1-cxA(x)}\right)&=\left[1+cxy\frac{A(x)-A(y)}{x-y}\right]_n\\ &=\left[c\frac{A(x)-A(y)}{x-y}\right]_{n-1}=c^{n-1}H^1_{n-1}(A),\end{aligned}$$

since $[1+xyD(x,y)]_n$ is the determinant of a block matrix of two blocks, with the first block $[1]$ and the second block $[D(x,y)]_{n-1}$. ∎

We now prove (6) and (9). First we set $c = u - 1$ and $A = f/x$ in (53), getting

$$V_n = \det\left(a_{i+j+1}\right)_{0\le i,j\le n-1} = H_n(f/x) = H_n\left(\frac{f/x}{1+(1-u)f}\right).$$

Next we show that

$$\frac{f/x}{1+(1-u)f} = \sum_{n=0}^{\infty} s_n(u)x^n, \tag{55}$$

where

$$s_n(u) = \sum_{k=0}^{n} \frac{k+1}{n+1}\binom{3n-k+1}{n-k}u^k. \tag{56}$$

We have

$$\begin{aligned}
\frac{f/x}{1+(1-u)f} &= \frac{f/x}{1+f}\cdot\frac{1}{1-uf/(1+f)}\\
&= \frac{(1+f)^2}{1-ux(1+f)^2},\ \text{since } f = x(1+f)^3,\\
&= \sum_{k=0}^{\infty} u^k x^k (1+f)^{2k+2}\\
&= \sum_{k=0}^{\infty} u^k x^k \sum_{m=0}^{\infty} \frac{2k+2}{3m+2k+2}\binom{3m+2k+2}{m}x^m,\ \text{by (31)},\\
&= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \frac{k+1}{n+1}\binom{3n-k+1}{n-k}u^k,
\end{aligned}$$

which proves (55). Then $s_n(1) = a_{n+1}$ from (55), $s_n(0) = \frac{1}{n+1}\binom{3n+1}{n}$ by setting $u = 0$ in (56), and $s_n(3) = \binom{3n+2}{n}$ follows from (32). This completes the proof of (6).

Next we prove (9), which by (55) is equivalent to

$$H_n\left(u^{-1} + \frac{f}{1+(1-u)f}\right) = U_n/u. \tag{57}$$

We have

$$u^{-1} + \frac{f}{1+(1-u)f} = \frac{u^{-1}}{1-ux(1+f)^2},$$

so by (54), the Hankel determinant is equal to $u^{-1}H^1_{n-1}((1+f)^2)$. In the notation of Section 3, this is $u^{-1}H^1_{n-1}(b)$, which by (43) is equal to $u^{-1}U_n$.

We also have an analogue of Theorem 4.2 for the Hankel determinants H^1_n.

Theorem 4.3. *Let $A(x)$ be a formal power series with $A(0) = 1$ and let $c \neq 1$ be a constant. Then we have*

$$H^1_n\left(\frac{A(x)}{1-cA(x)}\right) = (1-c)^{-2n}H^1_n(A) \tag{58}$$

Proof. We use the method of generating functions. By (46),

$$H_n^1\left(\frac{A(x)}{1-cA(x)}\right) = \left[\frac{\dfrac{A(x)}{1-cA(x)} - \dfrac{A(y)}{1-cA(y)}}{x-y}\right]_n$$
$$= \left[\frac{1}{(1-cA(x))(1-cA(y))}\frac{A(x)-A(y)}{x-y}\right]_n.$$

Since $(1-cA(x))^{-1}$ is a formal power series with constant term $(1-c)^{-1}$ when $c \neq 1$, we get

$$H_n^1\left(\frac{(1-c)^2A(x)}{1-cA(x)}\right) = \left[(1-c)^{-2}\frac{A(x)-A(y)}{x-y}\right]_n = (1-c)^{-2n}H_n^1(A). \qquad \square$$

5 A Hankel Determinant for the Number of Alternating Sign Matrices

Let $\mathcal{A}_n$ be the number of $n \times n$ alternating sign matrices. It is well-known that

$$\mathcal{A}_n = \prod_{k=0}^{n-1}\frac{(3k+1)!}{(n+k)!},$$

as conjectured by Mills Robbins and Rumsey [21] and proved by Zeilberger [31] and Kuperberg [18].

The numbers $\mathcal{A}_n$ also count totally symmetric, self-complementary plane partitions, as shown by Andrews [2]. We find, up to a power of 3, a Hankel determinant expression for $\mathcal{A}_n$.

Let

$$\hat{C}(x) = \frac{1-(1-9x)^{1/3}}{3x}. \tag{59}$$

The coefficients of $\hat{C}(x)$ are positive integers that are analogous to Catalan numbers. They have no known combinatorial interpretation and have been little studied, but they do appear in [20, Eq. 61].

Theorem 5.1. *The number of $n \times n$ alternating sign matrices is*

$$\mathcal{A}_n = 3^{-\binom{n}{2}}H_n(\hat{C}). \tag{60}$$

Proof. Let

$$D(x,y) = (x\hat{C}(x) - y\hat{C}(y))/(x-y)$$

be the generating function for the Hankel determinant $H_n(\hat{C})$. It is easy to see that $D(x/\sqrt{3}, y/\sqrt{3})$ is the generating function for $3^{-\binom{n}{2}}H_n(\hat{C})$. We make the substitution

$x \to x - \sqrt{3}x^2 + x^3$, $y \to y - \sqrt{3}y^2 + y^3$ in $D(x/\sqrt{3}, y/\sqrt{3})$, and simplify. The generating function becomes

$$\frac{1}{1-\sqrt{3}(x+y)+x^2+xy+y^2}.$$

Let $\omega = -\frac{1}{2} - \frac{\sqrt{-3}}{2}$ be a cube root of unity. Make another substitution $x \to -\sqrt{-1}x/(1+\omega x), y \to \sqrt{-1}y/(1+\omega^2 y)$, and simplify. The generating function becomes

$$\frac{(1+\omega x)^2\,(1+\omega^2 y)^2}{(1-xy)(1-x-y)}.$$

Dividing by $(1+\omega x)^2\,(1+\omega^2 y)^2$, the generating function becomes

$$\frac{1}{(1-x-y)(1-xy)}.$$

Multiplying by $(1-x+x^2)(1-y)/(1-x)$, we get

$$\frac{(1-x+x^2)(1-y)}{(1-x)(1-x-y)(1-xy)} = \frac{x}{y(1-x-y)} + \frac{1}{1-xy} - \frac{x}{y(1-x)}.$$

Expanding the right-hand side of the above equation, we get

$$H_n(\hat{C}) = 3^{\binom{n}{2}} \det\left(\binom{i+j}{i-1} + \delta_{i,j}\right)_{0\le i,j\le n-1},$$

where $\delta_{i,j}$ equals 1 if $i = j$ and 0 otherwise. The theorem then follows from a known formula for $\mathcal{A}_n$ [5, p. 22]. $\square$

Remark 5.2. *We have another determinant expression*

$$\mathcal{A}_n = \det\left(\binom{i+j}{i} - \delta_{i,j+1}\right),$$

since

$$\frac{1}{(1-x-y)(1-xy)} = \frac{1}{1-y+y^2}\left(\frac{1}{1-x-y} - \frac{y}{1-xy}\right).$$

There is a result similar to Theorem 5.1

$$\hat{C}_1(x) = \frac{1-(1-9x)^{2/3}}{3x}.$$

Let $\mathcal{A}'_n$ be the number of cyclically symmetric plane partitions in the n-cube. We have

Theorem 5.3.

$$\mathcal{A}'_n = 3^{-\binom{n}{2}} H_n(\hat{C}_1). \tag{61}$$

Proof. Let

$$D(x,y) = (x\hat{C}_1(x) - y\hat{C}_1(y))/(x-y)$$

be the generating function for the Hankel determinant $H_n(\hat{C}_1)$. Similarly $D(x/\sqrt{3}, y/\sqrt{3})$ is the generating function for $3^{-\binom{n}{2}} H_n(\hat{C}_1)$. We make the same substitution (as for $H_n(\hat{C})$) $x \to x - \sqrt{3}x^2 + x^3$, $y \to y - \sqrt{3}y^2 + y^3$, and simplify. The generating function becomes

$$\frac{2-\sqrt{3}(x+y)}{1-\sqrt{3}(x+y)+x^2+xy+y^2}.$$

Similarly, we make another substitution $x \to -\sqrt{-1}x/(1+\omega x), y \to \sqrt{-1}y/(1+\omega^2 y)$, and simplify. The generating function becomes

$$\frac{(2-x-y-xy)(1+\omega x)\,(1+\omega^2 y)}{(1-xy)(1-x-y)}.$$

Dividing by $(1+\omega x)\,(1+\omega^2 y)$, the generating function becomes

$$\frac{(2-x-y-xy)}{(1-x-y)(1-xy)} = \frac{1}{1-x-y} + \frac{1}{1-xy}.$$

So we have

$$H_n(\hat{C}_1) = 3^{\binom{n}{2}} \det\left(\binom{i+j}{i} + \delta_{i,j}\right)_{0\le i,j\le n-1},$$

which is equal to $3^{\binom{n}{2}}\mathcal{A}'_n$. (See [5, p. 177, (5.28)].) $\square$

Since $\hat{C}(x) = {}_2F_1\left(\frac{2}{3}, 1; 2 \mid 9x\right)$, we can find a continued fraction for $\hat{C}(x)$ by setting $a = \frac{2}{3}, b = 0, c = 1$ in Lemma 2.2, and thus evaluate the Hankel determinant for $\hat{C}(x)$ by Lemma 2.1. Similarly, since $\hat{C}_1(x) = 2\,{}_2F_1\left(\frac{1}{3}, 1; 2 \mid 9x\right)$, we can evaluate the Hankel determinant for $\hat{C}_1(x)$ be taking $a = \frac{1}{3}, b = 0, c = 1$ in Lemma 2.2.

The Hankel determinants $H_n(\hat{C})$ and $H_n(\hat{C}_1)$, can also be evaluated by applying a more general result (see, e.g., [16, Theorem 26, Eq. (3.12)]):

$$\det\left(\binom{A}{L_i+j}\right)_{1\le i,j\le n} = \frac{\prod_{1\le i<j\le n}(L_i-L_j)}{\prod_{1\le i\le n}(L_i+n)!}\frac{\prod_{1\le i\le n}(L_i+A+1)!}{\prod_{1\le i\le n}(A+1-i)!}, \tag{62}$$

where $L_1, \ldots, L_n$ and A are indeterminates, and the factorials are interpreted using gamma functions when necessary.

Thus these calculations give a simple method of evaluating the determinants

$$\det\left(\binom{i+j}{i-1} + \delta_{i,j}\right)_{0\le i,j\le n-1} \quad\text{and}\quad \det\left(\binom{i+j}{i} + \delta_{i,j}\right)_{0\le i,j\le n-1}.$$

For more information on similar determinants, see Krattenthaler [16, Theorems 32–35] [17, Section 5.5].

6 A Combinatorial Proof of (7)

For the reader's convenience, we restate equation (7) as follows:

$$\det\left(a_{i+j}\right)_{0\le i,j\le n-1} = \det\left(\binom{i+j}{2i-j}\right)_{0\le i,j\le n-1}. \tag{63}$$

Both sides of (63) have combinatorial meanings in terms of nonintersecting paths (see Gessel and Viennot [11]). The right-hand side counts $\mathcal{U}_R(n)$, the set of n-tuples of nonintersecting paths from $P'_0,\dots,P'_{n-1}$ to $Q'_0,\dots,Q'_{n-1}$, where $P'_i=(i,-2i)$ and $Q'_i=(2i,-i)$. For the paths to be nonintersecting, P'_i must go to Q'_i. See the right picture of Figure 1. Mills, Robbins, and Rumsey [15] in fact gave a bijection from the type (a) objects of Section 1 to such n-tuples of lattice paths.

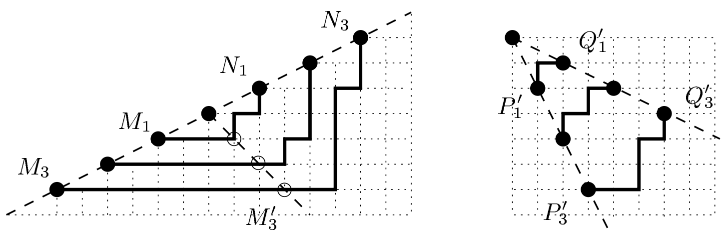

Figure 1: Lattice path interpretation of (63).

For the left-hand side, we notice that a_n counts the number of paths from $(0,0)$ to $(2n,n)$ that never go above the line $y=x/2$. See, e.g., [10]. It is easy to see that the left-hand side of (63) counts $\mathcal{U}_L(n)$, the set of n-tuples of nonintersecting paths that stay below the line $y=x/2$, from $M_0,\dots,M_{n-1}$ to $N_0,\dots,N_{n-1}$, where $M_i=(-2i,-i)$ and $N_i=(2i,i)$. For the paths to be nonintersecting, M_i must go to N_i. Moreover, from the left picture of Figure 1, we see that M_i can be replaced with $M'_i=(i,-i)$.

An interesting problem is to find a bijection from $\mathcal{U}_L(n)$ to $\mathcal{U}_R(n)$. Such a bijection will result in a combinatorial enumeration of the type (a) objects.

Both $\mathcal{U}_L(n)$ and $\mathcal{U}_R(n)$ can be easily converted into variations of plane partitions. But we have not found it helpful.

We find an alternative bijective proof of (63). The algebraic idea behind the proof is the following matrix identity that implies (63):

$$\left(a_{i+j}\right)_{0\le i,j\le n-1} = \left(\frac{3j+1}{3i+1}\binom{3i+1}{i-j}\right)_{0\le i,j\le n-1} \left(\binom{i+j}{2i-j}\right)_{0\le i,j\le n-1} \left(\frac{3i+1}{3j+1}\binom{3j+1}{j-i}\right)_{0\le i,j\le n-1}, \tag{64}$$

where

$$\frac{3i+1}{3j+1}\binom{3j+1}{j-i} = [x^i]gf^j$$

(See (30)). Note that the left (right) transformation matrix is a lower (upper) triangular matrix with diagonal entries 1. The matrix identity is obtained by carefully analyzing the transformation we performed in Section 4 when proving (7).

The bijective proof relies on a new interpretation of a_n in terms of certain paths that we call K-paths. The matrix identity (64) follows easily from the new interpretation. This gives a bijection from $\mathcal{U}_R(n)$ to $\mathcal{U}_K(n)$, the set of n-tuples of nonintersecting K-paths resulting from the new interpretation. The desired bijection could be completed by giving the bijection from $\mathcal{U}_K(n)$ to $\mathcal{U}_L(n)$. But we have not succeeded in this.

The new interpretation of a_n consists of three kinds of paths: normal paths, H_2-paths, and V_2-paths. A normal path has steps $(0,1)$ and $(1,0)$. A path is an H_2 path if each horizontal step is $(2,0)$ instead of $(1,0)$. By dividing each horizontal 2-step into two horizontal 1-steps, we can represent an H_2 path as a normal path. Similarly, a path is a V_2 path if each vertical step is $(0,2)$.

By reflecting in the line $y=-x$, we can convert an H_2 path into a V_2 path, or a V_2 path into an H_2 path. This bijection can convert any property of H_2-paths into a similar property of V_2-paths.

It is well-known that the number of paths that start at $(0,0)$, end at $(n,2n)$, and never go above the line $y=2x$ is $a_n=\frac{1}{2n+1}\binom{3n}{n}$. Replacing each horizontal step by two horizontal steps, it follows that:

Proposition 6.1. *The number of H_2-paths (or V_2-paths) that start at $(0,0)$, end at $(2n,2n)$ and never go above the diagonal equals a_n.*

Definition 6.2. *We call a path P a K-path if it satisfies the following four conditions.*

1. *The path P never goes above the diagonal.*
2. *The part of P that is below the line $y=-2x$ is a V_2 path.*
3. *The part of P between the two lines $y=-2x$ and $x=-2y$ is a normal path.*
4. *The part of P that is above the line $x=-2y$ is an H_2 path.*

From the definition, we see that a K-path can be uniquely decomposed into three kinds of paths: a V_2 path, followed by a normal path, followed by an H_2 path. Depending on its starting point, some of the paths may be empty. The normal path region is between the two lines $y=-2x$ and $x=-2y$. The steps occurring in a K-path are shown in Figure 2. We have

Theorem 6.3. *The number of K-paths from $(-2m,-2m)$ to $(2n,2n)$, where $m+n\geq 0$, is a_{m+n}.*

The proof of the theorem will be given later. From the new interpretation of a_n, U_n counts $\mathcal{U}_K(n)$, the set of n-tuples of nonintersecting K-paths from $P_0,\dots,P_{n-1}$ to $Q_0,\dots,Q_{n-1}$, where $P_i=(-2i,-2i)$, and $Q_i=(2i,2i)$ for $i=0,1,\dots,n-1$. See Figure 2. For the paths to be nonintersecting, P_i must go to Q_i. In such an n-tuples

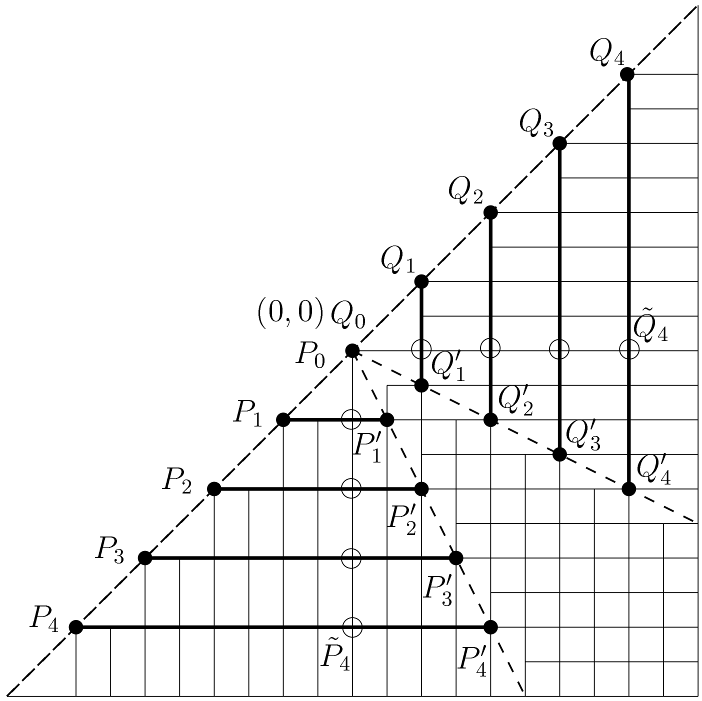

Figure 2: The grid for K-paths.

of nonintersecting K-paths, the path from P_i to Q_i must start with a path from $P_i = (-2i, -2i)$ to $P'_i = (i, -2i)$, and end with a path from $Q'_i = (2i, -i)$ to $Q_i = (2i, 2i)$. So $\mathcal{U}_K(n)$ is in natural bijection with $\mathcal{U}_R(n)$. If we count the number of K-paths according to their intersections with the lines $y = -2x$ and $x = -2y$, we get the matrix identity (64).

If P is a K-path from $(-2m, -2m)$ to $(2n, 2n)$ with $m \le 0$ (or $n \le 0$), then P is an H_2 (or a V_2)-path, and Theorem 6.3 follows from Proposition 6.1. So we can assume that m and n are both positive integers.

The idea of the proof of Theorem 6.3 is to show that the number of K-paths from $(-2m, -2m)$ to $(2n, 2n)$ is unchanged after sliding their starting and ending points along the diagonal by $(2, 2)$.

In fact, the following refinement is true. See Figure 3.

Lemma 6.4 (Sliding Lemma). *The number of K-paths from $(i-2, -2i-2)$ to $(2j, -j)$ equals the number of K-paths from $(i, -2i)$ to $(2j+2, -j+2)$.*

Proof. Let $N(i, j)$ be the number of K-paths from $A_i = (i-2, -2i-2)$ to $B_j = (2j, -j)$. It is clear that $N(i, j) = 0$ if $i < 0$ or $j < 0$.

By reflecting in the line $y = -x$, we can give a bijective proof of the following statement: The number of K-paths from $(i, -2i)$ to $(2j+2, -j+2)$ equals the number of K-paths from $(j-2, -2j-2)$ to $(2i, -i)$, which is $N(j, i)$. Therefore it suffices to show that $N(i, j) = N(j, i)$.

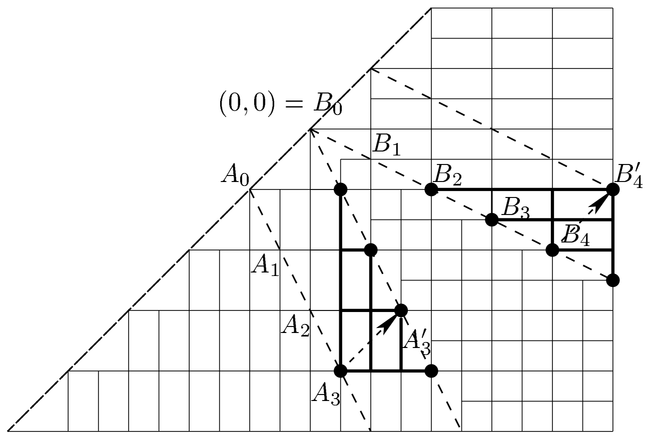

Figure 3: Picture for the sliding lemma.

The cases $i = 0$ and $i = 1$ correspond to starting at A_0 and A_1. From Figure 3, we can check that $N(i, j) = N(j, i)$ directly. We have

$i \backslash j$	0	1	2	3	4
0	1	2	1	0	0
1	2	5	9	5	1
2	1	9	$*$	$*$	$*$
3	0	5	$*$	$*$	$*$
4	0	1	$*$	$*$	$*$

and $N(i, j) = 0$ if one of i, j is 0 or 1 and the other is great than 4.

In the case $i \geq 2$, we count the number of K-paths from A_i to B_j according to its intersection with the line $y = -2x$. From Figure 3, we see that there are 4 possible intersection points. We have

$$\begin{aligned} N(i, j) = B(2j - i + 2, 2i - j - 4) + 3B(2j - i + 1, 2i - j - 2) \\ + 3B(2j - i, 2i - j) + B(2j - i - 1, 2i - j + 2), \end{aligned} \tag{65}$$

where $B(a, b) = \binom{a+b}{b}$.

Let $M(a, b)$ be defined by

$$M(a, b) = B(a + 2, b - 4) + 3B(a + 1, b - 2) + 3B(a, b) + B(a - 1, b + 2).$$

Then $N(i, j) = M(2j - i, 2i - j)$. We need to show that $M(a, b) = M(b, a)$ for $a + b \geq 2$, which implies $N(i, j) = N(j, i)$ for $i \geq 2$. Using the basic identity of binomial coefficients $B(c, d) = B(c - 1, d) + B(c, d - 1)$ for all integers c and d, when $a + b \geq 2$, we have

$$\begin{aligned} M(a, b) = B(a - 4, b + 2) + 3B(a - 3, b + 1) + 6B(a - 2, b) + 7B(a - 1, b - 1) \\ + 6B(a, b - 2) + 3B(a + 1, b - 3) + B(a + 2, b - 4). \end{aligned} \tag{66}$$

In the following Figure 4, every number we put at a point is the sum of the numbers at points that are to the left of it or under it. This corresponds to the formula $B(c,d) = B(c-1,d) + B(c,d-1)$.

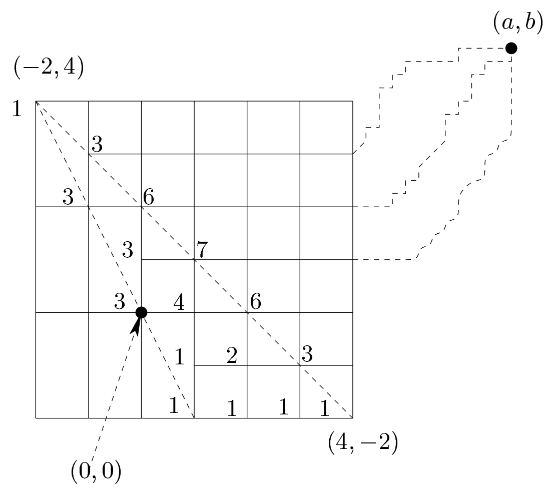

Figure 4: Proof of equation (66) by picture.

By the symmetry property $B(c,d) = B(d,c)$, we have $M(a,b) = M(b,a)$. This completes the proof. $\square$

Remark 6.5. *Observe that the symmetry property of the numbers* $(1,3,6,7,6,3,1)$ *along the diagonal in Figure 4 implies* (66). *A bijective proof of this symmetry will induce a bijective proof of* $N(i,j) = N(j,i)$, *and then a bijective proof of Lemma 6.4.*

Proof of Theorem 6.3. Let $G(m,n)$ be the number of K-paths starting at $(-2m,-2m)$ and ending at $(2n,2n)$. We will prove that $G(m-1,n+1) = G(m,n)$ for all $m > 0$. Then by induction, $G(m,n) = G(0,m+n) = a_{m+n}$.

We give the bijection as follows. Given a K-path P from $(-2m,-2m)$ to $(2n,2n)$, we separate it by the two lines $y+2 = -2(x+2)$ and $x = -2y$ into three parts: a V_2-path P_1, followed by a K-path P_2, followed by an H_2-path P_3.

Applying the bijection in the sliding lemma (Lemma 6.4) for P_2, we get P_2', a K-path starting on the line $y = -2x$, ending on the line $x-2 = -2(y-2)$. Then $P' = P_1P_2'P_3$ with starting point $(-2m+2,-2m+2)$ is the desired K-path.

A similar argument gives the inverse bijection. $\square$

This bijective proof of Theorem 6.3 is not ideal, though it is sufficient to prove the determinant formula (63). The proof relies on the sliding lemma, whose proof involves a case by case bijection that is not explicitly given. We would prefer a *natural* bijection for the sliding lemma that preserves the nonintersecting properties of K-paths. This is because such a bijection would give rise to a bijection from $\mathcal{U}_K(n)$ to $\mathcal{U}_L(n)$: we could slide the n-tuples of K-paths in $\mathcal{U}_K(n)$ so that all the paths are above the line $x = -2y$. Then the resulting paths would all be H_2-paths that can be easily converted into normal paths in $\mathcal{U}_L(n)$.

7 Trinomial Coefficients and a_n

In this section, we introduce T-paths that are counted by *trinomial coefficients*. The trinomial coefficient $T(a,b)$ is defined by

$$T(a,b) = [x^a y^b](x^2 + xy + y^2)^{\frac{a+b}{2}},$$

if $a+b$ is even, and $T(a,b) = 0$ otherwise.

The trinomial coefficients $T(a,b)$ have a simple combinatorial interpretation: We call a path P a T-path if each step of P is $(2,0)$ or $(1,1)$ or $(0,2)$. Then the number of T-paths that start at $(0,0)$ and end at (a,b) is $T(a,b)$. This follows easily from the definition of $T(a,b)$. See the following Figure 5, in which dots represent vertices of T-paths.

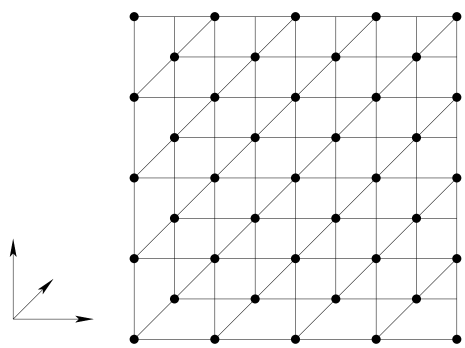

Figure 5: The grid for T-paths.

Using T-paths, we can give a better bijective proof of the sliding lemma. In addition, we find a new determinant identity (68).

Let $n = (a+b)/2$. We can get another formula for $T(a,b)$ in the following way. We write $(x^2+xy+y^2)^n = (x(x+y)+y^2)^n$ and use the binomial theorem twice:

$$\begin{aligned}(x(x+y)+y^2)^n &= \sum_{k=0}^{n} \binom{n}{k} y^{2(n-k)} x^k \sum_{l=0}^{k} \binom{k}{l} x^l y^{k-l} \\ &= \sum_{a=0}^{2n} \left(\sum_{k=0}^{a} \binom{n}{k} \binom{k}{a-k} \right) x^a y^{2n-a}.\end{aligned}$$

So

$$T(a,b) = \sum_{k=0}^{a} \binom{n}{k} \binom{k}{a-k}. \tag{67}$$

This algebraic fact gives another combinatorial explanation of $T(a,b)$:

Lemma 7.1. *The number of paths from $(0,-2m)$ to $(i,-i)$, in which the part below the line $y=-2x$ is a V_2 path, and the other part is a normal path, is equal to the number of T-paths from $(0,-2m)$ to $(i,-i)$, which is $T(i, 2m-i)$.*

Proof. For a given path P from $(0,-2m)$ to $(i,-i)$, with the part P_1 below the line $y=-2x$ a V_2 path, and the other part P_2 a normal path, it is clear that P_1 must end at a point $(j,-2j)$ for some $j\geq 0$, and this j is unique.

We observe that the number of horizontal steps in P_1 is j, which equals the total number of steps in P_2. Therefore, we can associate to each horizontal step in P_1 a step in P_2, with order preserved. We call this new path Q. Clearly, Q is a T-path, since each step of Q is a $(0,2)$-step, which is kept from P_1, or a $(1,1)$-step, by associating a vertical step in P_2 to a horizontal step in P_1, or a $(2,0)$-step, by associating a horizontal step in P_2 to a horizontal step in P_1. Since the above procedure is a rearrangement of the steps in P_1P_2, Q is a T-path from $(0,-2m)$ to $(i,-i)$. So Q is the desired T-path. The above procedure is clearly reversible. $\square$

By reflecting in $y=-x$, we get

Lemma 7.2. *The number of paths from $(i,-i)$ to $(0,2m)$, in which the part above the line $x=-2y$ is an H_2 path, and the other part is a normal path, is also $T(i,2m-i)$.*

Pictures for generalizations of these lemmas can be found in Figures 8 and 9. These lemmas correspond to the case $r=2$.

Theorem 7.3. *The number of K-paths from $(0,-2m)$ to $(2n,0)$ is equal to the number of T-paths from $(0,-2m)$ to $(2n,0)$, which is $T(2n,2m)$. Moreover, we have the following identity.*

$$U_n = \det\left(T(2i,2j)\right)_{0\leq i,j\leq n-1}. \tag{68}$$

Proof of Theorem 7.3. We can split any K-path from $(0,-2m)$ to $(2n,0)$ into two parts: one ends at $(i,-i)$ and the other starts at $(i,-i)$ for some $i\geq 0$ (this i is unique). Then using the two bijections in Lemmas 7.1 and 7.2, we have a bijective proof of the first part of the corollary.

We have shown in last section that U_n equals the number of n-tuples of nonintersecting K-paths from $P_0,\ldots,P_{n-1}$ to $Q_0,\ldots,Q_{n-1}$, where $P_i=(-2i,-2i)$, and $Q_i=(2i,2i)$ for $i=0,1,\ldots,n-1$. It is clear (see Figure 2) that it is still true if we replace P_i by $\tilde{P}_i=(0,-2i)$, and Q_i by $\tilde{Q}_i=(2i,0)$. But from the first part of this corollary, the number of K-paths from $\tilde{P}_i$ to $\tilde{Q}_j$ is $T(2j,2i)=T(2i,2j)$ for all $0\leq i,j\leq$. Then the identity (68) follows. $\square$

Remark 7.4. *The identity* (68) *has a generalization in Section 8. Note that U_n does not equal the number of n-tuples of nonintersecting T-paths from $P_0',\ldots,P_{n-1}'$ to $Q_0',\ldots,Q_{n-1}'$, because their steps can cross without meeting at a vertex of the T-paths.*

Definition 7.5. *We call a path P a KT-path if it satisfies the following conditions.*

1. *The path P never goes above the diagonal.*

2. *The part of P that is to the left of the line $x=0$ is a V_2 path.*

3. The part of P in the fourth quadrant is a T-path.

4. The part of P that is above the line $y = 0$ is an H_2 path.

Theorem 7.6. *The number of KT-paths from $(-2m, -2m)$ to $(2n, 2n)$ is a_{m+n} for all $m + n \geq 0$.*

We give three bijective proofs of this theorem. The first bijective proof establishes the bijection from K-paths to KT-paths. A sliding lemma for KT-paths will then yield a sliding lemma for K-paths. We find that it is much easier to slide KT-paths: we can slide slowly and we can also slide fast. We give the fast sliding in our bijection from KT-paths to V_2-paths. This is the second proof. The slow sliding will be given in Section 8 in a more general setting. This yields the third proof. We suspect that the sliding lemma for K-paths resulting from our second and third bijections are natural, i.e., preserve the nonintersecting property.

Bijection from KT-paths to K-paths. We first uniquely separate P, according to its intersections with the lines $x = 0$ and $y = 0$, into three parts, a V_2-path P_1, followed by a T-path P_2, followed by an H_2-path P_3, such that P_1 ends with a horizontal step and P_3 starts with a vertical step, except that P_1 and P_3 may be empty.

From Theorem 7.3, we can get a K-path P_2' from P_2 without changing the starting and ending points. Then $P' = P_1 P_2' P_3$ is the desired K-path. This procedure is clearly reversible. □

The next proof relies highly on Lemmas 7.1 and 7.2. The bijection ϕ_v for Lemma 7.1 maps a T-path P to a V_2-path P_V followed by a normal path P_N, in which the number of horizontal steps in P_V equals the total number of steps in P_N. Given the starting point $S(P)$ and ending point $E(P)$ of P, we can predict the position of $E(P_V) = S(P_N)$: $E(P_V)$ must lie on the line with slope -2 and passing through the point O, which is determined by the conditions that $O \to S(P)$ is vertical and the slope of $O \to E(P)$ is -1.

Similarly the bijection ϕ_h for Lemma 7.2 maps a T-path P to a normal path P_N followed by an H_2-path P_H with similar properties.

Fast sliding bijection from KT-paths to V_2-paths. Let P be a KT-path from $(-2m, -2m)$ to $(2n, 2n)$ with $m, n \geq 0$.

We first uniquely separate P, according to its intersections with the lines $x = 0$ and $y = 0$, into a V_2-path P_1, followed by a T-path P_2, followed by an H_2-path P_3, such that P_1 ends with a horizontal step and P_3 starts with a vertical step, except that P_1 and P_3 may be empty. We will map $P_2 P_3$ to a V_2 path with the same starting and ending points.

Suppose $S(P_2) = (0, -i)$ and $E(P_2) = (j, 0)$. Obviously we can assume $j > 0$ for otherwise P_2 is a V_2-path and P_3 is the empty path. Applying ϕ_v to P_2 gives us a V_2-path P_{2V} followed by a normal path P_{2N} with $S(P_{2N}) = (a, -2a + j)$ for some $a > 0$. See Figure 6, where we did not draw the paths explicitly.

Draw a vertical line at $S(P_{2N})$, which intersects the diagonal at (a, a). It is easy to check that the the total number of steps of P_{2N} is a. Factor the H_2-path P_3, according

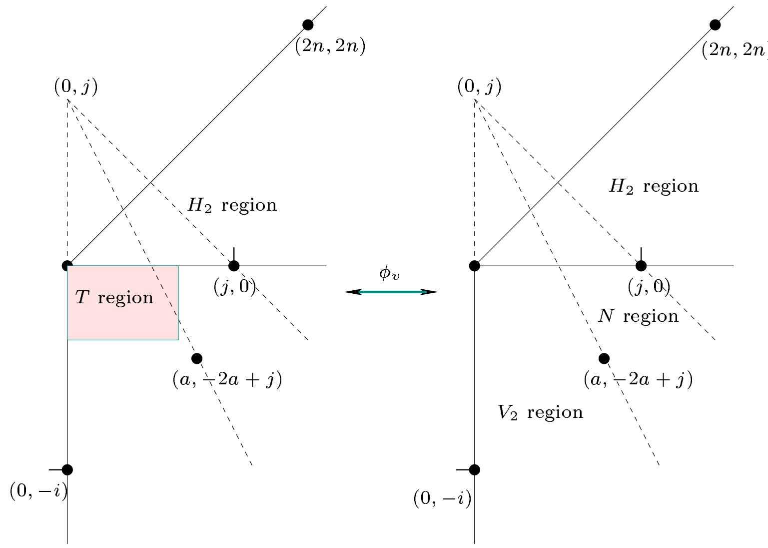

Figure 6: First step of the fast sliding.

to its intersection with the line $y = a$, into $P_{31}P_3'$ such that P_3' starts with a vertical step. The the number of vertical steps of P_{31} equals a. Applying ϕ_h^{-1} to $P_{2N}P_{31}$ gives us a T-path P_2'. See Figure 7.

Now we are left to map $P_{2V}P_2'P_3'$ to a V_2-path. If we slide down the path by (a, a), then we met the same situation as for the path P_2P_3. Repeat the above procedure we can finally obtain the desired V_2-path. The procedure is reversible because of the required conditions of ending with a horizontal step or starting with a vertical step, as shown in Figures 6 and 7. $\square$

8 Generalizations of K-paths and KT-paths

Let $g_n^{(r)} = \frac{1}{rn+1}\binom{(r+1)n}{n}$ be the number of $r+1$-ary trees with n nodes, and

$$g^{(r)}(x) = \sum_{n\geq 0} g_n^{(r)} x^n$$

be the generating function. Then $g^{(r)}(x)$ satisfies the following functional equation.

$$g^{(r)}(x) = 1 + x\left(g^{(r)}(x)\right)^{r+1}.$$

For $r = 1$, $g_n^{(1)}$ is the Catalan number. It is well-known that the Hankel determinants of the Catalan generating function are all 1. We have studied the the case $r = 2$. We wish to say something about the Hankel determinants of $g^{(r)}(x)$ for $r \geq 3$.

Since $H_n(g^{(r)}(x))$ does not factor for $r \geq 3$, a formula like (4) is unlikely. However, we find generalizations of (63) (which is the same as (7)), (64), and (68). They are given

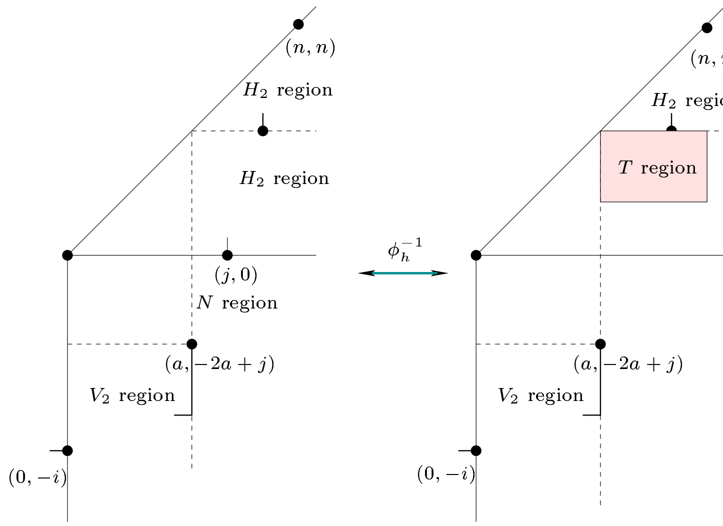

Figure 7: Second step of the fast sliding.

by (71), (72), and (73). Their algebraic proofs can be found in Section 9. Except for $H_n(g^{(r)}(x))$, we do not have nonintersecting paths interpretation of these determinants.

We have natural generalizations of the concepts in the last two sections. A path is an H_r path if each step is either $(r,0)$ or $(0,1)$. Similarly, a path is a V_r path if each step is either $(1,0)$ or $(0,r)$. The following is equivalent to a special case of a classical result given (without proof) by Barbier [4]. (See [12] for a new proof and further references.)

Proposition 8.1. *The number of V_r-paths (or H_r-paths) from $(0,0)$ to (rn,rn) that never go above the diagonal is $g_n^{(r)}$.*

A path is a $T^{(r)}$-path if each of its step is $(r,0)$, $(r-1,1)$, …, or $(0,r)$. For any path P, we denote by $S(P)$ the starting point, $E(P)$ the ending point, and $L(P)$ the number of steps in P.

For a $T^{(r)}$-path P with $S(P)=(0,0)$ and $L(P)=k$, $E(P)$ must lie on the line $y=-x+rk$. So to compute $L(P)$, we take the sum of the x-coordinate and y-coordinate of $E(P)-S(P)$, and divide by r. A normal path is also a $T^{(1)}$-path, a V_1 path, and an H_1 path, and a T-path is a $T^{(2)}$-path.

Let $T^{(r)}(a,b)$ be the number of $T^{(r)}$-paths from $(0,0)$ to (a,b). Then $T^{(r)}(a,b)=0$ if $a+b$ is not divisible by r, so we can suppose $a=ri-s$ and $b=rj+s$ for some i,j and $0\le s\le r-1$. We have

$$T^{(r)}(ri+s,rj-s)=[x^{ri+s}y^{rj-s}](x^r+x^{r-1}y+\cdots+y^r)^{i+j}.$$

Since the right-hand side of the above equation is homogeneous in x and y, we can write it in terms of one variable t, where $t=y/x$.

Let $\alpha=1+t+t^2+\cdots+t^r$ and $\beta=\alpha/t^r=1+t^{-1}+\cdots t^{-r}$. Then

$$T(ri+s,rj-s)=\mathrm{CT}\,t^s\alpha^i\beta^j=\mathrm{CT}\,t^{s-rj}\alpha^{i+j}, \tag{69}$$

where CT means to take the constant term of a Laurent polynomial of t.

Definition 8.2. *We call a path P a $K^{(r)}$-path if it satisfies the following conditions.*

1. *The path P never goes above the diagonal.*
2. *The part of P that is to the left of the line $y = -rx$ is a V_r path.*
3. *The part of P between the two lines $y = -rx$ and $x = -ry$ is a $T^{(r-1)}$-path.*
4. *The part of P that is above the line $x = -ry$ is an H_r path.*

Definition 8.3. *We call a path P a $KT^{(r)}$-path if it satisfies the following conditions.*

1. *The path P never goes above the diagonal.*
2. *The part of P that is to the left of the line $x = 0$ is a V_r path.*
3. *The part of P in the fourth quadrant is a $T^{(r)}$-path.*
4. *The part of P that is above the line $y = 0$ is an H_r path.*

For example, a K-path is a $K^{(2)}$-path, and a KT-path is a $KT^{(2)}$-path.

Let $\mathbb{K}(m, n, r)$ be the set of all $K^{(r)}$-paths from $(-mr, -mr)$ to (nr, nr). Let $\mathbb{T}(m, n, r, s)$ be the set of all $KT^{(r)}$-paths from $(s - mr, s - mr)$ to $(nr + s, nr + s)$. Now we can state our main results.

Theorem 8.4. *The cardinality of $\mathbb{K}(m, n, r)$ is $g^{(r)}_{n+m}$ for all m and n. The cardinality of $\mathbb{T}(m, n, r, s)$ is also $g^{(r)}_{n+m}$ for all m, n and s.*

As in the case $r = 2$, if $m \le 0$ (or $n \le 0$), then $K^{(r)}$-paths and $KT^{(r)}$-paths are in fact H_r-paths (or V_r-paths), and in these cases, Theorem 8.4 follows from Proposition 8.1. The idea of the proof of Theorem 8.4 is to show that $|\mathbb{T}(m, n, r, s)| = |\mathbb{T}(m, n, r, s-1)|$ for all $1 \le s \le r$. Then $|\mathbb{T}(m, n, r, s)| = g^{(r)}_{m+n}$ follows by induction. We will give a bijection from $\mathbb{T}(m, n, r, 0)$ to $\mathbb{K}(m, n, r)$.

The bijective proof we are going to give highly relies on the following lemma, especially on the bijection from $\mathbf{T}(i, j)$ to $\mathbf{V}(i, j)$, which is a generalization of Lemma 7.1.

Lemma 8.5. *The following four sets all have cardinality $T^{(r)}(j, ri - j)$.*

1. *The set $\mathbf{T}(i, j)$ of all $T^{(r)}$-paths from $(0, -ri)$ to $(j, -j)$.*
2. *The set $\mathbf{T}'(i, j)$ of all $T^{(r)}$-paths from $(j, -j)$ to $(ri, 0)$.*
3. *The set $\mathbf{V}(i, j)$ of all paths from $(0, -ri)$ to $(j, -j)$, with the part before the line $y = -rx$ a V_r path, and the part after the line $y = -rx$ a $T^{(r-1)}$-path.*
4. *The set $\mathbf{H}(i, j)$ of all paths from $(j, -j)$ to $(ri, 0)$ with the part before the line $x = -ry$ a $T^{(r-1)}$-path, and the part after the line $x = -ry$ an H_r path.*

Proof. We construct only the bijection from $\mathbf{T}(i,j)$ to $\mathbf{V}(i,j)$. The bijection from $\mathbf{T}'(i,j)$ to $\mathbf{H}(i,j)$ is similar. The bijection from $\mathbf{V}(i,j)$ to $\mathbf{H}(i,j)$ and the bijection from $\mathbf{T}(i,j)$ to $\mathbf{T}'(i,j)$ are given by reflecting in the line $y=-x$.

For any given path $T \in \mathbf{T}(i,j)$ the steps in T are $(r-k,k)$ for $k=0,1,\ldots,r$. We first replace all the steps in T that are not $(0,r)$ with steps $(r,0)$. Then we get a path T_1, with $E(T_1)$ on the line $y=-x$. Changing every $(r,0)$-step in T_1 into $(1,0)$ will give us a V_r path V_1 with $E(V_1)$ on the line $y=-rx$. By removing all the $(0,r)$-steps in T, and changing every $(r-k,k)$-step into $(r-k-1,k)$ for $k=0,1,\ldots r-1$, we get a $T^{(r-1)}$-path V_2. Then $V=V_1V_2 \in \mathbf{V}(i,j)$ is the desired path. One way to see that $E(V)=(j,-j)$ is that if we regard an $(r-k,k)$-step as a $(1,0)$ step followed by a $(r-k-1,k)$-step for $k=0,1,\ldots,r-1$, then the bijection we gave is just a rearrangement of the steps in T.

The inverse procedure is as follows. For a given $V' \in \mathbf{V}(i,j)$, the line $y=-rx$ divides V' into a V_r path V_1' followed by a $T^{(r-1)}$ path V_2'. Suppose $E(V_1')=(k',-rk')$ for some k'. We can see that the number of $(1,0)$-steps in V_1', which is k', equals the total number of steps in V_2', which is $((j-k')+(-j+rk'))/(r-1)=k'$. Then we can associate to each $(1,0)$ step in V_1' a step in V_2', with order preserved. This gives us a $T^{(r)}$-path $T' \in \mathbf{T}(i,j)$. In Figure 8, we give an example of the case $r=4$. The above two procedures are clearly

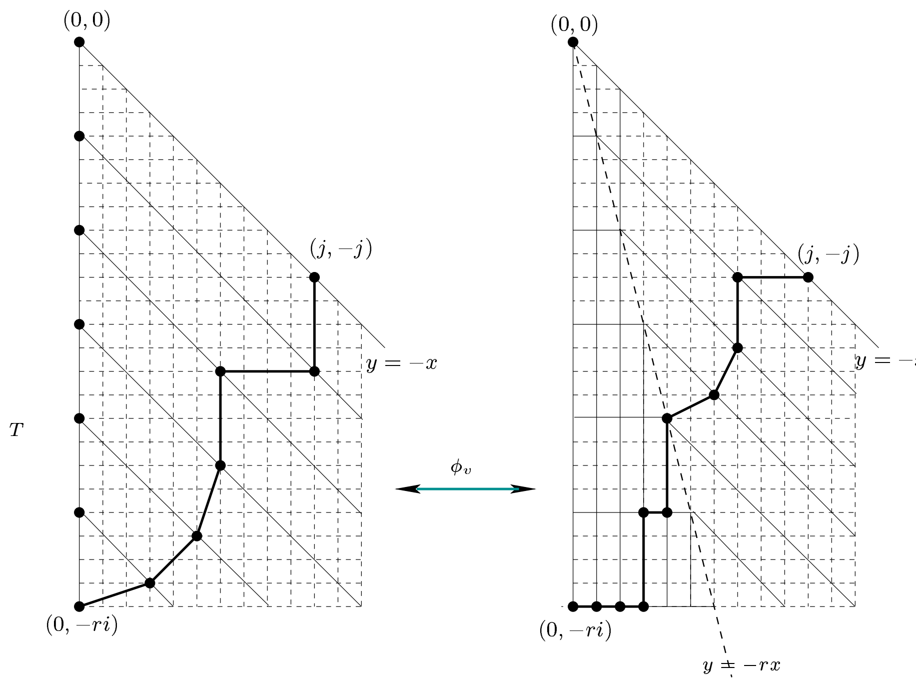

Figure 8: A $T^{(r)}$-path T and its image under ϕ_v.

inverse to each other. $\square$

Remark 8.6. *The bijection from $\mathbf{T}(i,j)$ to $\mathbf{V}(i,j)$ we gave originated from the following algebraic fact.*

For $r \geq 3$, there are many ways to group and expand the polynomial $(x^r + x^{r-1}y + \cdots + y^r)^m$. We find the following way has a nice combinatorial explanation.

$$
\begin{aligned}
(x(x^{r-1}+x^{r-2}y+\cdots y^{r-1})+y^r)^m &= \sum_{j=0}^{m}\binom{m}{j} y^{r(m-j)}x^j(x^{r-1}+x^{r-2}y+\cdots y^{r-1})^j \\
&= \sum_{i=0}^{rm}\left(\sum_{j=0}^{m}\binom{m}{j}T^{(r-1)}(rj-i,i-j)\right)x^iy^{rm-i}
\end{aligned}
$$

So we have

$$
T^{(r)}(i,rm-i)=\sum_{j=0}^{m}\binom{m}{j}T^{(r-1)}(rj-i,i-j). \tag{70}
$$

In Figure 8, $|\mathbf{V}(i,j)|$ *can be counted according to the intersection points of the paths in* $\mathbf{V}(i,j)$ *with the line* $y=-rx$*. This yields* (70).

We denote the bijection from $\mathbf{T}(i,j)$ to $\mathbf{V}(i,j)$ by ϕ_v, and the bijection from $\mathbf{T}'(i,j)$ to $\mathbf{H}(i,j)$ by ϕ_h. One thing we should mention is that neither ϕ_h nor ϕ_v changes the starting point and the ending point. The path in Figure 9 is obtained from the $T^{(r)}$-path T in Figure 8 by applying ϕ_h.

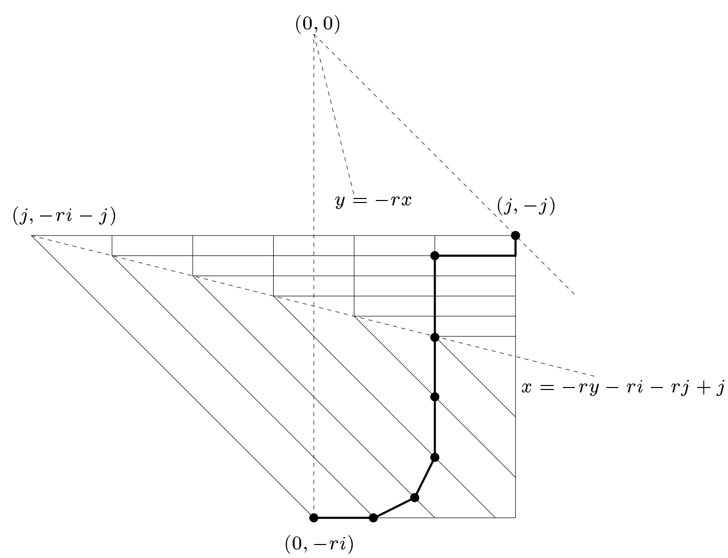

Figure 9: The image of T under ϕ_h.

Applying ϕ_v to a $T^{(r)}$ path will give us a V_r path followed by a $T^{(r-1)}$ path, in which the number of horizontal steps in the V_r path equals the total number of steps in the $T^{(r-1)}$ path. We can locate the ending point of the V_r path by the following three easy steps. (See Figure 8.)

1. Draw a vertical line at $S(T)$.

2. Draw a line of slope -1 at $E(T)$.

3. At the intersecting point of the above two lines, draw a line of slope $-r$. Then this is the line on which the ending point of the V_r path must lie.

We call the line obtained in the above three steps the *bisecting line* $\mathrm{BL}_v(T)$ of $\phi_v(T)$. For any $T^{(r)}$ path T, with $S(T) = (0, -ri)$ and $E(T) = (j, -j)$, $\mathrm{BL}_v(T)$ is $y = -rx$, which is independent of i and j.

A similar argument for ϕ_h holds for a $T^{(r)}$-path T'. The corresponding three steps are given as follows. (See Figure 9.)

1. Draw a horizontal line at $E(T')$.

2. Draw a line of slope -1 at $S(T')$.

3. At the intersecting point of the above two lines, draw a line of slope $-1/r$. Then this is the line on which the starting point of the H_r-path must lie.

We call the line obtained in the above three steps the bisecting line $\mathrm{BL}_h(T')$ of $\phi_h(T')$.

The following is a generalization of Theorem 7.3.

Lemma 8.7. *The number of each of the following four kinds of paths from* $(0, -ri + s)$ *to* $(rj + s, 0)$ *equals to* $T(rj + s, ri - s)$*.*

1. *All* $T^{(r)}$*-paths from* $(0, -ri + s)$ *to* $(rj + s, 0)$*.*

2. *All paths from* $(0, -ri+s)$ *to* $(rj+s, 0)$ *consisting of a* V_r *path, followed by a* $T^{(r-1)}$ *path, in which the number of* $(1, 0)$*-steps in the* V_r *path equals the total number of steps in the* $T^{(r-1)}$ *path.*

3. *All the paths from* $(0, -ri + s)$ *to* $(rj + s, 0)$ *consisting of a* $T^{(r-1)}$ *path, followed by an* H_r *path, in which the number of* $(0, 1)$*-steps in the* H_r *path equals the total number of steps in the* $T^{(r-1)}$ *path.*

4. *All the paths from* $(0, -ri+s)$ *to* $(rj+s, 0)$*, with the part before the line* $y = -rx+s$ *a* V_r *path, between the lines* $y = -rx + s$ *and* $x = -ry + s$ *a* $T^{(r-1)}$ *path, and the part after the line* $x = -ry + s$ *an* H_r *path.*

Proof. Part 1 follows from the definition of $T^{(r)}(rj + s, ri - s)$. Part 2 and part 3 are obvious by Lemma 8.5, so we need only prove part 4.

For a given $T^{(r)}$ path T from $(0, -ri + s)$ to $(rj + s, 0)$, we can uniquely separate it by the line $y = -x + s$ into a $T^{(r)}$ path T_1 followed by a $T^{(r)}$ path T_2. Applying ϕ_v on T_1, we get a V_r path T_{1V} followed by a $T^{(r-1)}$ path T_{1T}. Applying ϕ_h on T_2, we get a $T^{(r-1)}$ path T_{2T} followed by an H_r path T_{2H}. Using the three steps for finding $\mathrm{BL}_v(T_1)$, we see that $E(T_{1V})$ must lie on the line $y = -rx + s$, since the line $y = -x + s$ intersects the line $x = 0$ at $(0, s)$. Similarly, $S(T_{2T})$ must lie on the line $x = -ry + s$, since the line $y = -x + s$ intersect the line $y = 0$ at $(s, 0)$. Together with the fact that

$E(T_{1T}) = E(T_1) = S(T_2) = S(T_{2T})$, we see that $T_{1T}T_{2T}$ is also a $T^{(r-1)}$ path and the path $T_{1V}T_{1T}T_{2T}T_{2H}$ is the desired path.

The above procedure is clearly invertible. □

The bijection in the above proof will induce a bijection from $\mathbb{T}(m,n,r,0)$ to $\mathbb{K}(m,n,r)$. We will see this in the proof of Theorem 8.4.

For any $1 \le s \le r$, let $\mathbb{A}_s(i,j)$ be the set of $KT^{(r)}$-paths from $(0,-(ri-s))$ to $(rj+s,1)$, and $\mathbb{B}_s(i,j)$ be the set of $KT^{(r)}$-paths from $(-1,-(ri-s+1)$ to $(rj+s-1,0)$. Then we have the following lemma, which will induce the bijection from $\mathbb{T}(m,n,r,s)$ to $\mathbb{T}(m,n,r,s-1)$.

Lemma 8.8 (Slow Sliding Lemma). *There is a bijection from $\mathbb{A}_s(i,j)$ to $\mathbb{B}_s(i,j)$ for all i,j.*

We will give two proofs for this lemma. The algebraic proof will be given in the next section. The bijective proof is as follows.

Bijective proof of Lemma 8.8. For any given $P \in \mathbb{A}_s(i,j)$, we uniquely factor P, according to its intersections with the lines $y=0$ and $y=-x+s$, into $P_1P_2P_3$, where we require P_3 to start with a vertical step. In the left picture of Figure 10, we marked each intersection point by a $\circ$.

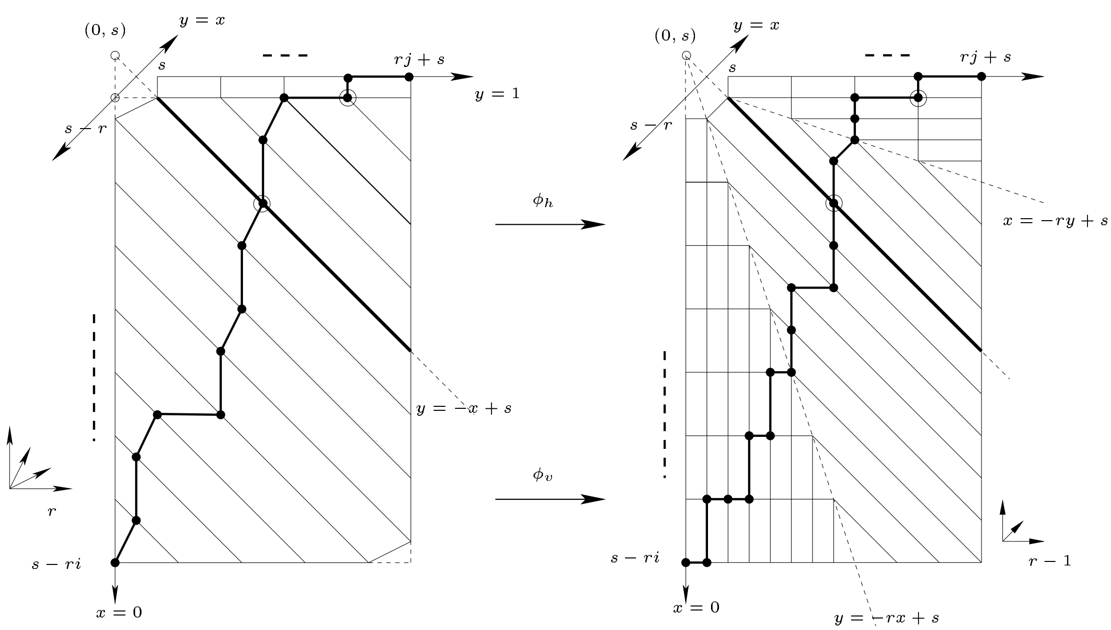

Figure 10: First step of the slow sliding bijection.

Now we apply ϕ_v to P_1 to obtain a V_r-path followed by a $T^{(r-1)}$-path, and apply ϕ_h to P_2 to obtain a $T^{(r-1)}$-path followed by an H_r-path. The bisection lines are $y=-rx+s$ and $x=-ry+s$, as drawn in the right picture of Figure 10.

Denote by P' the whole path obtained this way. We uniquely factor P', according to its intersections with the lines $x = 1$ and $y = -x + s - r + 1$ into $P_1'P_2'P_3'$, where we require P_1' to end with a horizontal step. In the left picture of Figure 11, we marked each intersection point by a $\square$.

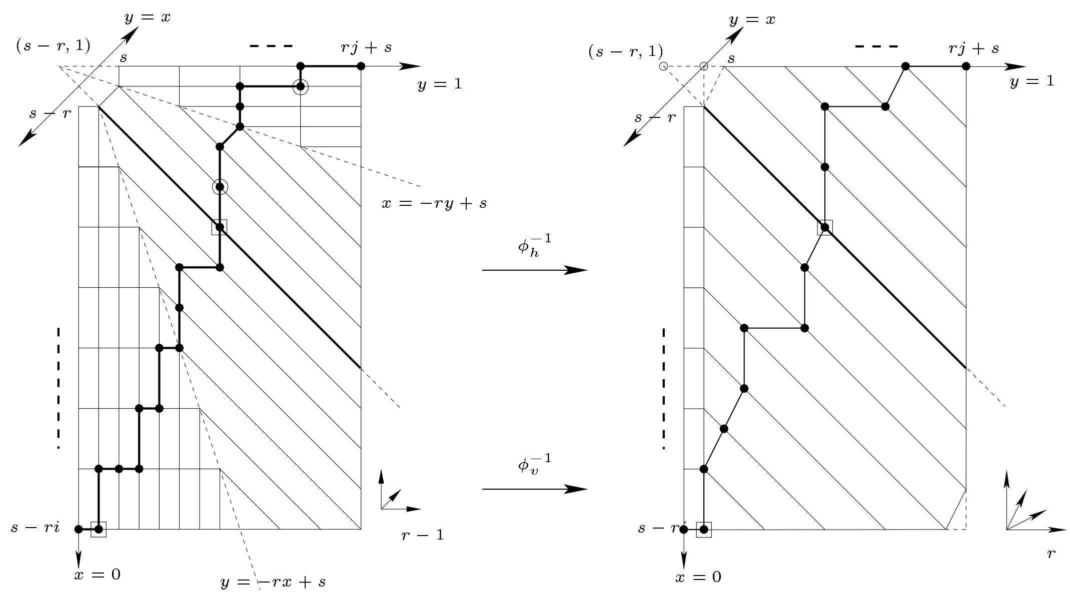

Figure 11: Second step of the slow sliding bijection.

Now we apply ϕ_v^{-1} to P_2' to obtain a $T^{(r)}$-path Q_2, and apply ϕ_h^{-1} to P_3' to obtain a $T^{(r)}$-path Q_3. See the right picture of Figure 11. We need to check that the three lines $y = -x + s - r + 1$, $x = -ry + s$, and $y = 1$ intersect at the point $(s - r, 1)$, so that $\phi_h^{-1}(P_3')$ is well defined.

Finally, let Q be obtained from $P_1'Q_2Q_3$ by sliding down by $(1, 1)$. Then Q is the desired path. Every step in the above procedure is invertible. This completes the proof. $\square$

Proof of Theorem 8.4. First we construct the bijection from $\mathbb{T}(m, n, r, 0)$ to $\mathbb{K}(m, n, r)$. This shows that $|\mathbb{T}(m, n, r, 0)| = |\mathbb{K}(m, n, r)|$.

Recall that any $P \in \mathbb{T}(m, n, r, 0)$ is a $KT^{(r)}$-path from $(-rm, -rm)$ to (rn, rn), and any $Q \in \mathbb{K}(m, n, r)$ is a K_r-path with the same ending points. We can uniquely factor P, according to its intersections with the lines $x = 0$ and $y = 0$, into $P_1P_2P_3$, such that P_1 is a V_r path ending with a horizonal step and P_3 is an H_r path starting with a vertical step, except that P_1 and P_3 may be empty. Applying the bijection of Lemma 8.7 part 4 to P_2, we get a K_r-path $Q_2 = Q_{2a}Q_{2b}Q_{2c}$ from $S(P_2)$ to $E(P_2)$, with Q_{2a} a V_r path ending on the line $y = -rx$, Q_{2b} a $T^{(r-1)}$ path ending on the line $x = -ry$, and Q_{2c} an H_r path. Then P_1Q_{2a} is a V_r path, and $Q_{2c}P_3$ is an H_r path. So $Q = P_1Q_2P_3 = (P_1Q_{2a})Q_{2b}(Q_{2c}P_3)$ belonging to $\mathbb{K}(m, n, r)$ is the desired path. The above procedure is clearly reversible.

Next we construct the bijection from $\mathbb{T}(m, n, r, s)$ to $\mathbb{T}(m, n, r, s - 1)$ for $1 \le s \le r$, which implies $|\mathbb{T}(m, n, r, s)| = |\mathbb{T}(m, n, r, s - 1)|$. Any $P \in \mathbb{T}(m, n, r, s)$ can be uniquely factored, according to its intersections with the lines $x = 0$ and $y = 1$, into $P_1P_2P_3$, such that P_1 is a V_r path ending with a horizonal step and P_3 is an H_r path starting with a vertical step, except that P_1 and P_3 may be empty. Then P_2 is a $KT^{(r)}$-path.

Applying the bijection of Lemma 8.8 to P_2, we get a $KT^{(r)}$-path $Q_2 = Q_{2a}Q_{2b}$, with Q_{2a} a V_r path starting on the line $x = -1$ and ending on the line $x = 0$, and Q_{2b} a T_r path ending on the line $y = 0$. Let Q_1 be obtained from P_1 by sliding down by $(1, 1)$, and Q_3 be obtained from P_3 by sliding down by $(1, 1)$. Then $Q = Q_1Q_2Q_3 = (Q_1Q_{2a})Q_{2b}Q_3$ is a path from $(s-1-mr, s-1-mr)$ to $(s-1+nr, s-1, nr)$ that never goes above the diagonal and with the part before the line $x = 0$ a V_r path, the part between the two lines $x = 0$ and $y = 0$ a T_r path, and the part after the line $y = 0$ an H_r path. Hence $Q \in \mathbb{T}(m, n, r, s-1)$ is the desired path. The above procedure is clearly reversible.

Finally, we use induction to conclude the theorem. By the second part, it is easy to see that $|\mathbb{T}(m, n, r, s)| = |\mathbb{T}(m, n, r, r)|$ for $0 \le s \le r$. But $\mathbb{T}(m, n, r, r)$ is in fact $\mathbb{T}(m-1, n+1, r, 0)$. Theorem 8.4 then follows by induction and the fact that $|\mathbb{T}(0, m + n, r, 0)| = g^{(r)}_{m+n}$. □

Remark 8.9. *We can also give a fast sliding bijection from* $\mathbb{T}(m, n, r, 0)$ *to* $\mathbb{T}(0, m + n, r, 0)$.

Recall that $g^{(r)}(x)$ is the generating function of $r+1$-ary trees. Let $f^{(r)}(x) = g^{(r)}(x)-1$. Then $f^{(r)}$ satisfies the following functional equation.

$$f^{(r)} = x(1 + f^{(r)})^{r+1}.$$

If we count $K^{(r)}$-paths from $(-mr, -mr)$ to (nr, nr) according to their intersections with the lines $y = -rx$ and $x = -ry$, we see that Theorem 8.4 yields the matrix identity

$$\begin{aligned}\left(g^{(r)}_{i+j}\right)_{0\le i,j\le n-1} &= \left([x^i]g^{(r)}(f^{(r)})^j\right)_{0\le i,j\le n-1} \\ &\qquad \left(T^{(r-1)}(rj-i, ri-j)\right)_{0\le i,j\le n-1}\left([x^j]g^{(r)}(f^{(r)})^i\right)_{0\le i,j\le n-1}, \end{aligned} \tag{71}$$

where

$$[x^i]g^{(r)}(f^{(r)})^j = \frac{(r+1)j+1}{(r+1)i+1}\binom{(r+1)i+1}{i-j}$$

is the number of V_r-paths from $(-ri, -ri)$ to $(j, -rj)$ that never go above the diagonal.

Since the transformation matrices in (71) are upper (or lower) triangular matrices with diagonal entries 1, we have

$$\det\left(g^{(r)}_{i+j}\right)_{0\le i,j\le n-1} = \det\left((T^{(r-1)}(rj-i, ri-j)\right)_{0\le i,j\le n-1} \tag{72}$$

A similar argument gives

$$\det\left(g^{(r)}_{i+j}\right)_{0\le i,j\le n-1} = \det\left(T^{(r)}(ri, rj)\right)_{0\le i,j\le n-1} = \det\left(T^{(r)}(ri-s, rj+s)\right)_{0\le i,j\le n-1}, \tag{73}$$

for any $0 \le s \le r-1$.

9 The Algebraic Proof

The scheme of our algebraic proof of Lemma 8.8 is by first representing our object as the constant term of a rational function, and then evaluating the constant term. This technique is well-known. For instance, Egorychev [7] gave many applications for evaluating combinatorial sums in the context of residues (equivalent to constant terms). We also use this method, together the method in Section 4, to give algebraic proofs of (72) and (73).

The only thing we need here is the following Proposition 9.1. Its proof is included since the idea of the proof applies to most of our examples. We will give a different algebraic proof of equations (48) and (52).

Let $B(x,y,t) \in \mathbb{Q}[t,t^{-1}][[x,y]]$. Then $B(x,y,t)$ can be written as

$$B(x,y,t) = \sum_{i,j=0}^{\infty} b_{ij}(t)x^i y^j,$$

where $b_{ij}(t) \in \mathbb{Q}[t,t^{-1}]$. Define

$$\mathrm{CT}\, B(x,y,t) = \sum_{i,j=0}^{\infty} (\mathrm{CT}\, b_{ij}(t))x^i y^j,$$

where $\mathrm{CT}\, b_{ij}(t)$ is the constant term of the Laurent polynomial $b_{ij}(t)$ in t.

The general problem in this section is to find the constant term of the function $(1 - P(t^{-1})x)^{-1}(1-Q(t)y)^{-1}$, for some specific $P(t), Q(t) \in \mathbb{Q}[t]$.

Proposition 9.1. *Let $P(t)$ and $Q(t)$ be polynomials in t, and let $a_{mn} = \mathrm{CT}\, P(t)^m Q(t^{-1})^n$. Then*

$$\sum_{m,n} a_{mn} x^m y^n$$

is a rational function in x and y.

Proof. We show that

$$\mathrm{CT}\, \frac{1}{(1-P(t)x)(1-Q(t^{-1})y)}$$

is rational, where we work in the ring $\mathbb{Q}[t,t^{-1}][[x,y]]$. We may assume that $P(t)$ has degree at at least 1. Let d be the degree of $Q(t)$. Let

$$F = \frac{1}{(1-P(t)x)(1-Q(t^{-1})y)} = \frac{t^d}{(1-P(t)x)(t^d - t^d Q(t^{-1})y)}.$$

Since $t^d Q(t^{-1})$ is a polynomial in t of degree at most d, and the degree of $P(t)$ is at least 1, F has a partial fraction expansion in t that may be written

$$F = \frac{1}{R(x,y)} \left(\frac{A(x,y,t)}{1-P(t)x} + \frac{B(x,y,t)}{t^d - t^d Q(t^{-1})y} \right) = \frac{1}{R(x,y)} \left(\frac{A(x,y,t)}{1-P(t)x} + \frac{B(x,y,t)t^{-d}}{1-Q(t^{-1})y} \right) \tag{74}$$

where $R(x,y)$ is a polynomial in x and y, $A(x,y,t)$ and $B(x,y,t)$ are polynomials in x, y, and t, and the degree of B in t is less than d.

Now the constant term in t of $B(x,y,t)t^{-d}/(1-Q(t^{-1})y)$ is 0 and the constant term in $A(x,y,t)/(1-P(t)x)$ is $A(x,y,0)/(1-P(0)x)$. We would like to conclude that

$$\mathrm{CT}\,F = \frac{A(x,y,0)}{R(x,y)(1-P(0)x)}. \tag{75}$$

However, we don't know that $1/R(x,y)$ has a power series expansion. To avoid this problem, we multiply (74) by $R(x,y)$ to get

$$R(x,y)F = \frac{A(x,y,t)}{1-P(t)x} + \frac{B(x,y,t)t^{-d}}{1-Q(t^{-1})y}.$$

Then

$$\mathrm{CT}\,R(x,y)F = \frac{A(x,y,0)}{1-P(0)x},$$

but since $\mathrm{CT}\,R(x,y)F = R(x,y)\,\mathrm{CT}\,F$, (75) follows. □

Since the main idea of this proof is a partial fraction decomposition, we call this method the *partial fraction method*. In the following examples, we use formula (69). Let $\alpha = 1+t+t^2+\cdots+t^r$ and $\beta = \alpha/t^r = 1+t^{-1}+\cdots t^{-r}$. Then

$$T(ri+s, rj-s) = \mathrm{CT}\,t^s\alpha^i\beta^j.$$

In particular,

$$\binom{m+n}{m} = [t^n](1+t)^{m+n} = \mathrm{CT}(1+t)^m(1+\frac{1}{t})^n.$$

Example 9.2. *A different proof of identity* (48)

$$\frac{1-xy}{1-xy^2-3xy-x^2y} = \sum_{i,j}\binom{i+j}{2i-j}x^iy^j.$$

Proof. We have

$$\begin{aligned}\sum_{i,j\geq 0}\binom{i+j}{2i-j}x^iy^j &= \sum_{i,j\geq 0}\left(\mathrm{CT}(1+t)^{2i-j}(1+t^{-1})^{2j-i}\right)x^iy^j\\ &= \mathrm{CT}\sum_{i,j\geq 0}\frac{(1+t)^{2i}}{(1+t^{-1})^i}x^i\cdot\frac{(1+t^{-1})^{2j}}{(1+t)^j}y^j\\ &= \mathrm{CT}\,\frac{1}{(1-tx-t^2x)(1-t^{-1}y-t^{-2}y)}\end{aligned}$$

Using Maple, we find the partial fraction expansion in t:

$$\frac{1}{(1-tx-t^2x)(1-t^{-1}y-t^{-2}y)} = \frac{1}{1-3xy-x^2y-xy^2} \left(\frac{1-xy+txy+tx^2y}{1-tx-t^2x} - \frac{y(1+t+tx-xy)}{t^2-ty-y} \right) \tag{76}$$

It is easy to see that

$$\mathrm{CT}\, \frac{1}{1-3xy-x^2y-xy^2} \frac{1-xy+txy+tx^2y}{1-tx-t^2x} = \frac{1-xy}{1-3xy-x^2y-xy^2},$$

obtained by setting $t=0$, since it is a formal power series in x and y, with coefficients in $\mathbb{Q}[t]$. Similarly,

$$\mathrm{CT}\, \frac{y(1+t+tx-xy)}{t^2-ty-y} = \mathrm{CT}\, t^{-1} \frac{y(1+t^{-1}+x-t^{-1}xy)}{1-t^{-1}y-t^{-2}y} = 0,$$

since it is a formal power series in x and y with coefficients in $t^{-1}\mathbb{Q}[t^{-1}]$.

Equation (48) then follows. □

Similarly, we can compute the generating function of $\binom{i+j+r}{2i-j}$ for nonnegative integer r, from which it is easy to deduce (52).

Example 9.3.

$$\sum_{i,j,r\geq 0} \binom{i+j+r}{2i-j} x^i y^j z^r = \frac{1-z-xy(1-2z-xz)}{(1-3xy-x^2y-xy^2)(1-2z+z^2-xz)}. \tag{77}$$

Proof. We have

$$\begin{aligned}
\sum_{i,j,r\geq 0} \binom{i+j+r}{2i-j} x^i y^j z^r &= \sum_{i,j,r\geq 0} \left(\mathrm{CT} (1+t)^{2i-j} (1+t^{-1})^{2j-i+u} \right) x^i y^j z^r \\
&= \mathrm{CT} \sum_{i,j,r\geq 0} \frac{(1+t)^{2i}}{(1+t^{-1})^i} x^i \cdot \frac{(1+t^{-1})^{2j}}{(1+t)^j} y^j \cdot (1+t^{-1})^r z^r \\
&= \mathrm{CT}\, \frac{1}{1-tx-tx^2} \cdot \frac{1}{1-t^{-1}y-t^{-2}y} \cdot \frac{1}{1-z-t^{-1}z}
\end{aligned}$$

By (76), this equals

$$\mathrm{CT}\, \frac{1}{1-3xy-x^2y-xy^2} \left(\frac{1-xy+txy+tx^2y}{1-tx-t^2x} - \frac{y(1+t+tx-xy)}{t^2-ty-y} \right) \frac{1}{1-z-t^{-1}z}.$$

Since $1/(1-z-t^{-1}z)$ is a formal power series in z with coefficients in $\mathbb{Q}[t^{-1}]$, we can discard the second part of (76) in our computation. So we have

$$\mathrm{CT}\, \frac{1}{1-3xy-x^2y-xy^2} \left(\frac{1-xy+txy+tx^2y}{1-tx-t^2x} \right) \frac{1}{1-z-t^{-1}z}.$$

Converting this into partial fraction in t, we get two parts, one with coefficients in $\mathbb{Q}[t]$, the other with coefficients in $t^{-1}\mathbb{Q}[t^{-1}]$. So we can discard the second part and set $t = 0$ to get equation (77). □

Algebraic Proof of Lemma 8.8. For s with $1 \le s \le r$ and any $P \in \mathbb{A}_s(i, j)$, we factor P, according to its intersection with the line $y = 0$, uniquely as P_1P_2, where P_1 is a $T^{(r)}$-path, and P_2 is an H_r path starting with a vertical step. Then $S(P_2) = (kr + s, 0)$, for some $0 \le k \le j$. See the left picture of Figure 12.

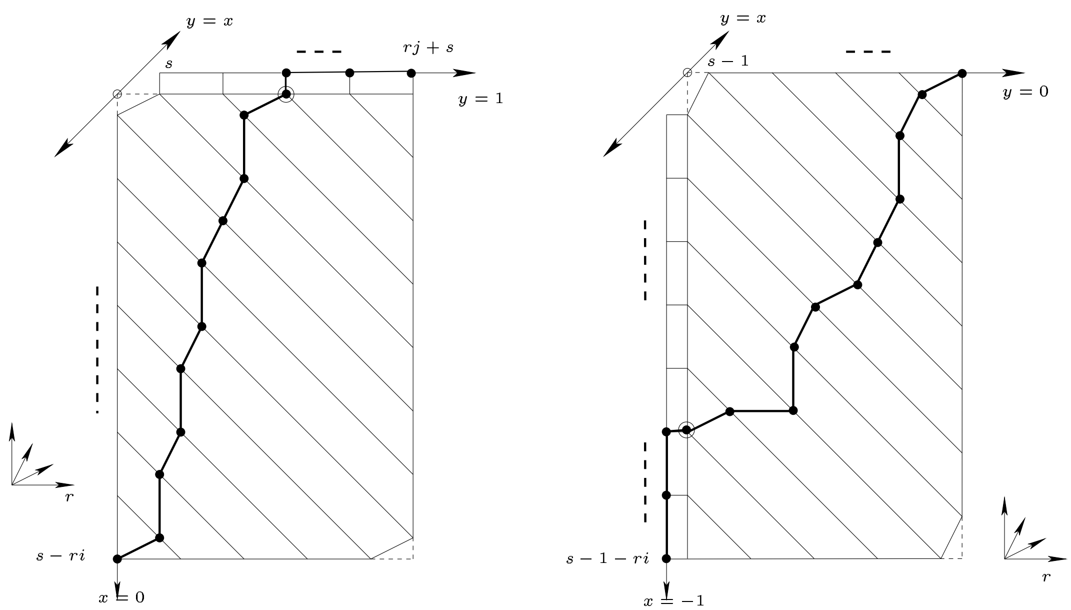

Figure 12: Example paths for $\mathbb{A}_s(i, j)$ and $\mathbb{B}_s(i, j)$.

For any k, there are $T^{(r)}(kr + s, ir - s)$ choices for P_1 since it is a $T^{(r)}$-path from $(0, -(ir-s))$ to $(kr+s, 0)$. There is only one choice for P_2 since it is a $(0, 1)$ step followed by a fixed number of $(r, 0)$ steps. Conversely, any such P_1P_2 stays below the diagonal and hence belongs to $\mathbb{A}_s(i, j)$. Summing on all possible k, we get a formula for $|\mathbb{A}_s(i, j)|$:

$$|\mathbb{A}_s(i, j)| = \sum_{k=0}^{j} T^{(r)}(kr + s, ir - s). \tag{78}$$

Multiplying both sides of equation (78) by x^iy^j, and summing on i and j, we have

$$\begin{aligned}\sum_{i,j=0}^{\infty}|\mathbb{A}_s(i,j)|x^iy^j &= \sum_{i,j=0}^{\infty}\sum_{k=1}^{i}T^{(r)}(kr+s,ir-s)x^iy^j\\ &= \sum_{i,j=0}^{\infty}\sum_{k=0}^{i}(\mathrm{CT}\,t^s\alpha^k\beta^i)x^iy^j\\ &= \mathrm{CT}\,t^s\sum_{j=0}^{\infty}\beta^ix^i\sum_{k=0}^{\infty}\alpha^ky^k\sum_{j\ge k}y^{j-k}\\ &= \mathrm{CT}\,t^s\frac{1}{(1-\alpha y)(1-\beta x)(1-y)}\end{aligned}$$

Similarly, any $Q\in\mathbb{B}_s(i,j)$ can be factored, according to its intersection with the line $x=0$, uniquely as Q_1Q_2, where Q_1 is a V_r path ending with a horizontal step and Q_2 is a $T^{(r)}$-path. Then $E(Q_1)=(0,-(k'r-s+1))$ for some $1\le k'\le i$. See the right picture of Figure 12. We see that even in the case $s=1$, k' cannot be zero, since otherwise Q_1 will go above the diagonal.

For any k', there is only one choice for Q_1 since it is a fixed number of $(0,r)$ steps followed by a $(1,0)$ step. There are $T^{(r)}(jr+s-1,k'r-s+1)$ choices for Q_2 since it is a $T^{(r)}$-path from $(0,-(k'r-s+1))$ to $(jr+s-1,0)$. Summing on all the possible k', we get a formula for $|\mathbb{B}_s(i,j)|$.

$$|\mathbb{B}_s(i,j)|=\sum_{k'=1}^{i}T^{(r)}(jr+s-1,k'r-s+1). \tag{79}$$

A similar computation shows that

$$\begin{aligned}\sum_{i,j=0}^{\infty}|\mathbb{B}_s(i,j)|x^iy^j &= \mathrm{CT}\,t^{s-1}\sum_{j=0}^{\infty}\alpha^jy^j\sum_{k'=1}^{\infty}\beta^{k'}x^{k'}\sum_{i\ge k'}x^{i-k'}\\ &= \mathrm{CT}\,t^{s-1}\frac{1}{1-\alpha y}\left(\frac{1}{1-\beta x}-1\right)\frac{1}{1-x}\\ &= \mathrm{CT}\,t^{s-1}\frac{1}{(1-\alpha y)(1-\beta x)(1-x)}-\mathrm{CT}\,t^{s-1}\frac{1}{(1-\alpha y)(1-x)}\\ &= \mathrm{CT}\,t^{s-1}\frac{1}{(1-\alpha y)(1-\beta x)(1-x)}-\delta_{s,1}\frac{1}{(1-y)(1-x)}.\end{aligned}$$

To compute the two generating functions does not seem easy, but their difference has a simple form.

$$\begin{aligned}&\mathrm{CT}\,t^s\frac{1}{1-\alpha y}\frac{1}{1-\beta x}\frac{1}{1-x}-\mathrm{CT}\,t^{s-1}\frac{1}{1-\alpha y}\frac{1}{1-\beta x}\frac{1}{1-y}\\ &\qquad\qquad= \mathrm{CT}\,t^{s-1}\frac{t-ty-1+x}{(1-x)(1-y)(1-\alpha y)(1-\beta x)}\end{aligned}$$

Direct computation shows that

$$\frac{t-ty-1+x}{(1-\alpha y)(1-\beta x)} = -\frac{x}{t^r(1-\beta x)} - \frac{1-y-t}{1-\alpha y}$$

It is easy to see that $-\dfrac{1-y-t}{1-\alpha y}$, belonging to $\mathbb{Q}[t][[y]]$, has constant term constant term -1 in t, and $-\dfrac{x}{t^r(1-\beta x)}$, belonging to $t^{-1}\mathbb{Q}[t^{-1}][[x]]$, has constant term 0. Hence

$$\mathrm{CT}\, t^{s-1} \frac{t-ty-1+x}{(1-x)(1-y)(1-\alpha y)(1-\beta x)} = -\delta_{s,1} \frac{1}{(1-x)(1-y)}.$$

Put the above altogether, we obtain

$$\sum_{i,j=0}^{\infty} (|\mathbb{A}_s(i,j)| - |\mathbb{B}_s(i,j)|) x^i y^j = 0. \qquad \square$$

Lemma 9.4.

$$\sum_{i,j\geq 0} T^{(r)}(ri,rj) x^i y^j = \frac{x(1-x)^{r-1} - y(1-y)^{r-1}}{x(1-x)^r - y(1-y)^r}. \tag{80}$$

Proof. Let $x = uv$ and $y = v$. Then

$$\begin{aligned} \sum_{i,j\geq 0} T^{(r)}(ri,rj) x^i y^j &= \sum_{i,j\geq 0} T^{(r)}(ri,rj) u^i v^{i+j} \\ &= \sum_{n\geq i\geq 0} T^{(r)}(ri, r(n-i)) u^i v^n \\ &= \sum_{n\geq i\geq 0} \mathrm{CT}\, \frac{\alpha^n}{t^{ri}} u^i v^n \\ &= \mathrm{CT}\, \frac{1}{1-ut^{-r}} \frac{1}{1-\alpha v}. \end{aligned}$$

We have

$$\begin{aligned}
\frac{1}{1-\alpha v} &= \frac{1-t}{(1-t)-v(1-t^{r+1})} \\
&= \frac{1}{1-v}\frac{1-t}{1-t\dfrac{1-t^r v}{1-v}} \\
&= \frac{1}{1-v}\frac{(1-t)\left(1+t\dfrac{1-t^r v}{1-v}+\cdots+\left(t\dfrac{1-t^r v}{1-v}\right)^{r-1}\right)}{1-t^r\left(t\dfrac{1-t^r v}{1-v}\right)^r} \\
&= \frac{1}{1-v}\frac{1-t^r\left(\dfrac{1-t^r v}{1-v}\right)^{r-1}}{1-t^r\left(t\dfrac{1-t^r v}{1-v}\right)^r} + \text{other terms}
\end{aligned}$$

Since the other terms contain only terms like $a_{rm+s}t^{rm+s}$ for $1 \le s \le r-1$, they do not contribute to the constant term in t. Let $z = t^r$. Then we have

$$\operatorname{CT} \frac{1}{1-ut^{-r}}\frac{1}{1-\alpha v} = \operatorname*{CT}_z \frac{1}{1-uz^{-1}}\frac{1}{1-v}\frac{1-z\left(\dfrac{1-zv}{1-v}\right)^{r-1}}{1-z\left(\dfrac{1-zv}{1-v}\right)^r},$$

where CT_z means to take the constant term of a function in z. Since the other part of the right side of the above equation is a formal power series in z, it is straightforward to obtain

$$\operatorname{CT} \frac{1}{1-ut^{-r}}\frac{1}{1-\alpha v} = \frac{1}{1-v}\frac{1-u\left(\dfrac{1-uv}{1-v}\right)^{r-1}}{1-u\left(\dfrac{1-uv}{1-v}\right)^r}.$$

Replacing u with x/y and v with y, we get formula (80) □

Theorem 9.5.

$$\det\left(g_{i+j}^{(r)}\right)_{0\le i,j\le n-1} = \det\left(T^{(r)}(ri,rj)\right)_{0\le i,j\le n-1} = \det\left(T^{(r-1)}(rj-i,ri-j)\right)_{0\le i,j\le n-1}.$$

Note that the identities in this theorem appeared in (72) and (73).

Proof. We use the technique of Section 4. The generating function for the first determinant is

$$\frac{xg^{(r)}(x)-yg^{(r)}(y)}{x-y}.$$

Since $f^{(r)}(x) = g^{(r)}(x) - 1$ is a formal power series in x satisfying

$$f^{(r)}(x) = x/\left(1 + f^{(r)}(x)\right)^{r+1},$$

we make the substitution $x \to x/(1+x)^{r+1}$ and $y \to y/(1+y)^{r+1}$. The generating function becomes

$$\frac{x(1+x)^{-r-1}(1+x) - y(1+y)^{-r-1}(1+y)}{x(1+x)^{-r-1} - y(1+y)^{-r-1}} = \frac{x(1+x)^{-r} - y(1+y)^{-r}}{x(1+x)^{-r-1} - y(1+y)^{-r-1}}.$$

Normalizing and dividing by $(1+x)(1+y)$, we get

$$\frac{x(1-x)^{r-1} - y(1-y)^{r-1}}{x(1-x)^{r} - y(1-y)^{r}}. \tag{81}$$

The first equality hence follows from Lemma 9.4.

To show the second equality, we compute the generating function for the third determinant.

$$\begin{aligned}
&\sum_{i,j\geq 0} T^{(r-1)}(rj-i, ri-j)x^i y^j \\
&\qquad = \sum_{i,j\geq 0} T^{(r-1)}((r-1)j + (j-i), (r-1)i - (j-i))x^i y^j \\
&\qquad = \sum_{i,j\geq 0} \mathrm{CT}\, t^{j-i}(1 + t + \cdots + t^{r-1})^j (1 + t^{-1} + \cdots + t^{-r+1})^i x^i y^j \\
&\qquad = \mathrm{CT} \sum_{i,j\geq 0} (t^{-1} + t^{-2} \cdots + t^{-r})^i x^i (t + t^2 + \cdots + t^r)^j y^j \\
&\qquad = \mathrm{CT}\, \frac{1}{1 - (t^{-1} + t^{-2} \cdots + t^{-r})x} \frac{1}{1 - (t + t^2 + \cdots + t^r)y}
\end{aligned} \tag{82}$$

Similarly, the generating function for the second determinant is

$$\mathrm{CT}\, \frac{1}{1 - (1 + t^{-1} + t^{-2} \cdots + t^{-r})x} \frac{1}{1 - (1 + t + t^2 + \cdots + t^r)y}. \tag{83}$$

The following computation shows that the (83) can be obtained from (82) by making the substitution $x \to x/(1-x)$ and $y \to y/(1-y)$, and then dividing by $(1-x)(1-y)$. This yields the second equality.

$$\begin{aligned}
&\mathrm{CT}\, \frac{1}{1 - (1 + t^{-1} + t^{-2} \cdots + t^{-r})x} \frac{1}{1 - (1 + t + t^2 + \cdots + t^r)y} \\
&= \mathrm{CT}\, \frac{1}{1 - x - (t^{-1} + t^{-2} \cdots + t^{-r})x} \frac{1}{1 - y - (t + t^2 + \cdots + t^r)y} \\
&= \mathrm{CT}\, \frac{1}{(1-x)(1-y)} \frac{1}{1 - (t^{-1} + t^{-2} \cdots + t^{-r})\frac{x}{1-x}} \frac{1}{1 - (t + t^2 + \cdots + t^r)\frac{y}{1-y}}. \qquad \square
\end{aligned}$$

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