On some Ramsey and Turán-type numbers for paths and cycles

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Abstract

For given graphs $G_1, G_2, ..., G_k$, where $k \geq 2$, the *multicolor Ramsey number* $R(G_1, G_2, ..., G_k)$ is the smallest integer n such that if we arbitrarily color the edges of the complete graph on n vertices with k colors, there is always a monochromatic copy of G_i colored with i, for some $1 \leq i \leq k$. Let P_k (resp. C_k) be the path (resp. cycle) on k vertices. In the paper we show that $R(P_3, C_k, C_k) = R(C_k, C_k) = 2k - 1$ for odd k. In addition, we provide the exact values for Ramsey numbers $R(P_4, P_4, C_k) = k + 2$ and $R(P_3, P_5, C_k) = k + 1$.

1 Introduction

In this paper all graphs considered are undirected, finite and contain neither loops nor multiple edges. Let G be such a graph. The vertex set of G is denoted by $V(G)$, the edge set of G by $E(G)$, and the number of edges in G by $e(G)$. C_m denotes the cycle of length m and P_m – the path on m vertices. For given graphs $G_1, G_2, ..., G_k, k \geq 2$, the *multicolor Ramsey number* $R(G_1, G_2, ..., G_k)$ is the smallest integer n such that if we arbitrarily color the edges of the complete graph of order n with k colors, then it always contains a monochromatic copy of G_i colored with i, for some $1 \leq i \leq k$. We only consider 3-color Ramsey numbers $R(G_1, G_2, G_3)$ (in other words we color the edges of K_n with colors red, blue and green). *The Turán number* $T(n, G)$ is the maximum number of edges in any n-vertex graph which does not contain a subgraph isomorphic to G. By $T'(n, G)$ we denote the maximum number of edges in any n-vertex non-bipartite graph which does not contain a subgraph isomorphic to G. A non-bipartite graph on n vertices is said to be *extremal with respect to* G if it does not contain a subgraph isomorphic to G and has exactly $T'(n, G)$ edges. By $T^*(n, G)$ we denote the maximum number of edges in any n-vertex bipartite graph which does not contain a subgraph isomorphic to G. For any $v \in V(G)$, by $r(v)$, $b(v)$ and $g(v)$ we denote the number of red, blue and green edges incident to v, respectively. The degree of vertex v will be denoted by $d(v)$ and the minimum degree of a vertex of G by $\delta(G)$. The open neighbourhood of vertex v is $N(v) = \{u \in V(G) | \{u, v\} \in E(G)\}$. $G_1 \cup G_2$ denotes the graph which consists of two disconnected subgraphs G_1 and G_2. kG stands for the graph consisting of k disconnected subgraphs G. We will use $G_1 + G_2$ to denote the join of G_1 and G_2, defined as $G_1 \cup G_2$ together with all edges between G_1 and G_2.

The remainder of this paper is organized as follows. Section 2 contains some facts on the numbers $T'(n, G)$, where G is a cycle. We first establish the exact value of $T'(n, C_k)$, where $k \leq n \leq 2k - 2$. Next, we continue in this fashion to obtain an upper bound for $T'(2k-1, C_k)$. Section 3 contains our main result that $R(P_3, C_k, C_k) = R(C_k, C_k) = 2k-1$, where C_k is the odd cycle on k vertices. The last Section 4 presents two new formulas for the following Ramsey numbers: $R(P_4, P_4, C_k) = k + 2$ and $R(P_3, P_5, C_k) = k + 1$.

2 Values of $T'(n, C_k)$

First, we present some facts which are often used in the paper.

Definition 1 *The* circumference $c(G)$ *of a graph* G *is the length of its longest cycle.*

Definition 2 *The* girth *of a graph* G *is the length of its shortest cycle.*

Definition 3 *A graph is called* weakly pancyclic *if it contains cycles of every length between the girth and the circumference.*

Theorem 4 (Brandt, [3]) *A non-bipartite graph* G *of order* n *and more than* $\frac{(n-1)^2}{4} + 1$ *edges contains all cycles of length between* 3 *and the length of the longest cycle* (*thus such a graph is weakly pancyclic of girth* 3).

Theorem 5 (Brandt, [4]) *Every non-bipartite graph* G *of order* n *with minimum degree* $\delta(G) \geq (n + 2)/3$ *is weakly pancyclic of girth* 3 *or* 4.

The following notation and terminology comes from [6].

For positive integers a and b define $r(a, b)$ as

$$r(a, b) = a - b\left\lfloor \frac{a}{b} \right\rfloor = a \bmod b.$$

For integers $n \geq k \geq 3$, define $w(n,k)$ as

$$w(n,k) = \frac{1}{2}(n-1)k - \frac{1}{2}r(k-r-1),$$

where $r = r(n-1, k-1)$.

Woodall's theorem [12] can then be written as follows.

Theorem 6 ([6]) *Let G be a graph on n vertices and m edges with $m \geq n$ and $c(G) = k$. Then*

$$m \leq w(n,k)$$

and this result is the best possible.

First, we state the following lemma.

Lemma 7 *If $n \geq 2k-3$ and $k \geq 1$, then $T^*(kK_2, n) = (k-1)n - (k-1)^2$.*

Proof. The proof is by induction on k. It is clear that $T^*(K_2, n) = 0$ for any integer n. It is easy to see that $K_{1,r}$ for $r \geq 1$ and K_3 are the only connected graphs which do not contain $K_2 \cup K_2$. Thus we obtain $T^*(2K_2, n) = n-1$ for all n, since K_3 is not bipartite.

Let G be any nonempty bipartite graph of order n, which does not contain kK_2. Choose any edge vw. Define H to be the subgraph induced by $V(G) - \{v, w\}$. Obviously H cannot contain $(k-1)K_2$, so by the induction hypothesis $e(H) \leq (k-2)(n-2) - (k-2)^2$. Since G is bipartite, so the number of edges with at least one vertex in $\{v, w\}$ is not greater than $n-1$. Thus we obtain $e(G) \leq (k-2)(n-2) - (k-2)^2 + (n-1) = (k-1)n - (k-1)^2$, which implies $T^*(kK_2, n) \leq (k-1)n - (k-1)^2$. The graph $K_{k-1,n-k+1}$ implies that $T^*(kK_2, n) \geq (k-1)n - (k-1)^2 = (k-1)(n-k+1)$. □

Lemma 8 *Let G be a bipartite graph of order $2k-2$ with $k^2 - 3k + 4$ edges, where k is odd and $k \geq 9$. Then any two vertices, which belong to different sides of the bipartition, are joined by a path of length $k-2$.*

Proof. Let $\{X, Y\}$ be the bipartition of G and choose any two vertices $x \in X$, $y \in Y$. Graph G can be seen as a complete bipartite graph without at most $k-3$ edges. Define $X' = (X \setminus \{x\}) \cap N(y)$ and $Y' = (Y \setminus \{y\}) \cap N(x)$. The number of edges in G guarantees that $|X'| \geq 1$, $|Y'| \geq 1$ and $|X'| + |Y'| \geq 2k - 4 - (k-3) = k-1$. Thus the complete bipartite graph with bipartition $\{X', Y'\}$ contains at least $k-2$ edges, so at least one of them, say vw, where $v \in X'$ and $w \in Y'$ must belong to G as well. In this way we obtain path $xwvy$, which is a path of length 3 joining x and y. Now we will show that any path of length at least 3 but shorter than $k-2$ which joins x and y can be extended by two additional vertices to a longer path joining x and y, which by induction completes the proof.

Assume that x and y are joined by a path P of length $k-p$, where $4 \leq p \leq k-3$. Define $G' = G[V(G) \setminus V(P)]$. We have $e(G') = e(G) - e(P) - |\{vw \in E(G) : v \in P, w \in$

$G'\}| \geq k^2-3k+4-(k-p+1)^2/4-(k-p+1)(k+p-3)/2$. From Lemma 7 we have $T^*((p/2+1)K_2, k+p-3) = (p^2+2kp-6p)/4$. One can easily verify that this implies $e(G') \geq T^*((p/2+1)K_2, k+p-3)$ and thus G' contains $p/2+1$ independent edges. Assume that there is no path of order $k-p+2$ joining x and y in graph G. In this case any edge from G' can be connected to at most $(k-p+1)/2$ vertices from P or in other words cannot be connected to at least $(k-p+1)/2$ vertices from P. So we have $e(G) \leq e(K_{k-1,k-1}) - |\{vw \notin E(G) : v \in P, w \in G'\}| \leq (k-1)^2-(p/2+1)(k-p+1)/2 = k^2-(10+p)k/4+(p^2+p+2)/4 < k^2-3k+4 = e(G)$, a contradiction. Hence there must be a path of order $k-p+2$ joining x and y in graph G. $\square$

Theorem 9 *For odd integers $k \geq 5$*

$$T'(n, C_k) = w(n, k-1),$$

where $k \leq n \leq 2k-2$.

Proof. The last part of the thesis of Theorem 6 implies that $T'(n, C_k) \geq w(n, k-1)$. Let us suppose that there exists a non-bipartite C_k-free graph G' on n vertices which has more than $w(n, k-1)$ edges. Observe that $w(n,k)$ is not a decreasing function of k and of n, i.e. $w(n,k_1) \geq w(n,k_2)$ if $k_1 > k_2$ and $w(n_1,k) \geq w(n_2,k)$ if $n_1 > n_2$. Then, graph G' must contain a cycle of length greater than k. Now, we prove that $w(n,k-1)+1 > \frac{(n-1)^2}{4}+1$. The maximal possible value of n is $2k-2$. Then, the left-hand side is equal to k^2-3k+4 and the right-hand side is equal to $k^2-3k+\frac{13}{4}$, so by Brandt's theorem graph G' contains C_k. For the case $n = 2k-3$ we obtain that $r(n-1, k-2) = 0$ and $w(n,k-1)+1 > \frac{(n-1)^2}{4}+1$, and G' also contains a cycle of length k. For the case $n \leq 2k-4$ we have that $r(n-1,k-2) = n-(k-1)$. Then, $w(n,k-1)+1 = \frac{1}{2}n^2+k^2-kn-3k+\frac{3}{2}n+3$ and the inequality $w(n,k-1)+1 > \frac{(n-1)^2}{4}+1$ implies the following inequality: $\frac{n^2}{4}+n(2-k)+k^2+\frac{7}{4} > 3k$. The minimal value of the left-hand side holds for $n = 2k-4$ and it is equal to $4k-2.25$, so for $k \geq 3$ graph G' contains a cycle of length k. $\square$

Theorem 10 *For odd integers $k \geq 9$*

$$T'(2k-1, C_k) \leq \frac{(2k-2)^2}{4} - 1 = (k-1)^2 - 1.$$

Proof. Let G be a non-bipartite graph of order $2k-1$. By Theorem 4 and by property $w(2k-1,k-1) = k^2-3k+5 < \frac{(2k-2)^2}{4}+2$ we obtain that if G has at least $\frac{(2k-2)^2}{4}+2$ edges, then it contains C_k.

Assume that G has $\frac{(2k-2)^2}{4}+1 = k^2-2k+2$ edges. Suppose that there is a vertex $v \in V(G)$ such that $d(v) \leq k-2$. If $G-v$ is a non-bipartite subgraph, we immediately obtain a contradiction with $T'(2k-2, C_k) = k^2-3k+3$, so $G-v$ must be bipartite. It is clear that vertex v must be joined to at least one vertex in each side of the bipartition of $G-v$. Applying Lemma 8 we find a cycle C_k in graph G, so we have that $\delta(G) = k-1$. In this case, the number of edges of graph G is at least $\frac{(2k-1)(k-1)}{2} = k^2 - \frac{3}{2}k + \frac{1}{2} > k^2-2k+2$, a contradiction. These observations lead us to the conclusion that a non-bipartite graph G on $2k-1$ vertices and $\frac{(2k-2)^2}{4}+1$ edges must contain a cycle C_k.

Consider the last case when G has $(k-1)^2$ edges. Since $w(2k-1, k-1) < (k-1)^2$ for $k > 4$ and $w(k,n)$ is a non-decreasing function of k and n, graph G must contain a cycle of length at least k. It follows that $\delta(G) \geq k-2$. We obtain this property using the same arguments as those in the previous case. Since $k-2 \geq (2k+1)/3$ for $k \geq 7$, then by Theorem 5 graph G is weakly pancyclic of girth 3 or 4, so it contains a cycle of length k. $\square$

Finally, for the sake of completeness we recall a few Turán numbers for short paths. In 1975 Faudree and Schelp proved

Theorem 11 ([9]) *If G is a graph with $|V(G)| = kt + r$, $0 \leq r < k$, containing no path on $k+1$ vertices, then $|E(G)| \leq t\binom{k}{2} + \binom{r}{2}$ with equality if and only if G is either $(tK_k) \cup K_r$ or $((t-l-1)K_k) \cup (K_{(k-1)/2} + \overline{K}_{(k+1)/2+ik+r})$ for some l, $0 \leq l < t$, when k is odd, $t > 0$, and $r = (k \pm 1)/2$.*

It is easy to check that we obtain the following

Corollary 12 *For all integers $n \geq 3$*

$$T(n, P_3) = \left\lfloor \frac{n}{2} \right\rfloor$$

$$T(n, P_4) = \begin{cases} n & \text{if } n \equiv 0 \mod 3 \\ n-1 & \text{otherwise.} \end{cases}$$

$$T(n, P_5) = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \mod 4 \\ \frac{3n}{2} - 2 & \text{if } n \equiv 2 \mod 4 \\ \frac{3n}{2} - \frac{3}{2} & \text{otherwise} \end{cases}$$

3 Ramsey numbers for odd cycles

In 1973 Bondy and Erdős proved that

Theorem 13 ([2]) *For odd integers $k \geq 5$*

$$R(C_k, C_k) = 2k-1$$

In 1983 Burr and Erdős gave the following Ramsey number.

Theorem 14 ([5])

$$R(P_3, C_3, C_3) = 11$$

In 2005 the first author determined two further numbers of this type.

Theorem 15 ([8])

$$R(P_3, C_5, C_5) = 9$$

$$R(P_3, C_7, C_7) = 13$$

Now, we prove our the main result of the paper.

Theorem 16 *For odd integers* $k \geq 9$

$$R(P_3, C_k, C_k) = R(C_k, C_k) = 2k - 1$$

Proof. Let the complete graph G on $2k - 2$ vertices be colored with two colors, say blue and green, as follows: the vertex set $V(G)$ of G is the disjoint union of subsets G_1 and G_2, each of order $k - 1$ and completely colored blue. All edges between G_1 and G_2 are colored green. This coloring contains neither monochromatic (blue or green) cycle C_k nor a monochromatic (red) path of length 2. We conclude that $R(P_3, C_k, C_k) \geq 2k - 1$.

Assume that the complete graph K_{2k-1} is 3-colored with colors red, blue and green. By Corollary 12, in order to avoid a red P_3, there must be at most $k - 1$ red edges. Suppose that K_{2k-1} contains at most $k - 1$ red edges and contains neither a blue nor a green C_k. Since the number of blue and green edges is greater or equal to $\binom{2k-1}{2} - (k-1) = 2(k-1)^2$, at least one of the blue or green color classes (say blue) contains at least $(k - 1)^2$ edges. If the blue color class is bipartite, then one of the partition sets has at least k vertices. Since $R(P_3, C_k) = k$ for $k \geq 5$ [11], the graph induced by this partition has to contain a red P_3 or a green C_k, so blue edges enforce a non-bipartite subgraph of order $2k - 1$ with at least $(k - 1)^2$ edges which by Theorem 10 contains a blue C_k. $\square$

4 The Ramsey numbers $R(P_l, P_m, C_k)$

This section makes some observations on 3-color Ramsey numbers for two short paths and one cycle of arbitrary length.

In [1] we find two values of Ramsey numbers: $R(P_4, P_4, C_3) = 9$ and $R(P_4, P_4, C_4) = 7$. By using simple combinatorial properties (without the aid of computer calculations) we proved: $R(P_4, P_4, C_5) = 9$ and $R(P_4, P_4, C_6) = 8$ (see [7] for details).

Theorem 17

$$R(P_4, P_4, C_7) = 9.$$

Proof. The proof of $R(P_4, P_4, C_7) \geq 9$ is very simple, so it is left to the reader. Let the vertices of K_9 be labeled $1, 2, \ldots, 9$. Since $R(P_4, P_4, C_6) = 8$, we can assume $1, 2, 3, 4, 5, 6$ to be the vertices of green C_6. If the subgraph induced by green edges of K_9 is bipartite, then since $R(P_4, P_4) = 5$, we immediately obtain a red or a blue P_4. Since $T(9, P_4) = 9$, the number of green edges is at least $18 > \frac{(9-1)^2}{4} + 1$, so the non-bipartite subgraph induced by green edges of K_9 is weakly pancyclic. Since $R(P_4, P_4, C_3) = 9$, this subgraph contains green cycles of every length between 3 and the green circumference. Avoiding a green cycle C_7 we know that the number of green edges from vertices $7, 8, 9$ to the green cycle is at most 3. We have to consider the two following cases.

1. *There is a vertex $v \in \{7, 8, 9\}$ which has three green edges to the vertices of green cycle C_6.* We can assume that the edges $\{1, 7\}$, $\{3, 7\}$, $\{5, 7\}$ are green. In this case the edges $\{2, 4\}$, $\{4, 6\}$, $\{2, 6\}$ are red or blue. Without loss of generality we can assume that $\{2, 4\}$ and $\{4, 6\}$ are red. This enforces $\{2, 7\}$, $\{6, 7\}$ to be blue and $\{2, 8\}$, $\{6, 8\}$ to be green, and we obtain a green cycle of length 8 and then a green C_7.

2. *There is a vertex $v \in \{7, 8, 9\}$ which has two green edges to the vertices of green cycle C_6.* We have to consider two subcases.

   (i) *The edges $\{1, 7\}$, $\{3, 7\}$ are green and $\{2, 7\}$, $\{4, 7\}$, $\{5, 7\}$, $\{6, 7\}$ are red or blue.* This enforces $\{2, 6\}$ and $\{2, 4\}$ to be red or blue. We obtain two situations. In the first, if edge $\{2, 6\}$ is red and $\{2, 4\}$ blue, then we can assume that edge $\{2, 7\}$ is blue, then $\{5, 7\}$ is red and we obtain a red or a blue P_4 with edge $\{6, 7\}$. In the second, if edges $\{2, 6\}$ and $\{2, 4\}$ are red, then $\{4, 7\}$, $\{6, 7\}$ are blue and $\{4, 8\}$, $\{6, 8\}$, $\{4, 9\}$, $\{6, 9\}$ are green. Edge $\{2, 6\}$ cannot be green. If edge $\{5, 8\}$ is red, then we obtain a blue P_4: $2 - 5 - 7 - 6$ and if $\{5, 8\}$ is blue, then we have a red P_4: $6 - 2 - 5 - 7$.

   (ii) *The edges $\{1, 7\}$, $\{4, 7\}$ are green and $\{2, 7\}$, $\{3, 7\}$, $\{5, 7\}$, $\{6, 7\}$ are red or blue.* Then vertex 8 and vertex 9 have green edges to at most one vertex from $\{2, 3, 5, 6\}$, otherwise we have either the situation considered in (i) or a green cycle of length 8. By simple considering red and blue edges from $\{7, 8, 9\}$ to $\{2, 3, 5, 6\}$, we obtain a red or a blue P_4.

We obtain that there are at least 15 non-green edges from $\{7, 8, 9\}$ to the vertices of the green C_6. We can assume that there are at least 8 blue edges among them and we immediately have a blue P_4. □

Theorem 18 *For all integers $k \geq 6$*

$$R(P_4, P_4, C_k) = k + 2.$$

Proof. The critical coloring which gives us the lower bound $k+2$ is easy to obtain, so we only give a proof for the upper bound. This proof can be easily deduced from Turán numbers and the theorems given above. By Theorem 9 and Corollary 12 we obtain that $T'(k+2, C_k) = \frac{1}{2}k^2 - \frac{3}{2}k + 7$ for $k \geq 5$ and $T(k+2, P_4) \leq k+2$. It is easy to check that $T'(k+2, C_k)$ is greater than the maximal number of edges in a bipartite graph on $k+2$ vertices, thus $T(k+2, C_k) = T'(k+2, C_k)$. Suppose that we have a 3-coloring of the complete graph K_{k+2}. This graph has $\frac{1}{2}k^2 + \frac{3}{2}k + 1$ edges. Note that $T(k+2, C_k) + 2T(k+2, P_4) \leq \frac{1}{2}k^2 + \frac{1}{2}k + 11 < \frac{1}{2}k^2 + \frac{3}{2}k + 1$ for all $k > 10$. If $k \in \{8, 9, 10\}$, we obtain that $T(k+2, C_k) + 2T(k+2, P_4) \leq \binom{k+2}{2}$ with equality for $k = 8$ and $k = 10$, so $R(P_4, P_4, C_9) = 11$. By Theorem 11 we know the properties of the extremal graphs with respect to P_4 and by Theorem 9 and [6] we can describe the extremal graphs with respect to C_k, so it is easy to check that the theorem holds for the remaining cases when $k \in \{8, 10\}$. $\square$

The following lemma will be useful in further considerations.

Lemma 19 *Suppose that graph G has $k+1$ vertices and it contains a cycle C_k and suppose that we have a vertex $v \notin V(C_k)$, which is joined by r edges to C_k, where $2 \leq r \leq k$. Then one of the following two possibilities holds:*

(i) *G contains a cycle C_{k+1}.*

(ii) *$G' = G[V(C_k)]$ contains at most $\frac{k(k-1)}{2} - \frac{r(r-1)}{2}$ edges.*

Proof. Let $C = x_1 x_2 x_3 \ldots x_k$ be a cycle C_k and $v \notin V(C)$ be a vertex, which is joined by $d(v) = r$ edges to C, where $2 \leq r \leq k$. First, if $r \geq \lceil \frac{k}{2} \rceil$, then we immediately have a cycle C_{k+1} and (i) follows. Assume that $2 \leq r \leq \lceil \frac{k}{2} - 1 \rceil$. Let the vertices of C, which are joined by an edge to vertex v, be labeled $p_{i_1}, p_{i_2}, \ldots, p_{i_r}$. If any two of them are adjacent, then we obtain the cycle C_{k+1} and (i) follows. Otherwise, consider the following vertices: $p_{i_1+1}, p_{i_2+1}, \ldots, p_{i_r+1}$. In order to avoid a cycle C_{k+1}, these vertices must be mutually nonadjacent and G' contains at most $\frac{k(k-1)}{2} - \frac{r(r-1)}{2}$ edges. $\square$

Theorem 20 *For all integers $k \geq 8$*

$$R(P_3, P_5, C_k) = k + 1.$$

Proof. A critical coloring which gives us the lower bound $k+1$ is very simple, so all we need is the upper bound. It is easy to see that simply using Turán numbers does not give us the proof. Indeed, the sum $T(k+1, P_3) + T(k+1, P_5) + T(k+1, C_n)$ is far greater than the maximal number of edges in the complete graph on $k+1$ vertices. Suppose that we have a 3-coloring of K_{k+1} with colors red, blue and green which neither contains a red P_3, nor a blue P_5, nor a green C_k. K_{k+1} has to contain a green cycle C_{k-1}. Indeed, suppose that this is not the case. Since $T(k+1, P_3) + T(k+1, P_5) + T(k+1, C_{k-1}) < \binom{k+1}{2}$ for $k > 11$, we obtain either a red P_3 or a blue P_5. For the case of $k \in \{8, 9, 10, 11\}$ we use the properties of the extremal graphs with respect to P_3 and P_5 and we also obtain either a red P_3 or a blue P_5. Let the vertices of K_{k+1} be labeled $v_0, v_1, ..., v_k$. We can assume the first $k-1$ vertices to be the vertices of green C_{k-1}. It is easy to see that $b(v_{k-1})$ and $b(v_k)$ are greater or equal to $k - \lfloor (k-1)/2 \rfloor - 1$. Note that in order to avoid a blue P_5 we obtain that the vertices v_{k-1} and v_k have no common vertex which belongs to $V(C_{k-1})$ and which is joined by a blue edge to them. If the vertex v_{k-1} or v_k is joined by at least 4 green edges to the vertices of C_{k-1}, then by Lemma 19 and $R(P_3, P_5) = 5$ we have a blue P_5. If v_{k-1} and v_k are joined by at most 3 green edges to the vertices of C_{k-1}, then by Lemma 19 and $R(P_3, P_4) = 4$ we obtain a blue P_4. If $k \geq 9$ then we also have a blue P_5. In the case $k = 8$ by simple considering possible colorings of the edges of v_{k-1} and v_k we obtain either a red P_3, or a blue P_5, or else a green C_k. □

References

[1] Arste J., Klamroth K., Mengersen I.: Three color Ramsey numbers for small graphs, *Util. Math.* **49** (1996) 85–96.

[2] Bondy J.A., Erdős P.: Ramsey numbers for cycles in graphs, *J. Combin. Theory Ser. B* **14** (1973) 46–54.

[3] Brandt S.: A sufficient condition for all short cycles, *Disc. Appl. Math.* **79** (1997) 63–66

[4] Brandt S.: Sufficient conditions for graphs to contain all subgraphs of a given type, *Ph.D. Thesis*, Freie Universität Berlin, 1994.

[5] Burr A., Erdős P.: Generalizations of a Ramsey-theoretic result of Chvatal, *J. Graph Theory* **7** (1983) 39–51.

[6] Caccetta L., Vijayan K.: Maximal cycles in graphs, *Disc. Math.* **98** (1991) 1–7.

[7] Dzido T.: Computer experience from calculating some 3-color Ramsey numbers, *Technical Report of Gdańsk University of Technology, ETI Faculty* **18/03** (2003).

[8] Dzido T.: Multicolor Ramsey numbers for paths and cycles, *Discuss. Math. Graph Theory* **25** (2005) 57–65.

[9] Faudree R.J., Schelp R.H.: Path Ramsey numbers in multicolorings, *J. Combin. Theory Ser. B* **19** (1975) 150–160.

[10] Greenwood R.E., Gleason A.M.: Combinatorial relations and chromatic graphs, *Canad. J. Math.* **7** (1955) 1–7.

[11] Radziszowski S.P.: Small Ramsey numbers, *Electronic Journal of Combinatorics*, Dynamic Survey 1, revision #10, July 2004, *http://www.combinatorics.org*.

[12] Woodall D.R.: Maximal circuits of graphs I, *Acta Math. Acad. Sci. Hungar.* **28** (1976) 77–80.