Total domination and matching numbers in claw-free graphs

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Abstract

A set M of edges of a graph G is a matching if no two edges in M are incident to the same vertex. The matching number of G is the maximum cardinality of a matching of G. A set S of vertices in G is a total dominating set of G if every vertex of G is adjacent to some vertex in S. The minimum cardinality of a total dominating set of G is the total domination number of G. If G does not contain $K_{1,3}$ as an induced subgraph, then G is said to be claw-free. We observe that the total domination number of every claw-free graph with minimum degree at least three is bounded above by its matching number. In this paper, we use transversals in hypergraphs to characterize connected claw-free graphs with minimum degree at least three that have equal total domination and matching numbers.

Keywords: claw-free, matching number, total domination number

1 Introduction

Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [3] and is now well studied in graph theory. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [5, 6].

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Let $G = (V, E)$ be a graph with vertex set V and edge set E. A set $S \subseteq V$ is a *total dominating set*, abbreviated TDS, of G if every vertex in V is adjacent to a vertex in S. Every graph without isolated vertices has a TDS, since $S = V$ is such a set. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS of G. A TDS of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set.

Two edges in a graph G are *independent* if they are not adjacent in G. A set of pairwise independent edges of G is called a *matching* in G, while a matching of maximum cardinality is a *maximum matching*. The number of edges in a maximum matching of G is called the *matching number* of G which we denote by $\alpha'(G)$. A *perfect matching* in G is a matching with the property that every vertex is incident with an edge of the matching. Matchings in graphs are extensively studied in the literature (see, for example, the survey articles by Plummer [10] and Pulleyblank [11]).

For notation and graph theory terminology we in general follow [5]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order $n(G) = |V|$ and edge set E of size $m(G) = |E|$, and let v be a vertex in V. The *open neighborhood* of v in G is $N(v) = \{u \in V \mid uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$ and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. If $Y \subseteq V$, then the set S is said to *dominate* the set Y if $Y \subseteq N[S]$, while S *totally dominates* Y if $Y \subseteq N(S)$.

Throughout this paper, we only consider finite, simple undirected graphs without isolated vertices. For a subset $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. A vertex of degree k we call a *degree-k vertex*. We denote the minimum degree of the graph G by $\delta(G)$ and its maximum degree by $\Delta(G)$. A graph G is *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$. A graph is *cubic* if every vertex has degree 3, while we say that a graph is *almost cubic* if it has one vertex of degree 4 and all other vertices of degree 3.

The *transversal number* $\tau(H)$ of a hypergraph H is the minimum number of vertices meeting every edge. For a graph $G = (V, E)$, we denote by H_G the open neighborhood hypergraph, abbreviated ONH, of G; that is, H_G is the hypergraph with vertex set $V(H_G) = V$ and with edge set $E(H_G) = \{N_G(x) \mid x \in V(G)\}$ consisting of the open neighborhoods of vertices of V in G. We observe that $\gamma_t(G) = \tau(H_G)$.

A hypergraph H is said to be k-uniform if every edge of H has size k. We call an edge of H that contains ℓ vertices an ℓ-edge. If H has vertex set V and $X \subseteq V$, we denote by $H \setminus X$ the induced subhypergraph on $V \setminus X$; that is, we delete all the vertices of X, and all the edges having a vertex in X. We denote the degree of v in a hypergraph H by $d_H(v)$, or simply by $d(v)$ if H is clear from context. The hypergraph H is said to be *regular* if every vertex of H has the same degree.

2 Known Hypergraph Results

2.1 Hypergraph Results

Chvátal and McDiarmid [2] and Tuza [15] independently established the following result about transversals in hypergraphs (see also [14] for a short proof of this result).

Theorem 1 ([2, 15]) *If H is a hypergraph on n vertices and m edges with all edges of size at least three, then $4\tau(H) \le n + m$.*

We shall need the following definition.

Definition 1 *Let $i, j \ge 0$ be arbitrary integers. Let $H_{i,j}^{4edge}$ be the hypergraph defined as follows. Let the vertex set and edge set of $H_{i,j}^{4edge}$ be defined as follows.*

$$\begin{aligned}
V(H_{i,j}^{4edge}) &= \{u, x_0, x_1, \ldots, x_i, y_0, y_1, \ldots, y_i, w_0, w_1, \ldots, w_j, z_0, z_1, \ldots, z_j\}, \\
E_1 &= \bigcup_{a=1}^{i} \{\{x_{a-1}, x_a, y_a\}, \{y_{a-1}, x_a, y_a\}\}, \\
E_2 &= \bigcup_{b=1}^{j} \{\{w_{b-1}, w_b, z_b\}, \{z_{b-1}, w_b, z_b\}\}, \\
E(H_{i,j}^{4edge}) &= \{\{u, x_0, y_0\}, \{u, w_0, z_0\}, \{x_0, y_0, z_0, w_0\}\} \cup E_1 \cup E_2.
\end{aligned}$$

Let

$$H^{4edge} = \bigcup_{i \ge 0} \bigcup_{j \ge 0} \{H_{i,j}^{4edge}\}.$$

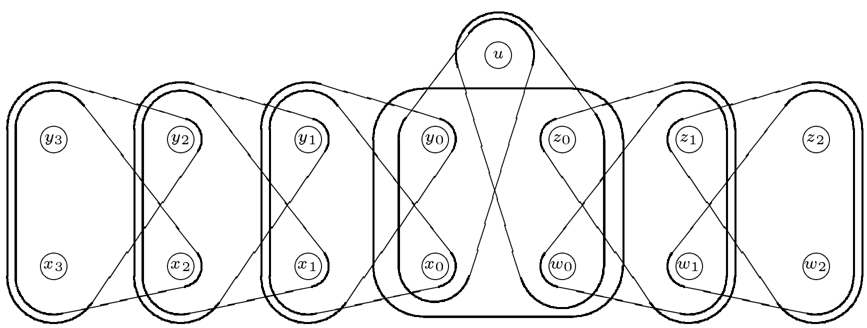


Figure 1: The hypergraph $H_{3,2}^{4edge}$.

In Figure 1 we give an example of a hypergraph in the family H^{4edge}.

We shall need the following result from [8].

Theorem 2 ([8]) *Let H be a connected hypergraph on n vertices and m edges where all edges contain at least three vertices. If H is not 3-uniform and $4\tau(H) = n + m$, then $H \in H^{4edge}$.*

2.2 Known Graph Results

As an immediate consequence of Theorem 1, we have that the total domination number of a graph with minimum degree at least 3 is at most one-half its order.

Theorem 3 *If G is a graph of order n with $\delta(G) \geq 3$, then $\gamma_t(G) \leq n/2$.*

Proof. The ONH hypergraph H_G of G has n vertices and n edges with all edges of size at least three. By Theorem 1, there exists a transversal in H_G of size at most $(n+n)/4 = n/2$. Hence, $\gamma_t(G) = \tau(H_G) \leq n/2$. □

We remark that Archdeacon et al. [1] recently found an elegant one page graph theoretic proof of Theorem 3.

The connected claw-free cubic graphs achieving equality in Theorem 3 are characterized in [4] and contain at most eight vertices.

G_1 :

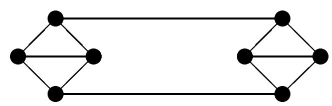


Figure 2: A claw-free cubic graph G_1 with $\gamma_t(G_1) = n/2$.

Theorem 4 ([4]) *If G is a connected claw-free cubic graph of order n, then $\gamma_t(G) \leq \lfloor n/2 \rfloor$ with equality if and only if $G = K_4$ or $G = G_1$ where G_1 is the graph shown in Figure 2.*

We now turn our attention to matchings in claw-free graphs. The following result was established independently by Las Vergnas [9] and Sumner [12, 13].

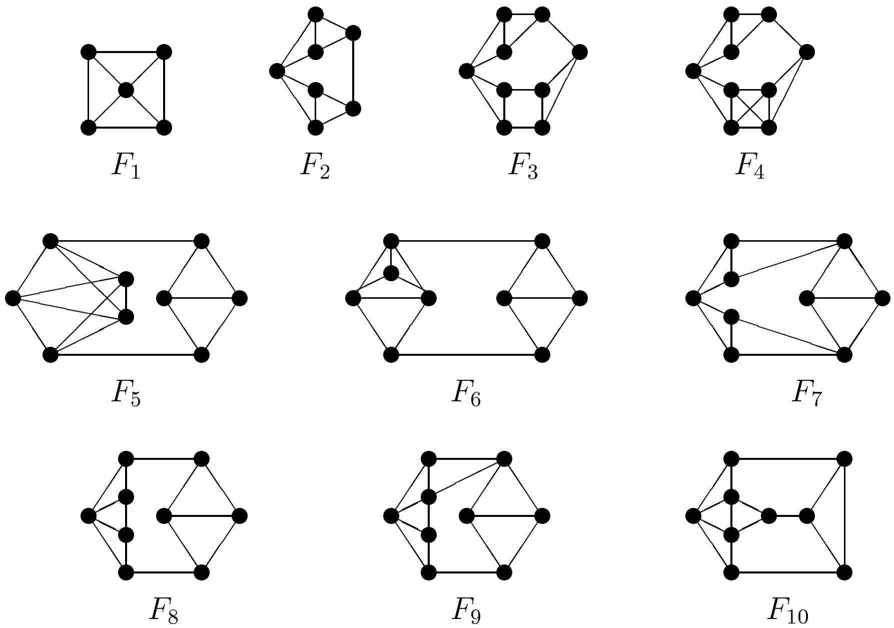
Theorem 5 ([9, 12, 13]) *Every claw-free graph of even order has a perfect matching.*

As a consequence of Theorem 5, we have the following result which was observed in [7].

Theorem 6 *If G is a claw-free graph of order n, then $\alpha'(G) = \lfloor n/2 \rfloor$.*

As a consequence of Theorems 3 and 6, it follows that the total domination number of every claw-free graph with minimum degree at least three is bounded above by its matching number. This result was first observed in [7].

Theorem 7 ([7]) *For every claw-free graph G with $\delta(G) \geq 3$, $\gamma_t(G) \leq \alpha'(G)$.*



3 Main Result

Our aim in this paper is to characterize the connected claw-free graphs with minimum degree at least three that achieve equality in the bound of Theorem 7. For this purpose, we define a collection $\mathcal{F}$ of connected claw-free graphs with minimum degree three and maximum degree four that have equal total domination and matching numbers. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_{12}\}$ be the collection of twelve graphs shown in Figure 3.

We shall prove:

Theorem 8 *Let G be a connected claw-free graph with $\delta(G) \geq 3$. Then, $\gamma_t(G) = \alpha'(G)$ if and only if $G \in \mathcal{F} \cup \{K_4, K_5 - e, K_5, G_1\}$.*

4 Proof of Theorem 8

The sufficiency is straightforward to verify. As a consequence of Theorem 4, the graph K_4 and the graph G_1 of Figure 2 are the only connected claw-free cubic graphs that achieve equality in the bound of Theorem 7. Hence it remains for us to characterize the

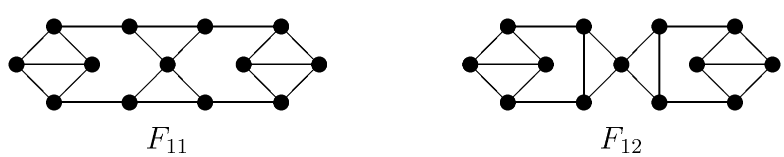


Figure 3: The collection $\mathcal{F}$ of twelve graphs.

connected claw-free graphs with minimum degree at least three that are not cubic and achieve equality in the bound of Theorem 7. We shall prove:

Theorem 9 *If G is a connected claw-free graph with minimum degree at least three and maximum degree at least four satisfying $\gamma_t(G) = \alpha'(G)$, then $G \in \mathcal{F} \cup \{K_5 - e, K_5\}$.*

Proof. Let $G = (V, E)$ have order n. By Theorem 6, $\alpha'(G) = \lfloor n/2 \rfloor$. If T is a transversal of H_G, then $|T| \geq \tau(H_G) = \gamma_t(G) = \alpha'(G)$. Hence we have the following observation.

Observation 1 *Every transversal in H_G has size at least $\lfloor n/2 \rfloor$.*

We shall frequently use the following observation, which is an application of Theorem 1.

Observation 2 *If $V' \subset V$ and $H' = H_G \setminus V'$ is a subhypergraph of H_G of order n' and size m' in which every edge has size at least 3, then there exists a transversal T' of H' such that $|T'| \leq (m' + n')/4$.*

Let v be a vertex of maximum degree in G, and so $d(v) = \Delta(G) \geq 4$.

Observation 3 *n is odd.*

Proof. If n is even, then in Observation 2, taking $V' = \{v\}$, we have $n' = n-1$, $m' \leq n-4$ and $|T'| \leq (2n-5)/4$. Thus, $T = T' \cup \{v\}$ is a transversal of H_G of size less than $\lfloor n/2 \rfloor$, contradicting Observation 1. Hence, n is odd. □

As a consequence of Theorem 2, we have the following observation.

Observation 4 $\Delta(G) = 4$.

Proof. Suppose that $\Delta(G) \geq 5$. Let $H' = H_G \setminus \{v\}$. Then, H' has order $n' = n - 1$ and size $m' \leq n - 5$, and every edge of H' contains at least three vertices. Since H' contains the edge $N(v)$, H' has at least one edge of size five or more. Hence, by Theorem 2, $\gamma_t(G) \leq \tau(H') + 1 \leq (n' + m' - 1)/4 + 1 \leq (2n - 3)/4$. The desired result now follows from the fact that n is odd. □

By Observations 3 and 4, G contains an odd number of degree-4 vertices. Furthermore, by Theorem 6, $\alpha'(G) = (n-1)/2$ and, by Observation 1, every transversal in H_G has size at least $(n-1)/2$. As a consequence of Theorem 2, we have the following result.

Observation 5 *Every two degree-4 vertices in G are at distance at most 2 apart.*

Proof. Suppose that G contains two degree-4 vertices, say u and v, at distance at least 3 apart. Let $H' = H_G \setminus \{u, v\}$. Then, H' has order $n' = n - 2$ and size $m' = n - 8$, and every edge of H' is a 3-edge or a 4-edge. Further, H' has at least two 4-edges, namely $N(u)$ and $N(v)$. Let H_v be the component of H' containing the 4-edge $N(v)$ (possibly, $H_v = H'$). Let $N(v) = \{v_1, v_2, v_3, v_4\}$.

Suppose $H_v \in H^{4edge}$. Then, H_v contains an edge $\{v_1, v_2, v_5\}$ containing v_1 and v_2, and an edge $\{v_3, v_4, v_5\}$ containing v_3 and v_4. Since the edges $N(v_i)$, $1 \le i \le 4$, are deleted from H_G when constructing H', there must exist vertices v_6 and v_7 in H_v such that in the graph G, $N(v_6) = \{v_1, v_2, v_5\}$ and $N(v_7) = \{v_3, v_4, v_5\}$. Thus in G, v_5v_6 and v_5v_7 are edges. Let $w \in N(v_5) \setminus \{v_6, v_7\}$. By the claw-freeness of G, we must have that wv_6 or wv_7 is an edge, implying that $w \in N(v)$. We may assume that $w = v_1$. Thus since $H_v \in H^{4edge}$, $N(v_5) = \{v_1, v_6, v_7\}$ and there exists a vertex v_8 such that in G, $N(v_8) = \{v_2, v_6, v_7\}$. But then $d(v_6) \ge 4$, contradicting our earlier observation that $N(v_6) = \{v_1, v_2, v_5\}$. Hence, $H_v \notin H^{4edge}$.

By Theorem 2, $4\tau(H_v) \le |V(H_v)| + |E(H_v)| - 1$. Applying Theorems 1 and 2 to every other component of H', if any, it follows that $4\tau(H') \le n' + m' - 1 = 2n - 11$. However if T' is a transversal of H', then $T' \cup \{u, v\}$ is a TDS of G, and so $\gamma_t(G) \le \tau(H') + 2 \le (2n-3)/4$, a contradiction. □

Let $V = \{v, v_1, v_2, \ldots, v_{n-1}\}$. For $i = 1, 2, \ldots, n-1$, let $V_i = \{v_1, v_2, \ldots, v_i\}$. We may assume that $N(v) = V_4$. Let $G_v = G[V_4]$. If $n = 5$, then $G_v \in \{C_4, K_4 - e, K_4\}$ in which case $G \in \{F_1, K_5 - e, K_5\}$ Hence we may assume that $n \ge 7$. Thus, G_v contains at most five edges and, since G is claw-free, G_v contains at least two edges.

Observation 6 *If $G_v = K_4 - e$, then $G = F_5$.*

Proof. We may assume that v_3v_4 is the edge missing in G_v and that $d(v_4) = 4$. If $d(v_3) = 3$, then in Observation 2, taking $V' = N[v]$, we have $n' = n - 5$, $m' = n - 6$ and $|T'| \le (2n-11)/4$. Thus, $T = T' \cup \{v, v_4\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, $d(v_3) = 4$.

Let $G' = G - v$. Then, G' is a claw-free graph with $\delta(G') \ge 3$ of even order $n' = n - 1$. If $\gamma_t(G') < n'/2$, then $\gamma_t(G') \le (n'-2)/2$. However every TDS of G' contains a vertex from the set $\{2, 3, 4\}$ (in order to totally dominate v_1) and is therefore also a TDS of G, implying that $\gamma_t(G) \le (n-3)/2$, a contradiction. Hence, $\gamma_t(G') \ge n'/2$. Thus by Theorem 4, $G' = G_1$ and so $G = F_5$. □

By Observation 6, we may assume that the subgraph induced by the neighborhood of every degree-4 vertex is not $K_4 - e$.

Observation 7 *The subgraph induced by the neighborhood of every degree-4 vertex is not a 4-cycle.*

Proof. Suppose $G_v = C_4$. We may assume that G_v is given by the cycle v_1, v_2, v_3, v_4, v_1. Since $n \ge 7$, we may assume that $d(v_1) = 4$ and that $v_1v_5 \in E(G)$. Since G is claw-free, we may further assume that $v_2v_5 \in E(G)$. If v_3v_5 or v_4v_5 is an edge, then $n = 6$, a contradiction. Hence neither v_3v_5 nor v_4v_5 is an edge.

If $d(v_3) = 3$, then in Observation 2, taking $V' = N[v]$, we have $n' = n - 5$, $m' = n - 6$ and $|T'| \le (2n-11)/4$. Thus, $T = T' \cup \{v_1, v_4\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, $d(v_3) = 4$. In Observation 2, taking $V' = V_3 \cup \{v\}$, we have $n' = n-4$, $m' = n-7$ and $|T'| \le (2n-11)/4$. Thus, $T = T' \cup \{v_2, v_3\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. □

Observation 8 *If $G_v = K_1 \cup C_3$, then $G = F_6$.*

Proof. Suppose that $G_v = K_1 \cup C_3$, where v_1 is the isolated vertex of G_v. If at least two vertices in $N(v) \setminus \{v_1\}$ have degree 3, say v_2 and v_3, then in Observation 2, taking $V' = V_3 \cup \{v\}$, we have $n' = n-4$, $m' \le n-7$ and $|T'| \le (2n-11)/4$. Thus, $T = T' \cup \{v, v_1\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence at most one vertex in $N(v) \setminus \{v_1\}$ has degree 3. We proceed further with the following claim.

Claim 1 *One vertex in $N(v) \setminus \{v_1\}$ has degree* 3.

Proof. Suppose, to the contrary, that each vertex in $\{v_2, v_3, v_4\}$ has degree 4. Let $\{v_5, v_6\} \subseteq N(v_1) \setminus \{v\}$. Then, v_5v_6 is an edge. Suppose there is an edge joining $\{v_2, v_3, v_4\}$ and $\{v_5, v_6\}$, say v_2v_5. Then in Observation 2, taking $V' = V_2 \cup \{v, v_5\}$, we have $n' = n-4$, $m' \le n-7$ and $|T'| \le (2n-11)/4$. Thus, $T = T' \cup \{v, v_1\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, there is no edge joining $\{v_2, v_3, v_4\}$ and $\{v_5, v_6\}$. For $i = 2, 3, 4$, let $N(v_i) \setminus N[v] = \{v'_i\}$.

Case 1. $v'_i = v'_j$ for some i and j, where $2 \le i < j \le 4$. We may assume that $i = 2$ and $j = 3$, and that $v_7 = v'_2$. If $v'_4 = v_7$, then we contradict our assumption that the subgraph induced by the neighborhood of every degree-4 vertex is not $K_4 - e$. Hence, $v'_4 \neq v_7$. We may assume that $v'_4 = v_8$, and so $N(v_4) = \{v, v_2, v_3, v_8\}$.

Suppose that v_7 is adjacent to v_5 or v_6, say v_5. If $d(v_1) = 4$ or $d(v_5) = 4$, then in Observation 2, taking $V' = V_3 \cup \{v, v_5, v_7\}$, we have $n' = n-6$, $m' \le n-9$ and $|T'| \le (2n-15)/4$. Thus, $T = T' \cup \{v, v_1, v_5\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, $d(v_1) = d(v_5) = 3$. If v_6v_7 is an edge, then taking $V' = (V_7 \setminus \{v_4\}) \cup \{v\}$, we have $n' = n-7$, $m' = n-8$ and $|T'| \le (2n-15)/4$. Thus, $T = T' \cup \{v, v_1, v_5\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_6v_7 is not an edge, implying that $d(v_7) = 3$. If v_6v_8 is not an edge, then in Observation 2, taking $V' = V_5 \cup \{v, v_7, v_8\}$, we have $n' = n-8$, $m' \le n-11$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v_1, v_4, v_5, v_8\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_6v_8 is an edge. Therefore in Observation 2, taking $V' = V_8 \cup \{v\}$, we have $n' = n-9$, $m' \le n-10$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v_1, v_4, v_5, v_8\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_7 is adjacent to neither v_5 nor v_6.

Suppose that v_7v_8 is an edge. Let v_9 be the common neighbor of v_7 and v_8, which exists as G is claw-free. In Observation 2, taking $V' = V_4 \cup \{v, v_9\}$, we have $n' = n-6$, $m' \le n-9$ and $|T'| \le (2n-15)/4$. Thus, $T = T' \cup \{v, v_1, v_9\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_7v_8 is not an edge.

Suppose that v_8 is adjacent to v_5 or v_6, say v_5. In Observation 2, taking $V' = V_8 \cup \{v\}$, we have $n' = n-9$, $m' \le n-10$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v_2, v_5, v_6, v_7\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_8 is adjacent to neither v_5 nor v_6.

Suppose that v_7 and v_8 have a common neighbor, say v_9. In Observation 2, taking $V' = V_4 \cup \{v, v_9\}$, we have $n' = n-6$, $m' \le n-9$ and $|T'| \le (2n-15)/4$. Thus, $T = T' \cup \{v, v_1, v_9\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting

Observation 1. Hence, v_7 and v_8 have no common neighbor. Let $v_9 \in N(v_7) \setminus \{v_2, v_3\}$ and let $\{v_{10}, v_{11}\} \subseteq N(v_8)$.

Suppose that v_9 is adjacent to v_{10} or v_{11}, say v_{10}. In Observation 2, taking $V' = (V_{10} \setminus \{v_5, v_6\}) \cup \{v\}$, we have $n' = n - 9$, $m' \leq n - 12$ and $|T'| \leq (2n - 21)/4$. Thus, $T = T' \cup \{v, v_1, v_9, v_{10}\}$ is a transversal of H_G of size at most $(2n - 5)/4$, contradicting Observation 1. Hence, v_9 is adjacent to neither v_{10} nor v_{11}. Suppose that v_9 is adjacent to v_5 or v_6, say v_5. In Observation 2, taking $V' = (V_9 \setminus \{v_6\}) \cup \{v\}$, we have $n' = n - 9$, $m' \leq n - 12$ and $|T'| \leq (2n - 21)/4$. Thus, $T = T' \cup \{v_4, v_5, v_8, v_9\}$ is a transversal of H_G of size at most $(2n - 5)/4$, contradicting Observation 1. Hence, v_9 is adjacent to neither v_5 nor v_6. Thus in Observation 2, taking $V' = (V_9 \setminus \{v_1, v_6, v_7\}) \cup \{v\}$, we have $n' = n - 7$, $m' \leq n - 12$ and $|T'| \leq (2n - 19)/4$. Thus, $T = T' \cup \{v_4, v_5, v_8, v_9\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. We conclude that $v'_i \neq v'_j$ for $2 \leq i < j \leq 4$.

Case 2. $v'_i \neq v'_j$ for $2 \leq i < j \leq 4$. For $i \in \{2, 3, 4\}$, let $v'_i = v_{i+5}$. Thus, v_2v_7, v_3v_8 and v_4v_9 are edges.

Suppose that there is an edge joining $\{v_5, v_6\}$ and $\{v_7, v_8, v_9\}$, say v_5v_7. If v_6v_7 is an edge, then in Observation 2, taking $V' = (V_8 \setminus \{v_4\}) \cup \{v\}$, we have $n' = n - 8$, $m' \leq n - 11$ and $|T'| \leq (2n - 19)/4$. Thus, $T = T' \cup \{v_3, v_5, v_7, v_8\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, v_6v_7 is not an edge. If v_8 or v_9, say v_8, is a common neighbor of v_5 and v_7, then in Observation 2, taking $V' = (V_9 \setminus \{v_6\}) \cup \{v\}$, we have $n' = n - 9$, $m' \leq n - 10$ and $|T'| \leq (2n - 19)/4$. Thus, $T = T' \cup \{v_1, v_4, v_5, v_9\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence we may assume that v_{10} is the common neighbor of v_5 and v_7. But then in Observation 2, taking $V' = (V_5 \setminus \{v_1\}) \cup \{v\}$, we have $n' = n - 5$, $m' \leq n - 10$ and $|T'| \leq (2n - 15)/4$. Thus, $T = T' \cup \{v_3, v_4, v_5\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence there is no edge joining $\{v_5, v_6\}$ and $\{v_7, v_8, v_9\}$.

Suppose that $\{v_7, v_8, v_9\}$ is not an independent set. We may assume that v_7v_8 is an edge. Then in Observation 2, taking $V' = (V_8 \setminus \{v_1, v_4, v_6\}) \cup \{v\}$, we have $n' = n - 6$, $m' \leq n - 10$ and $|T'| \leq (2n - 16)/4$. Thus, $T = T' \cup \{v_2, v_5, v_7\}$ is a transversal of H_G of size at most $(2n - 4)/4$, contradicting Observation 1. Hence, $\{v_7, v_8, v_9\}$ is an independent set.

Suppose that two vertices in $\{v_7, v_8, v_9\}$ have a common neighbor. We may assume that v_7 and v_8 have a common neighbor, say v_{10}. Then in Observation 2, taking $V' = V_3 \cup \{v, v_{10}\}$, we have $n' = n - 5$, $m' \leq n - 10$ and $|T'| \leq (2n - 15)/4$. Thus, $T = T' \cup \{v, v_1, v_{10}\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence no two vertices in $\{v_7, v_8, v_9\}$ have a common neighbor. Let $\{v_{10}, v_{11}\} \subseteq N(v_7)$, $\{v_{12}, v_{13}\} \subseteq N(v_8)$ and $\{v_{14}, v_{15}\} \subseteq N(v_9)$. Then, $v_{10}v_{11}$, $v_{12}v_{13}$ and $v_{14}v_{15}$ are all edges.

Suppose there is an edge joining two triangles each of which contain a vertex from $\{v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}\}$. We may assume that $v_{10}v_{12}$ is an edge. Then in Observation 2, taking $V' = V_3 \cup \{v, v_7, v_8, v_{10}, v_{12}\}$, we have $n' = n - 8$, $m' \leq n - 13$ and $|T'| \leq (2n - 21)/4$. Thus, $T = T' \cup \{v, v_1, v_{10}, v_{12}\}$ is a transversal of H_G of size at most $(2n - 5)/4$, contradicting Observation 1. Hence there is no edge joining two triangles each of which contain a vertex from $\{v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}\}$.

Suppose there is an edge joining $\{v_5, v_6\}$ and $\{v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}\}$, say v_5v_{10}. Then in Observation 2, taking $V' = (V_8 \setminus \{v_4\}) \cup \{v, v_{10}, v_{14}\}$, we have $n' = n - 10$, $m' \leq n - 15$ and $|T'| \leq (2n-25)/4$. Thus, $T = T' \cup \{v_3, v_5, v_8, v_{10}, v_{14}\}$ is a transversal of H_G of size at most $(2n-5)/4$, contradicting Observation 1. Hence there is no edge joining $\{v_5, v_6\}$ and $\{v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}\}$. Then in Observation 2, taking $V' = V_4 \cup \{v, v_{10}, v_{12}, v_{14}\}$, we have $n' = n-8$, $m' \leq n-15$ and $|T'| \leq (2n-23)/4$. Thus, $T = T' \cup \{v, v_1, v_{10}, v_{12}, v_{14}\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. This completes the proof of Claim 1. □

By Claim 1, one vertex in $N(v) \setminus \{v_1\}$ has degree 3. We may assume that $d(v_2) = 3$. Then, $d(v_3) = d(v_4) = 4$. If $d(v_1) = 4$, then in Observation 2, taking $V' = V_2 \cup \{v\}$, we have $n' = n-3$, $m' = n-8$ and $|T'| \leq (2n-11)/4$. Thus, $T = T' \cup \{v, v_1\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, $d(v_1) = 3$. We may assume that $N(v_1) = \{v, v_5, v_6\}$. Thus, v_5v_6 is an edge.

Suppose there is an edge joining $\{v_3, v_4\}$ and $\{v_5, v_6\}$, say v_3v_5. Then in Observation 2, taking $V' = V_3 \cup \{v\}$, we have $n' = n-4$, $m' = n-7$ and $|T'| \leq (2n-11)/4$. Thus, $T = T' \cup \{v, v_1\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, there is no edge joining $\{v_3, v_4\}$ and $\{v_5, v_6\}$. Let $N(v_3) = \{v, v_2, v_4, v_7\}$.

Claim 2 *v_4v_7 is an edge.*

Proof. Suppose, to the contrary, that v_4v_7 is not an edge. Let $N(v_4) = \{v, v_2, v_3, v_8\}$. Suppose there is an edge joining $\{v_5, v_6\}$ and $\{v_7, v_8\}$, say v_5v_7. If v_6v_7 is an edge, then in Observation 2, taking $V' = (V_7 \setminus \{v_4\}) \cup \{v\}$, we have $n' = n-7$, $m' \leq n-8$ and $|T'| \leq (2n-15)/4$. If v_6v_7 is not an edge, then there is a common neighbor of v_5 and v_7 (which may possibly be v_8), and in Observation 2, taking $V' = V_3 \cup \{v, v_5, v_7\}$, we have $n' = n-6$, $m' = n-9$ and $|T'| \leq (2n-15)/4$. In both cases, $T = T' \cup \{v, v_1, v_5\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, there is no edge joining $\{v_5, v_6\}$ and $\{v_7, v_8\}$.

Since each of v_5 and v_6 is at distance 3 from a degree-4 vertex (namely, v_3 and v_4), $d(v_5) = d(v_6) = 3$ by Observation 5. Further for $i \geq 9$, $d(v, v_i) \geq 3$, and so, by Observation 5, $d(v_i) = 3$.

Suppose that v_7v_8 is an edge. Let v_9 be a common neighbor of v_7 and v_8. In Observation 2, taking $V' = V_4 \cup \{v, v_9\}$, we have $n' = n-6$, $m' \leq n-9$ and $|T'| \leq (2n-15)/4$. Thus, $T = T' \cup \{v, v_1, v_9\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_7v_8 is not an edge.

Suppose $d(v_7) = 4$. Then $N[v_7] \setminus \{v_3\}$ induces a clique K_4. Let $v_7' \in N[v_7] \setminus \{v_3\}$. Then, $N[v_7'] = N[v_7] \setminus \{v_3\}$. In Observation 2, taking $V' = N[v] \cup N[v_7]$, we have $n' = n-9$, $m' = n-11$ and $|T'| \leq (2n-20)/4$. Thus, $T = T' \cup \{v, v_1, v_7, v_7'\}$ is a transversal of H_G of size at most $(2n-4)/4$, contradicting Observation 1. Hence, $d(v_7) = 3$. Similarly, $d(v_8) = 3$.

Let $N(v_7) = \{v_3, v_9, v_{10}\}$. Then, v_9v_{10} is an edge. Suppose v_8 is adjacent to v_9 or v_{10}, say v_9. Then, $N(v_9) = \{v_7, v_8, v_{10}\}$. By the claw-freeness of G, v_8v_{10} is an edge and $N(v_8) = \{v_4, v_9, v_{10}\}$. In Observation 2, taking $V' = V_4 \cup \{v, v_7, v_8, v_9, v_{10}\}$, we have

$n' = n - 9$, $m' = n - 11$ and $|T'| \le (2n - 20)/4$. Thus, $T = T' \cup \{v, v_1, v_7, v_9\}$ is a transversal of H_G of size at most $(2n - 4)/4$, contradicting Observation 1. Hence, v_8 is adjacent to neither v_9 nor v_{10}. Let $N(v_8) = \{v_3, v_{11}, v_{12}\}$. Then, $v_{11}v_{12}$ is an edge.

Suppose that there is an edge joining $\{v_5, v_6\}$ and $\{v_9, v_{10}, v_{11}, v_{12}\}$, say v_5v_9. In Observation 2, taking $V' = V_5 \cup \{v, v_7, v_9\}$, we have $n' = n - 8$, $m' = n - 11$ and $|T'| \le (2n - 19)/4$. Thus, $T = T' \cup \{v, v_4, v_5, v_9\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, there is no edge joining $\{v_5, v_6\}$ and $\{v_9, v_{10}, v_{11}, v_{12}\}$.

Suppose that there is an edge joining $\{v_9, v_{10}\}$ and $\{v_{11}, v_{12}\}$, say v_9v_{11}. In Observation 2, taking $V' = V_4 \cup \{v, v_7, v_8, v_9, v_{11}\}$, we have $n' = n - 9$, $m' = n - 13$ and $|T'| \le (2n - 22)/4$. Thus, $T = T' \cup \{v, v_1, v_9, v_{11}\}$ is a transversal of H_G of size at most $(2n - 6)/4$, contradicting Observation 1. Hence, there is no edge joining $\{v_9, v_{10}\}$ and $\{v_{11}, v_{12}\}$. Thus in Observation 2, taking $V' = V_4 \cup \{v, v_9, v_{11}\}$, we have $n' = n - 7$, $m' \le n - 12$ and $|T'| \le (2n - 19)/4$. Thus, $T = T' \cup \{v, v_1, v_9, v_{11}\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. This completes the proof of Claim 2. $\Box$

By Claim 2, v_4v_7 is an edge. If v_7 is adjacent to v_5 or v_6, say v_5v_7, then in Observation 2, taking $V' = V_5 \cup \{v, v_7\}$, we have $n' = n - 7$, $m' \le n - 8$ and $|T'| \le (2n - 15)/4$. Thus, $T = T' \cup \{v, v_1, v_5\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, v_7 is adjacent to neither v_5 nor v_6. Thus each of v_5 and v_6 is at distance 3 from a degree-4 vertex (namely, v_3 and v_4), and so $d(v_5) = d(v_6) = 3$ by Observation 5.

Let $v_8 \in N(v_7) \setminus \{v_3, v_4\}$. If $N(v_8) \neq \{v_5, v_6, v_7\}$, then in Observation 2, taking $V' = V_4 \cup \{v, v_8\}$, we have $n' = n - 6$, $m' \le n - 9$ and $|T'| \le (2n - 15)/4$. Thus, $T = T' \cup \{v, v_1, v_8\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, $N(v_8) = \{v_5, v_6, v_7\}$, implying that $G = F_6$. This completes the proof of Observation 8. $\Box$

By Observation 8, we may assume that the subgraph induced by the neighborhood of every degree-4 vertex is not $K_1 \cup C_3$.

Observation 9 *If $G_v = K_{1,3} + e$, then $G = F_4$.*

Proof. We may assume that v_1 is the degree-1 vertex in G_v and that v_1v_2 is an edge. Thus, v_2, v_3, v_4, v_2 is a cycle. If $d(v_3) = d(v_4) = 3$, then in Observation 2, taking $V' = V_4 \cup \{v\}$, we have $n' = n - 5$, $m' \le n - 6$ and $|T'| \le (2n - 11)/4$. Thus, $T = T' \cup \{v, v_1\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, we may assume that $d(v_3) = 4$. Let $N(v_3) = \{v, v_2, v_4, v_5\}$.

If v_1v_5 is an edge, then in Observation 2, taking $V' = V_3 \cup \{v, v_5\}$, we have $n' = n - 5$, $m' \le n - 6$ and $|T'| \le (2n - 11)/4$. Thus, $T = T' \cup \{v, v_1\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, v_1v_5 is not an edge. But then v_4v_5 must be an edge, for otherwise $G[N(v_3)] = K_1 \cup C_3$, contrary to assumption. Let $v_6 \in N(v_5) \setminus \{v_3, v_4\}$.

Suppose that v_1 and v_5 have a common neighbor. We may assume that v_1v_6 is an edge. Then in Observation 2, taking $V' = V_6 \cup \{v\}$, we have $n' = n - 7$, $m' \leq n - 8$ and $|T'| \leq (2n - 15)/4$. Thus, $T = T' \cup \{v, v_1, v_6\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_1 and v_5 have no common neighbor. In particular, v_1v_6 is not an edge. Thus, $d(v, v_6) = 3$, and so, by Observation 5, $d(v_6) = 3$. Let $v_7 \in N(v_1) \setminus \{v, v_2\}$. Then, v_5v_7 is not an edge. Thus, $d(v_3, v_7) = 3$, and so, by Observation 5, $d(v_7) = 3$.

If v_6v_7 is not an edge, then in Observation 2, taking $V' = V_4 \cup \{v, v_6\}$, we have $n' = n - 6$, $m' \leq n - 9$ and $|T'| \leq (2n - 15)/4$. Thus, $T = T' \cup \{v, v_1, v_6\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, v_6v_7 is an edge.

Suppose that $d(v_1) = d(v_5) = 3$. Then, v_6 and v_7 have a common neighbor, v_8 say. In Observation 2, taking $V' = V_5 \cup \{v, v_8\}$, we have $n' = n - 7$, $m' \leq n - 9$ and $|T'| \leq (2n - 16)/4$. Thus, $T = T' \cup \{v, v_4, v_8\}$ is a transversal of H_G of size at most $(2n - 4)/4$, contradicting Observation 1. Hence, by symmetry, we may assume that $d(v_1) = 4$. Let $N(v_1) = \{v, v_2, v_7, v_8\}$. Then, v_7v_8 is an edge. As shown with the vertex v_7, we must have that v_6v_8 is an edges and $d(v_8) = 3$. But then, $G = F_4$. □

By Observation 9, we may assume that the subgraph induced by the neighborhood of every degree-4 vertex is not $K_{1,3} + e$.

Observation 10 *If $G_v = P_4$, then $G \in \{F_8, F_9, F_{10}\}$.*

Proof. We may assume that G_v is given by the path v_1, v_2, v_3, v_4. We desired result now follows from Claim 3 and Claim 4.

Claim 3 *If $d(v_2) = d(v_3) = 3$, then $G = F_8$.*

Proof. Suppose that v_1 or v_4 has degree 4. We may assume that $d(v_1) = 4$. In Observation 2, taking $V' = V_3 \cup \{v\}$, we have $n' = n-4$, $m' \leq n-7$ and $|T'| \leq (2n-11)/4$. Thus, $T = T' \cup \{v, v_1\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, $d(v_1) = d(v_4) = 3$. Thus, since G is claw-free, v_1 and v_4 have no common neighbor other than v. Let $N(v_1) = \{v, v_2, v_5\}$ and $N(v_4) = \{v, v_3, v_6\}$. For $i \geq 7$, the vertex v_i is at distance at least 3 from the degree-4 vertex v, and so, by Observation 5, $d(v_i) = 3$.

If v_5v_6 is an edge, then in Observation 2, taking $V' = V_6 \cup \{v\}$, we have $n' = n - 7$, $m' \leq n - 8$ and $|T'| \leq (2n - 15)/4$. Thus, $T = T' \cup \{v, v_1, v_5\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, v_5v_6 is not an edge.

By our assumption that the subgraph induced by the neighborhood of every degree-4 vertex is not $K_1 \cup C_3$, we have that $d(v_5) = d(v_6) = 3$. Let $N(v_5) = \{v_1, v_7, v_8\}$. Then, $v_7v_8 \in E$. If v_6v_7 is not an edge, then in Observation 2, taking $V' = N[v] \cup \{v_7\}$, we have $n' = n - 6$, $m' \leq n - 9$ and $|T'| \leq (2n - 15)/4$. Thus, $T = T' \cup \{v, v_4, v_7\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, v_6v_7 is an edge. Thus, by the claw-freeness of G, v_6v_8 is an edge. Thus, $G = F_8$. □

Claim 4 *If v_2 or v_3 has degree 4, then $G \in \{F_9, F_{10}\}$.*

Proof. We may assume that $d(v_2) = 4$. Let $N(v_2) = \{v, v_1, v_3, v_5\}$. Since G is claw-free, v_1v_5 or v_3v_5 is an edge. We consider two cases.

Case 1. v_3v_5 is an edge. Then, v_1v_5 is not an edge, for otherwise, $N(v_2)$ induces a 4-cycle, contradicting Observation 7. Similarly, v_4v_5 is not an edge. Let $v_6 \in N(v_5)\setminus\{v_2, v_3\}$.

Case 1.1. v_5 has a common neighbor with v_1 or with v_4 that does not belong to $N(v)$. We may assume that v_1v_6 is an edge. Suppose that $d(v_6) = 4$. Let $v_7 \in N(v_6) \setminus \{v_1, v_5\}$. On the one hand, if v_4v_7 is not an edge, then in Observation 2, taking $V' = V_3 \cup \{v, v_5, v_6\}$, we have $n' = n - 6$, $m' \leq n - 9$ and $|T'| \leq (2n - 15)/4$. On the other hand, if v_4v_7 is an edge, then in Observation 2, taking $V' = V_6 \cup \{v\}$, we have $n' = n - 7$, $m' \leq n - 8$ and $|T'| \leq (2n - 15)/4$. In both cases, $T = T' \cup \{v, v_1, v_6\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, $d(v_6) = 3$ and $N(v_6) = \{v_1, v_5, v_7\}$. Then, v_1v_7 or v_5v_7 is an edge.

If v_1v_7 and v_5v_7 are edges, then in Observation 2, taking $V' = (V_7 \setminus \{v_4\}) \cup \{v\}$, we have $n' = n - 7$, $m' \leq n - 8$ and $|T'| \leq (2n - 15)/4$. Thus, $T = T' \cup \{v, v_1, v_7\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, either v_1v_7 or v_5v_7 is an edge (but not both).

Suppose v_5v_7 is an edge. Then, $d(v_1) = 3$. If v_4v_7 is an edge, then in Observation 2, taking $V' = V_7 \cup \{v\}$, we have $n' = n - 8$, $m' \leq n - 8$ and $|T'| \leq (2n - 16)/4$. Thus, $T = T' \cup \{v_2, v_5, v_7\}$ is a transversal of H_G of size at most $(2n - 4)/4$, contradicting Observation 1. Hence, v_4v_7 is not an edge. Thus, $d(v, v_7) = 3$, and so by Observation 5, $d(v_7) = 3$. Let $N(v_7) = \{v_5, v_6, v_8\}$. In Observation 2, taking $V' = V_8 \cup \{v\}$, we have $n' = n - 9$, $m' \leq n - 10$ and $|T'| \leq (2n - 19)/4$. Thus, $T = T' \cup \{v, v_4, v_7, v_8\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_5v_7 is not an edge. Thus, v_1v_7 is an edge and $d(v_5) = 3$.

If v_4v_7 is an edge, then in Observation 2, taking $V' = V_7 \cup \{v\}$, we have $n' = n - 8$, $m' \leq n - 8$ and $|T'| \leq (2n - 16)/4$. Thus, $T = T' \cup \{v_3, v_4, v_7\}$ is a transversal of H_G of size at most $(2n - 4)/4$, contradicting Observation 1. Hence, v_4v_7 is not an edge. If $d(v_4) = d(v_7) = 3$, then since G is claw-free, v_4 and v_7 have no common neighbor. Thus in Observation 2, taking $V' = (V_5 \setminus \{v_1\}) \cup \{v, v_7\}$, we have $n' = n - 6$, $m' \leq n - 9$ and $|T'| \leq (2n - 13)/4$. Thus, $T = T' \cup \{v_2, v_3, v_7\}$ is a transversal of H_G of size at most $(2n - 5)/4$, contradicting Observation 1.

Case 1.2. v_5 has no common neighbor with v_1 or with v_4 that does not belong to $N(v)$. In particular, neither v_1v_6 nor v_4v_6 is an edge. Thus, $d(v, v_6) = 3$, and so, by Observation 5, $d(v_6) = 3$.

Case 1.2.1 v_6 has a common neighbor with v_1 or v_4. We may assume that v_1 and v_6 have a common neighbor, v_7 say. By Case 1.1, v_5v_7 is not an edge. By the claw-freeness of G, v_4v_7 is not an edge. Thus, $d(v_3, v_7) = 3$, and so, by Observation 5, $d(v_7) = 3$. Let $N(v_7) = \{v_1, v_6, v_8\}$. Then, v_1v_8 or v_6v_8 is an edge. If both v_1v_8 and v_6v_8 are edges, then in Observation 2, taking $V' = V_8 \cup \{v\}$, we have $n' = n - 9$, $m' \leq n - 10$ and $|T'| \leq (2n - 19)/4$. Thus, $T = T' \cup \{v, v_4, v_5, v_6\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence either v_1v_8 or v_6v_8 is an edge (but not both).

Suppose v_1v_8 is an edge. Let $N(v_6) = \{v_5, v_7, v_9\}$. Then, v_5v_9 or v_7v_9 is an edge. If v_4v_9 is an edge, then in Observation 2, taking $V' = V_7 \cup \{v, v_9\}$, we have $n' = n - 9$, $m' \leq n - 10$ and $|T'| \leq (2n - 19)/4$. If v_8v_9 is an edge, then in Observation 2, taking $V' = (V_9 \setminus \{v_4\}) \cup \{v\}$, we have $n' = n - 9$, $m' \leq n - 10$ and $|T'| \leq (2n - 19)/4$. If v_9 is adjacent to vertex v_i, where $i \geq 10$, then taking $V' = V_7 \cup \{v, v_9\}$, we have $n' = n - 8$, $m' \leq n - 11$ and $|T'| \leq (2n - 19)/4$. In all three cases, $T = T' \cup \{v, v_1, v_6, v_9\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence we must have that $d(v_9) = 3$ and $N(v_9) = \{v_5, v_6, v_7\}$. But then in Observation 2, taking $V' = V_7 \cup \{v, v_9\}$, we have $n' = n - 9$, $m' \leq n - 10$ and $|T'| \leq (2n - 19)/4$. Thus, $T = T' \cup \{v, v_4, v_7, v_9\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, v_1v_8 is not an edge, implying that v_6v_8 is an edge. We may assume that $d(v_1) = 3$ for otherwise if v_1 and v_7 have a common neighbor (not adjacent with v_6), then as shown earlier we reach a contradiction.

If v_4v_8 is not an edge, then in Observation 2, taking $V' = V_8 \cup \{v\}$, we have $n' = n-9$, $m' \leq n - 10$ and $|T'| \leq (2n - 19)/4$. Thus, $T = T' \cup \{v, v_4, v_6, v_8\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, v_4v_8 is an edge and $d(v_6) = 3$. But then $G = F_{10}$.

Case 1.2.2 v_6 has no common neighbor with v_1 or v_4. If $d(v_1) = 4$ or if $d(v_6) = 4$, then in Observation 2, taking $V' = V_3 \cup \{v, v_6\}$, we have $n' = n - 5$, $m' \leq n - 10$ and $|T'| \leq (2n - 15)/4$. Thus, $T = T' \cup \{v, v_1, v_6\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, $d(v_1) = d(v_6) = 3$. Similarly, $d(v_4) = 3$. Thus by the claw-freeness of G, v is the only common neighbor of v_1 and v_4. It follows that for $i \geq 6$, the vertex v_i is at distance at least 3 from at least one vertex in $\{v, v_2, v_3\}$, and so, by Observation 5, $d(v_i) = 3$.

Suppose that $d(v_5) = 4$. Let $N(v_5) = \{v_2, v_3, v_6, v_7\}$. Then, v_6v_7 is an edge. Let $N(v_6) = \{v_5, v_7, v_8\}$. Then, v_1v_8 and v_4v_8 are not edges. Suppose v_7v_8 is an edge, i.e., if $N(v_7) = \{v_5, v_6, v_8\}$. Since G is claw-free, and $d(v_1) = d(v_8) = 3$, v_1 and v_8 have no common neighbor. Thus in Observation 2, taking $V' = (V_8 \setminus \{v_4\}) \cup \{v\}$, we have $n' = n - 8$, $m' \leq n - 11$ and $|T'| \leq (2n - 19)/4$. Thus, $T = T' \cup \{v, v_1, v_6, v_8\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, v_7v_8 is not an edge. In Observation 2, taking $V' = (V_6 \setminus \{v_4\}) \cup \{v, v_8\}$, we have $n' = n - 7$, $m' \leq n - 11$ and $|T'| \leq (2n - 19)/4$. Thus, $T = T' \cup \{v, v_1, v_6, v_8\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, $d(v_5) = 3$, i.e., $N(v_5) = \{v_2, v_3, v_6\}$. Let $N(v_6) = \{v_5, v_7, v_8\}$. Then, v_7v_8 is an edge, and there is no edge joining $\{v_1, v_4\}$ and $\{v_7, v_8\}$.

Suppose that a vertex in $\{v_1, v_4\}$ has a common neighbor with a vertex in $\{v_7, v_8\}$. We may assume that v_1 and v_7 have a common neighbor, say v_{10}. By the claw-freeness of G, $N(v_{10}) = \{v_1, v_7, v_8\}$. Thus in Observation 2, taking $V' = (V_{10} \setminus \{v_4\}) \cup \{v\}$, we have $n' = n - 9$, $m' \leq n - 10$ and $|T'| \leq (2n - 19)/4$. Thus, $T = T' \cup \{v, v_1, v_6, v_7\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence no vertex in $\{v_1, v_4\}$ has a common neighbor with a vertex in $\{v_7, v_8\}$.

Let $N(v_7) = \{v_6, v_8, v_9\}$. If v_8v_9 is an edge, then in Observation 2, taking $V' = (V_8 \setminus \{v_4\}) \cup \{v\}$, we have $n' = n - 8$, $m' \leq n - 11$ and $|T'| \leq (2n - 19)/4$. Thus,

$T = T' \cup \{v, v_1, v_6, v_7\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_8v_9 is not an edge. Let $N(v_8) = \{v_6, v_7, v_{10}\}$.

If v_9v_{10} is an edge, then in Observation 2, taking $V' = (V_{10} \setminus \{v_1, v_4\}) \cup \{v\}$, we have $n' = n - 10$, $m' = n - 12$ and $|T'| \leq (2n-22)/4$. Thus, $T = T' \cup \{v_2, v_3, v_7, v_9\}$ is a transversal of H_G of size at most $(2n-6)/4$, contradicting Observation 1. Hence, v_9v_{10} is not an edge. Let $N(v_9) = \{v_7, v_{11}, v_{12}\}$.

Suppose v_{10} is not adjacent to v_{11} or v_{12}. Let $N(v_{10}) = \{v_8, v_{13}, v_{14}\}$. If there is an edge joining $\{v_{11}, v_{12}\}$ and $\{v_{13}, v_{14}\}$, say $v_{11}v_{13}$ is an edge, then in Observation 2, taking $V' = (V_{11} \setminus V_4) \cup \{v, v_{13}\}$, we have $n' = n-8$, $m' = n-12$ and $|T'| \leq (2n-20)/4$. Thus, $T = T' \cup \{v_5, v_6, v_{11}, v_{13}\}$ is a transversal of H_G of size at most $(2n-4)/4$, contradicting Observation 1. Hence there is no edge joining $\{v_{11}, v_{12}\}$ and $\{v_{13}, v_{14}\}$. Suppose that there is an edge joining $\{v_1, v_4\}$ and $\{v_{11}, v_{12}, v_{13}, v_{14}\}$, say v_1v_{11}. Then in Observation 2, taking $V' = (V_9 \setminus \{v_4\}) \cup \{v, v_{11}\}$, we have $n' = n-10$, $m' = n-13$ and $|T'| \leq (2n-23)/4$. Thus, $T = T' \cup \{v, v_1, v_6, v_8, v_{11}\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence there is no edge joining $\{v_1, v_4\}$ and $\{v_{11}, v_{12}, v_{13}, v_{14}\}$. Now at least one of v_{11} or v_{13}, say v_{11}, has no common neighbor with v_1. Therefore in Observation 2, taking $V' = (V_8 \setminus \{v_4\}) \cup \{v, v_{11}\}$, we have $n' = n-9$, $m' = n-14$ and $|T'| \leq (2n-23)/4$. Thus, $T = T' \cup \{v, v_1, v_6, v_8, v_{11}\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_{10} is adjacent to v_{11} or v_{12}. Thus, by the claw-freeness of G, $N(v_{10}) = \{v_8, v_{11}, v_{12}\}$.

By the claw-freeness of G, v is the only common neighbor of v_1 and v_4. Let $N(v_1) = \{v, v_2, v_{13}\}$ and $N(v_2) = \{v, v_3, v_{14}\}$. If $v_{13}v_{14}$ is an edge, then in Observation 2, taking $V' = (V_{13} \setminus \{v_1\}) \cup \{v\}$, we have $n' = n - 13$, $m' = n - 15$ and $|T'| \leq (2n-28)/4$. Thus, $T = T' \cup \{v_3, v_5, v_9, v_{10}, v_{11}, v_{14}\}$ is a transversal of H_G of size at most $(2n-4)/4$, contradicting Observation 1. Hence, $v_{13}v_{14}$ is not an edge. Let $N(v_{13}) = \{v_1, v_{15}, v_{16}\}$.

If v_{14} is adjacent to v_{15} or v_{16}, then $N(v_{14}) = \{v_4, v_{15}, v_{16}\}$ and the graph G is fully described (and has order $n = 17$). But then $\{v_2, v_5, v_9, v_{10}, v_{11}, v_{14}, v_{15}\}$, for example, is a TDS of G, and so $\gamma_t(G) \leq 7 = (n-3)/2$, a contradiction. Hence, v_{14} is adjacent to neither v_{15} nor v_{16}. Let $N(v_{14}) = \{v_4, v_{17}, v_{18}\}$. Then in Observation 2, taking $V' = (V_{15} \setminus \{v_{13}\}) \cup \{v\}$, we have $n' = n - 15$, $m' \leq n - 17$ and $|T'| \leq (2n-32)/4$. Thus, $T = T' \cup \{v_2, v_5, v_9, v_{10}, v_{11}, v_{14}, v_{15}\}$ is a transversal of H_G of size at most $(2n-4)/4$, contradicting Observation 1.

Case 2. v_3v_5 is not an edge. Then, v_1v_5 is an edge since G is claw-free.

Suppose that v_4v_5 is an edge. Suppose $d(v_1) = 4$. Let $N(v_1) = \{v, v_2, v_5, v_6\}$. Then, $N(v_5) = \{v_1, v_2, v_4, v_6\}$. Thus in Observation 2, taking $V' = (V_5 \setminus \{v_3\}) \cup \{v\}$, we have $n' = n-5$, $m' = n-7$ and $|T'| \leq (2n-12)/4$. Thus, $T = T' \cup \{v, v_1\}$ is a transversal of H_G of size at most $(2n-4)/4$, contradicting Observation 1. Hence, $d(v_1) = 3$. Thus in Observation 2, taking $V' = (V_5 \setminus \{v_3\}) \cup \{v\}$, we have $n' = n-5$, $m' \leq n-6$ and $|T'| \leq (2n-11)/4$. Thus, $T = T' \cup \{v, v_4\}$ is a transversal of H_G of size at most $(2n-4)/4$, contradicting Observation 1. Hence, v_4v_5 is not an edge.

Case 2.1 $d(v_1) = 3$. If $d(v_3) = 4$, then in Observation 2, taking $V' = V_3 \cup \{v\}$, we have $n' = n-4$, $m' \leq n-7$ and $|T'| \leq (2n-11)/4$. Thus, $T = T' \cup \{v_2, v_3\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, $d(v_3) = 3$. Let

$v_6 \in N(v_5) \setminus \{v_1, v_2\}$.

Suppose that v_4 and v_5 have a common neighbor. We may assume that v_4v_6 is an edge. Then in Observation 2, taking $V' = V_6 \cup \{v\}$, we have $n' = n-7$, $m' \le n-8$ and $|T'| \le (2n-15)/4$. Thus, $T = T' \cup \{v, v_4, v_6\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_4 and v_5 have no common neighbor. It follows that for $i \ge 6$, the vertex v_i is at distance at least 3 from at least one of v and v_2, and so, by Observation 5, $d(v_i) = 3$. In particular, $d(v_6) = 3$.

If v_4 and v_6 have no common neighbor, then in Observation 2, taking $V' = V_4 \cup \{v, v_6\}$, we have $n' = n-6$, $m' \le n-9$ and $|T'| \le (2n-15)/4$. Thus, $T = T' \cup \{v, v_4, v_6\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_4 and v_6 have a common neighbor, v_7 say. Since v_4 and v_5 have no common neighbor, v_5v_7 is not an edge.

Let $N(v_7) = \{v_4, v_6, v_8\}$. If v_8 is adjacent to a vertex not in $\{v_4, v_5, v_6, v_7\}$, then in Observation 2, taking $V' = V_8 \cup \{v\}$, we have $n' = n-9$, $m' \le n-10$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v_2, v_5, v_7, v_8\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, $N(v_8) \subseteq \{v_4, v_5, v_6, v_7\}$.

On the one hand, if v_5v_8 is not an edge, then $N(v_8) = \{v_4, v_6, v_7\}$ and $d(v_5) = 3$. But then $G = F_9$. On the other hand, if v_5v_8 is an edge, then since v_4 and v_5 have no common neighbor, $N(v_8) = \{v_5, v_6, v_7\}$. If now $d(v_4) = 4$, then in Observation 2, taking $V' = V_8 \cup \{v\}$, we have $n' = n-9$, $m' \le n-10$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v_1, v_4, v_5, v_7\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, $d(v_4) = 3$, and so $G = F_9$.

Case 2.2 $d(v_1) = 4$. Let $N(v_1) = \{v, v_2, v_5, v_6\}$. Then, v_5v_6 is an edge. If $d(v_5) = 3$, then in Observation 2, taking $V' = V_2 \cup \{v, v_5\}$, we have $n' = n-4$, $m' \le n-7$ and $|T'| \le (2n-11)/4$. Thus, $T = T' \cup \{v, v_1\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, $d(v_5) = 4$. Let $N(v_5) = \{v_1, v_2, v_6, v_7\}$. Then, v_6v_7 is an edge.

If $d(v_3) = 3$, then in Observation 2, taking $V' = V_3 \cup \{v\}$, we have $n' = n-4$, $m' \le n-7$ and $|T'| \le (2n-11)/4$. Thus, $T = T' \cup \{v, v_1\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, $d(v_3) = 4$. Since G is claw-free, v_3v_6 is not an edge. If v_3v_7 is an edge, then so too is v_4v_7. But then in Observation 2, taking $V' = V_5 \cup \{v, v_7\}$, we have $n' = n-7$, $m' \le n-8$ and $|T'| \le (2n-15)/4$. Thus, $T = T' \cup \{v, v_1, v_4\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_3v_7 is not an edge. Let $N(v_3) = \{v, v_2, v_4, v_8\}$. Then, v_4v_8 is an edge.

If $d(v_4) = 3$ or if v_4 is adjacent to v_6 or v_7, then in Observation 2, taking $V' = V_5 \cup \{v\}$, we have $n' = n-6$, $m' \le n-9$ and $|T'| \le (2n-15)/4$. Thus, $T = T' \cup \{v_2, v_3, v_5\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, $d(v_4) = 4$ and neither v_4v_6 nor v_4v_7 is an edge. Let $N(v_4) = \{v, v_3, v_8, v_9\}$. Then, v_8v_9 is an edge. In Observation 2, taking $V' = (V_9 \setminus \{v_7\}) \cup \{v\}$, we have $n' = n-9$, $m' \le n-10$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v_1, v_4, v_6, v_9\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. This completes the proof of Claim 4 and of Observation 10. $\square$

By Observation 10, we may assume that the subgraph induced by the neighborhood of every degree-4 vertex is not isomorphic to P_4. This, together with our earlier assumptions, implies the following observation.

Observation 11 *The subgraph induced by the neighborhood of every degree-*4 *vertex is isomorphic to* $2K_2$.

Since G is claw-free, we have the following observation.

Observation 12 *If* u *and* w *are adjacent vertices that do not have exactly one common neighbor, then* $d(u) = d(w) = 3$.

Proof. Suppose, to the contrary, that $d(u) = 4$. If u and w have no common neighbor, then $N(u)$ induces a subgraph isomorphic to $K_1 \cup C_3$, while if u and w have at least two common neighbors, then $N(u)$ induces a subgraph that contains a path P_3, contrary to assumption. □

By Observation 11, $G_v = 2K_2$. We may assume that v_1v_2 and v_3v_4 are edges.

Observation 13 *If two vertices in* $N(v)$ *have a common neighbor different from* v*, then* $G \in \{F_2, F_3, F_7\}$.

Proof. We may assume that v_1 and v_2 have a common neighbor v_5 different from v. By Observation 12, $d(v_1) = d(v_2) = 3$.

If v_5 is adjacent to v_3 or v_4, say to v_3, then in Observation 2, taking $V' = V_3 \cup \{v, v_5\}$, we have $n' = n-5$, $m' = n-6$ and $|T'| \le (2n-11)/4$. Thus, $T = T' \cup \{v, v_3\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, $N(v_5) \cap N(v) = V_2$.

Case 1. $d(v_5) = 3$. Let $N(v_5) = \{v_1, v_2, v_6\}$. By Observation 12, $d(v_6) = 3$. If $N(v_6) = \{v_3, v_4, v_5\}$, then $G = F_2$. Hence we may assume that v_6 is not adjacent with both v_3 or v_4, say v_4v_6 is not an edge.

Suppose v_3v_6 is an edge. Let $N(v_6) = \{v_3, v_5, v_7\}$. Then, v_3v_7 is an edge. By Observation 12, v_4v_7 is not an edge. Let $v_8 \in N(v_7) \setminus \{v_3, v_6\}$. If v_4v_8 is an edge, then in Observation 2, taking $V' = V_8 \cup \{v\}$, we have $n' = n-9$, $m' \le n-10$ and $|T'| \le (2n-19)/4$. If v_4v_8 is not an edge, then in Observation 2, taking $V' = (V_8 \setminus \{v_4\}) \cup \{v\}$, we have $n' = n-8$, $m' = n-11$ and $|T'| \le (2n-19)/4$. In both cases, $T = T' \cup \{v, v_1, v_7, v_8\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_3v_6 is not an edge. Thus, v and v_6 have no common neighbor. Let $N(v_6) = \{v_5, v_7, v_8\}$. Then, $v_7v_8 \in E$.

Case 1.1 There is an edge joining $\{v_3, v_4\}$ and $\{v_7, v_8\}$. We may assume that v_3v_7 is an edge.

If v_4v_7 is an edge, then by Observation 12, $d(v_3) = d(v_4) = 3$. In Observation 2, taking $V' = V_8 \cup \{v\}$, we have $n' = n-9$, $m' \le n-10$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v_1, v_5, v_7, v_8\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_4v_7 is not an edge.

If v_3v_8 is an edge, then by Observation 12, $d(v_7) = d(v_8) = 3$. Thus in Observation 2, taking $V' = V_8 \cup \{v\}$, we have $n' = n - 9$, $m' \le n - 10$ and $|T'| \le (2n - 19)/4$. Thus, $T = T' \cup \{v_1, v_3, v_4, v_5\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, v_3v_8 is not an edge.

Suppose that $d(v_3) = 3$. Then, $d(v_7) = 3$. If v_4v_8 is not an edge or if $d(v_4) = 4$, then in Observation 2, taking $V' = V_8 \cup \{v\}$, we have $n' = n-9$, $m' \le n-10$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v, v_4, v_6, v_8\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_4v_8 is an edge and $d(v_4) = 3$, implying that $d(v_8) = 3$ and $G = F_3$. Hence we may assume that $d(v_3) = 4$. Similarly, we may assume that $d(v_4) = 4$.

Let $N(v_3) = \{v, v_4, v_7, v_9\}$. Then, v_7v_9 is an edge, and so $d(v_7) = 4$. By Observation 12, v_9 is adjacent to neither v_4 nor v_8. In Observation 2, taking $V' = V_7 \cup \{v\}$, we have $n' = n - 8$, $m' \le n - 11$ and $|T'| \le (2n - 19)/4$. Thus, $T = T' \cup \{v, v_4, v_6, v_7\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1.

Case 1.2 There is no edge joining $\{v_3, v_4\}$ and $\{v_7, v_8\}$. Then both v_7 and v_8 are at distance at least 3 from v, and so, by Observation 5, $d(v_7) = d(v_8) = 3$.

Suppose v_7 and v_8 have a common neighbor v_9, different from v_6. By Observation 12, $d(v_7) = d(v_8) = 3$. If $N(v_9) \cap \{v_3, v_4\} = \emptyset$, then in Observation 2, taking $V' = (V_9 \setminus \{v_3, v_4\}) \cup \{v\}$, we have $n' = n - 8$, $m' = n - 11$ and $|T'| \le (2n - 19)/4$. Thus, $T = T' \cup \{v, v_1, v_7, v_9\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence we may assume that v_3v_9 in an edge. But then in Observation 2, taking $V' = (V_9 \setminus \{v_4\}) \cup \{v\}$, we have $n' = n - 9$, $m' \le n - 10$ and $|T'| \le (2n - 19)/4$. Thus, $T = T' \cup \{v, v_1, v_7, v_9\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence we may assume that v_6 is the only common neighbor of v_7 and v_8.

Let $N(v_7) = \{v_6, v_8, v_9\}$. Then, v_8v_9 is not an edge. By Observation 12, $d(v_9) = 3$. If v_3v_9 is an edge, then in Observation 2, taking $V' = (V_9 \setminus \{v_4, v_8\}) \cup \{v\}$, we have $n' = n-8$, $m' = n-11$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v, v_1, v_7, v_9\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_3v_9 is not an edge. Similarly, v_4v_9 is not an edge. But then in Observation 2, taking $V' = (V_9 \setminus \{v_3, v_4, v_8\}) \cup \{v\}$, we have $n' = n - 7$, $m' \le n - 12$ and $|T'| \le (2n - 19)/4$. Thus, $T = T' \cup \{v, v_1, v_7, v_9\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1.

Case 2. $d(v_5) = 4$. Let $N(v_5) = \{v_1, v_2, v_6, v_7\}$. Then, v_6v_7 is an edge.

Case 2.1. There is an edge joining $\{v_3, v_4\}$ and $\{v_6, v_7\}$. We may assume that v_3v_6 is an edge. If v_4v_6 is an edge, then by Observation 12, $d(v_3) = d(v_4) = 3$. Thus in Observation 2, taking $V' = V_6 \cup \{v\}$, we have $n' = n-7$, $m' \le n-8$ and $|T'| \le (2n-15)/4$. Thus, $T = T' \cup \{v_1, v_5, v_6\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_4v_6 is not an edge.

If $d(v_6) = 4$, then in Observation 2, taking $V' = (V_6 \setminus \{v_4\}) \cup \{v\}$, we have $n' = n-6$, $m' \le n - 9$ and $|T'| \le (2n - 15)/4$. Thus, $T = T' \cup \{v_3, v_5, v_6\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, $d(v_6) = 3$. Therefore, $d(v_3) = 3$.

If v_4v_7 is an edge, then in Observation 2, taking $V' = V_7 \cup \{v\}$, we have $n' = n - 8$, $m' \le n - 8$ and $|T'| \le (2n - 16)/4$. Thus, $T = T' \cup \{v, v_4, v_7\}$ is a transversal of H_G of size at most $(2n - 4)/4$, contradicting Observation 1. Hence, v_4v_7 is not an edge.

If v_4 and v_7 have a common neighbor, say v_8, then in Observation 2, taking $V' = V_8 \cup \{v\}$, we have $n' = n-9$, $m' \le n-10$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v, v_4, v_7, v_8\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_4 and v_7 have no common neighbor.

Let $v_9 \in N(v_4) \setminus \{v, v_3\}$. In Observation 2, taking $V' = V_7 \cup \{v, v_9\}$, we have $n' = n-9$, $m' \le n-10$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v_4, v_5, v_7, v_9\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1.

Case 2.2. There is no edge joining $\{v_3, v_4\}$ and $\{v_6, v_7\}$. Then both v_6 and v_7 are at distance 3 from v, and so, by Observation 5, $d(v_6) = d(v_7) = 3$. Let $N(v_6) = \{v_5, v_7, v_8\}$.

Suppose that there is a vertex that is a common neighbor of a vertex in $\{v_3, v_4\}$ and a vertex in $\{v_6, v_7\}$. We may assume that v_3v_8 is an edge. Suppose v_7v_8 is not an edge. Then, by Observation 12, $d(v_8) = 3$ and v_3 and v_8 have a common neighbor. If v_4v_8 is an edge, then by Observation 12, $d(v_3) = d(v_4) = 3$. Let $N(v_7) = \{v_5, v_6, v_9\}$. Then in Observation 2, taking $V' = V_9 \cup \{v\}$, we have $n' = n-10$, $m' \le n-12$ and $|T'| \le (2n-22)/4$. Thus, $T = T' \cup \{v, v_3, v_7, v_9\}$ is a transversal of H_G of size at most $(2n-6)/4$, contradicting Observation 1. Hence, v_4v_8 is not an edge. But then in Observation 2, taking $V' = V_6 \cup \{v, v_8\}$, we have $n' = n-8$, $m' \le n-11$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v, v_4, v_6, v_8\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_7v_8 is an edge. If v_4v_8 is not an edge, then in Observation 2, taking $V' = V_8 \cup \{v\}$, we have $n' = n-9$, $m' \le n-10$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v, v_4, v_6, v_8\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_4v_8 is an edge, and so $G = F_7$.

Hence we may assume that no vertex is a common neighbor of a vertex in $\{v_3, v_4\}$ and a vertex in $\{v_6, v_7\}$, for otherwise $G = F_7$. Thus, $d(v, v_8) \ge 3$, and so, by Observation 5, $d(v_8) = 3$.

Suppose that v_3 or v_4, say v_3, has degree 3. Then in Observation 2, taking $V' = V_2 \cup \{v, v_3, v_6\}$, we have $n' = n-5$, $m' \le n-10$ and $|T'| \le (2n-15)/4$. Thus, $T = T' \cup \{v, v_3, v_6\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, $d(v_3) = d(v_4) = 3$.

If v_3 and v_4 have a common neighbor different from v, then in Observation 2, taking $V' = V_4 \cup \{v, v_6\}$, we have $n' = n-6$, $m' \le n-9$ and $|T'| \le (2n-15)/4$. Thus, $T = T' \cup \{v, v_3, v_6\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v is the only common neighbor of v_3 and v_4. Let $N(v_3) = \{v, v_4, v_9\}$. By Observation 12, $d(v_9) = 3$.

If v_7v_8 is an edge, then in Observation 2, taking $V' = V_7 \setminus \{v_4\}$, we have $n' = n-6$, $m' \le n-9$ and $|T'| \le (2n-15)/4$. Thus, $T = T' \cup \{v_3, v_5, v_6\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_7v_8 is not an edge. If v_8v_9 is not an edge, then in Observation 2, taking $V' = V_3 \cup \{v, v_5, v_6, v_9\}$, we have $n' = n-7$, $m' = n-12$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v_3, v_5, v_6, v_9\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_8v_9 is an edge.

Let v_{10} be the common neighbor of v_8 and v_9, and so $N(v_8) = \{v_6, v_9, v_{10}\}$ and $N(v_9) = \{v_3, v_8, v_{10}\}$. Then in Observation 2, taking $V' = (V_9 \setminus \{v_4\}) \cup \{v\}$, we have $n' = n-9$, $m' \le n-11$ and $|T'| \le (2n-20)/4$. Thus, $T = T' \cup \{v_3, v_5, v_7, v_9\}$ is a transversal of

H_G of size at most $(2n-4)/4$, contradicting Observation 1. This completes the proof of Observation 13. $\square$

By Observation 13, we may assume that no two vertices in $N(v)$ have a common neighbor different from v. Thus, $N(v_i) \cap N(v_j) = \{v\}$ for $1 \le i < j \le 4$. For $i = 1, 2, 3, 4$, let v_{i+4} be the neighbor of v_i not in $N[v]$. Thus, $\{v_1v_5, v_2v_6, v_3v_7, v_4v_8\} \subset E$.

Observation 14 *There is no 4-cycle containing both v_1 and v_2 or containing both v_3 and v_4.*

Proof. Suppose, to the contrary, that there is a 4-cycle containing both v_1 and v_2 or containing both v_3 and v_4. By symmetry, we may assume there is a 4-cycle containing both v_1 and v_2 and that v_5v_6 is an edge.

Case 1. v_1 or v_2 has degree 4. We may assume that $d(v_1) = 4$. Let $N(v_1) = \{v, v_2, v_5, v_9\}$. Then, v_5v_9 is an edge. If v_6v_9 is an edge, then by Observation 12, $d(v_5) = d(v_9) = 3$. Thus in Observation 2, taking $V' = V_2 \cup \{v_5, v_6, v_9\}$, we have $n' = n - 5$, $m' = n - 6$ and $|T'| \le (2n - 11)/4$. Thus, $T = T' \cup \{v_2, v_6\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, v_6v_9 is not an edge. Every neighbor of v_6, different from v_2 and v_5, is adjacent to v_2 or v_5.

Suppose that v_6 is adjacent to v_7 or v_8, say v_7. Then, v_5v_7 is an edge. If v_3 or v_4 or v_6 has degree 4, then in Observation 2, taking $V' = V_7 \cup \{v\}$, we have $n' = n - 8$, $m' \le n - 11$ and $|T'| \le (2n - 19)/4$. Thus, $T = T' \cup \{v_3, v_4, v_5, v_6\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, $d(v_3) = d(v_4) = d(v_6) = 3$. Thus in Observation 2, taking $V' = (V_7 \setminus \{v_4\}) \cup \{v\}$, we have $n' = n - 7$, $m' = n - 9$ and $|T'| \le (2n - 16)/4$. Thus, $T = T' \cup \{v, v_1, v_5\}$ is a transversal of H_G of size at most $(2n - 4)/4$, contradicting Observation 1. Hence, v_6 is adjacent to neither v_7 nor v_8. By the claw-freeness of G, v_5 is also adjacent to neither v_7 nor v_8.

If one of v_3 or v_4, say v_3, has degree 4 or if $d(v_6) = 4$, then in Observation 2, taking $V' = V_3 \cup \{v_5, v_6\}$, we have $n' = n - 5$, $m' \le n - 10$ and $|T'| \le (2n - 15)/4$. Thus, $T = T' \cup \{v_3, v_5, v_6\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, $d(v_3) = d(v_4) = d(v_6) = 3$. Thus, by Observation 12, $d(v_7) = d(v_8) = 3$. Let $N(v_6) = \{v_2, v_5, v_{10}\}$.

If v_7v_8 is an edge, then in Observation 2, taking $V' = V_8 \cup \{v\}$, we have $n' = n - 9$, $m' \le n - 11$ and $|T'| \le (2n - 20)/4$. Thus, $T = T' \cup \{v_3, v_5, v_6, v_7\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, v_7v_8 is not an edge.

If v_7 or v_8, say v_7, is adjacent to neither v_9 nor v_{10}, then in Observation 2, taking $V' = (V_7 \setminus \{v_4\}) \cup \{v\}$, we have $n' = n - 7$, $m' = n - 12$ and $|T'| \le (2n - 19)/4$. Thus, $T = T' \cup \{v_3, v_5, v_6, v_7\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence each of v_7 and v_8 is adjacent to at least one of v_9 and v_{10}. By the claw-freeness of G, each of v_7 and v_8 is adjacent to at most one of v_9 and v_{10}. Hence we may assume that $N(v_7) = \{v_3, v_9, v_{11}\}$ and $N(v_8) = \{v_4, v_{10}, v_{12}\}$. Thus, v_9v_{11} is an edge and $v_{10}v_{12}$ is an edge. In Observation 2, taking $V' = V_{10} \cup \{v\}$, we have $n' = n - 11$, $m' \le n - 13$ and $|T'| \le (2n - 24)/4$. Thus, $T = T' \cup \{v, v_1, v_8, v_9, v_{10}\}$ is a transversal of H_G of size at most $(2n - 4)/4$, contradicting Observation 1.

Case 2. $d(v_1) = d(v_2) = 3$. Thus by Observation 12, $d(v_5) = d(v_6) = 3$.

If v_7 or v_8, say v_7, is the common neighbor of v_5 and v_6, then in Observation 2, taking $V' = (V_7 \setminus \{v_4\}) \cup \{v\}$, we have $n' = n - 7$, $m' \le n - 8$ and $|T'| \le (2n - 15)/4$. Thus, $T = T' \cup \{v, v_3, v_7\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence neither v_7 nor v_8 is the common neighbor of v_5 and v_6. Let v_9 be the common neighbor of v_5 and v_6.

Suppose that v_9 has a common neighbor with v_3 or v_4. We may assume that v_7v_9 is an edge. Then in Observation 2, taking $V' = V_7 \cup \{v, v_9\}$, we have $n' = n - 9$, $m' \le n - 10$ and $|T'| \le (2n - 19)/4$. Thus, $T = T' \cup \{v, v_4, v_7, v_9\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, v_9 has no common neighbor with v_3 or v_4. In particular, v_9 is adjacent to neither v_7 nor v_8. Thus, $d(v, v_9) = 3$, and so, by Observation 5, $d(v_9) = 3$. Let $N(v_9) = \{v_5, v_6, v_{10}\}$. Then in Observation 2, taking $V' = V_3 \cup \{v, v_5, v_6, v_9, v_{10}\}$, we have $n' = n - 8$, $m' \le n - 11$ and $|T'| \le (2n - 19)/4$. Thus, $T = T' \cup \{v, v_3, v_9, v_{10}\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1.

Since both Case 1 and Case 2 produce a contradiction, we conclude that v_5v_6 is not an edge, i.e., there is no 4-cycle containing both v_1 and v_2 or containing both v_3 and v_4. $\square$

Observation 15 $\{v_5, v_6, v_7, v_8\}$ *is an independent set.*

Proof. Assume, to the contrary, that $\{v_5, v_6, v_7, v_8\}$ is not an independent set. Then, by Observation 14, there is an edge joining a vertex in $\{v_5, v_6\}$ with a vertex in $\{v_7, v_8\}$. We may assume that $v_6v_7 \in E$. We show first that v_6 and v_7 have a common neighbor.

Claim 5 v_6 *and* v_7 *have a common neighbor.*

Proof. Suppose, to the contrary, that v_6 and v_7 have no common neighbor. Then, by Observation 12, $d(v_6) = d(v_7) = 3$. Let $N(v_6) = \{v_2, v_7, v_9\}$ and let $N(v_7) = \{v_3, v_6, v_{10}\}$. Then, v_2v_9 and v_3v_{10} are edges, and $d(v_2) = d(v_3) = 4$. By Observation 14, v_5v_9 is not an edge and v_8v_{10} is not an edge.

If v_5v_{10} is an edge, then in Observation 2, taking $V' = (V_7 \setminus \{v_4\}) \cup \{v, v_{10}\}$, we have $n' = n - 8$, $m' \le n - 11$ and $|T'| \le (2n - 19)/4$. Thus, $T = T' \cup \{v, v_2, v_5, v_{10}\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, v_5v_{10} is not an edge. Similarly, v_8v_9 is not an edge.

Suppose v_9v_{10} is an edge. If $d(v_1) = 4$ or if $d(v_9) = 4$ or if $d(v_{10}) = 4$, then in Observation 2, taking $V' = V_3 \cup \{v, v_6, v_7, v_9, v_{10}\}$, we have $n' = n - 8$, $m' \le n - 11$ and $|T'| \le (2n - 19)/4$. Thus, $T = T' \cup \{v, v_1, v_9, v_{10}\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, $d(v_1) = d(v_9) = d(v_{10}) = 3$. Similarly, $d(v_4) = 3$. But then in Observation 2, taking $V' = (V_{10} \setminus \{v_5, v_8\}) \cup \{v\}$, we have $n' = n - 9$, $m' \le n - 11$ and $|T'| \le (2n - 20)/4$. Thus, $T = T' \cup \{v_1, v_2, v_3, v_4\}$ is a transversal of H_G of size at most $(2n - 4)/4$, contradicting Observation 1. Hence, v_9v_{10} is not an edge.

Suppose v_5v_8 is an edge. Suppose v_5 and v_8 have a common neighbor, say v_{11}. Then, in Observation 2, taking $V' = (V_7 \setminus \{v_5\}) \cup \{v, v_{11}\}$, we have $n' = n - 8$, $m' \le n - 11$

and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v, v_2, v_3, v_{11}\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_5 and v_8 have no common neighbor. Thus by Observation 12, $d(v_5) = d(v_8) = 3$. Let $N(v_5) = \{v_1, v_8, v_{11}\}$ and let $N(v_8) = \{v_4, v_5, v_{12}\}$. Thus, v_1v_{11} and v_4v_{12} are edges, and $d(v_1) = d(v_4) = 4$. A similar argument to the one that show that v_9v_{10} is not an edge, shows that $v_{11}v_{12}$ is not an edge.

By Observation 14, we know that neither v_9v_{11} nor $v_{10}v_{12}$ is an edge. Suppose that v_9v_{12} or $v_{10}v_{11}$ is an edge. By symmetry, we may assume v_9v_{12} is an edge. In Observation 2, taking $V' = V_9 \cup \{v, v_{12}\}$, we have $n' = n-11$, $m' \le n-13$ and $|T'| \le (2n-24)/4$. Thus, $T = T' \cup \{v, v_1, v_3, v_9, v_{12}\}$ is a transversal of H_G of size at most $(2n-4)/4$, contradicting Observation 1. Hence, neither v_9v_{12} nor $v_{10}v_{11}$ is an edge. Thus, $\{v_9, v_{10}, v_{11}, v_{12}\}$ is an independent set.

Let $v_{13} \in N(v_{12})$. Since $\{v_9, v_{10}, v_{11}, v_{12}\}$ is an independent set, v_{13} is adjacent to at most two vertices in $\{v_9, v_{10}, v_{11}, v_{12}\}$. Thus, v_{13} has at least one neighbor not in the set $\{v_9, v_{10}, v_{11}, v_{12}\}$. Therefore in Observation 2, taking $V' = V_8 \cup \{v, v_{11}, v_{12}, v_{13}\}$, we have $n' = n-12$, $m' \le n-15$ and $|T'| \le (2n-27)/4$. Thus, $T = T' \cup \{v_1, v_6, v_7, v_{11}, v_{12}, v_{13}\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. This completes the proof of Claim 5. □

By Claim 5, v_6 and v_7 have a common neighbor, say v_9. By Observation 14, v_9 is adjacent to neither v_1 nor v_4. Thus, $d(v, v_9) = 3$, and so, by Observation 4, $d(v_9) = 3$.

Suppose that $d(v_2) = d(v_3) = 3$. Then, by Observation 12, $d(v_6) = d(v_7) = 3$. Suppose that v_9 is adjacent to v_5 or v_8, say v_5. Then in Observation 2, taking $V' = V_7 \cup \{v, v_9\}$, we have $n' = n-9$, $m' \le n-10$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v, v_4, v_5, v_9\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, neither v_5v_9 nor v_8v_9 is an edge. Let $N(v_9) = \{v_6, v_7, v_{10}\}$. In Observation 2, taking $V' = V_3 \cup \{v, v_6, v_7, v_9, v_{10}\}$, we have $n' = n-8$, $m' \le n-11$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v, v_1, v_9, v_{10}\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence at least one of v_2 and v_3 has degree 4.

If v_2v_9 is an edge, then by Observation 12, $d(v_6) = d(v_9) = 3$, and so in Observation 2, taking $V' = (V_7 \setminus \{v_4\}) \cup \{v, v_9\}$, we have $n' = n-8$, $m' \le n-11$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v_1, v_3, v_5, v_7\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_2v_9 is not an edge. Similarly, v_3v_9 is not an edge. Thus, if $d(v_2) = 4$, then v_2 and v_6 have a common neighbor which is different from v_9, while if $d(v_3) = 4$, then v_3 and v_7 have a common neighbor which is different from v_9. In particular, $d(v_6) = 4$ or $d(v_7) = 4$.

Suppose v_9 is adjacent to v_5 or v_8, say v_5. By Observation 12, $d(v_5) = 3$. Hence, $d(v_1) = 4$ and v_1 and v_5 have a common neighbor. In Observation 2, taking $V' = (V_7 \setminus \{v_4\}) \cup \{v, v_9\}$, we have $n' = n-8$, $m' \le n-11$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v, v_1, v_6, v_7\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_9 is adjacent to neither v_5 nor v_8. Let $N(v_9) = \{v_6, v_7, v_{10}\}$. By Observation 12, $d(v_{10}) = 3$.

If v_{10} is adjacent to v_2 or v_3, say v_2, then v_6v_{10} is an edge, and so $N(v_6)$ induces a subgraph that contains a P_4, contradicting Observation 11. Hence, v_{10} is adjacent to neither v_2 nor v_3. If v_{10} is adjacent to v_1 or v_4, say v_1, then v_5v_{10} is an edge. But then in Observation 2, taking $V' = V_3 \cup \{v, v_6, v_7, v_9, v_{10}\}$, we have $n' = n-8$, $m' \le n-11$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v, v_1, v_6, v_7\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_{10} is adjacent to no vertex in $N(v)$.

If v_{10} is adjacent to v_5 or v_8, say v_5, then v_5 and v_{10} have a common neighbor. In Observation 2, taking $V' = (V_{10} \setminus \{v_4\}) \cup \{v\}$, we have $n' = n-10$, $m' \le n-14$ and $|T'| \le (2n-24)/4$. Thus, $T = T' \cup \{v_1, v_5, v_6, v_7, v_8\}$ is a transversal of H_G of size at most $(2n-4)/4$, contradicting Observation 1. Hence, v_{10} is adjacent to neither v_5 nor v_8. Let $N(v_{10}) = \{v_9, v_{11}, v_{12}\}$. Then, $v_{11}v_{12}$ is an edge.

Suppose there is an edge joining a vertex in $\{v_2, v_3\}$ with a vertex in $\{v_{11}, v_{12}\}$. We may assume v_2v_{11} is an edge. Then v_6v_{11} is an edge and in Observation 2, taking $V' = (V_{11} \setminus \{v_1, v_4, v_5, v_8\}) \cup \{v\}$, we have $n' = n-8$, $m' \le n-11$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v, v_3, v_{10}, v_{11}\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, there is no edge joining $\{v_2, v_3\}$ and $\{v_{11}, v_{12}\}$.

Suppose there is an edge joining a vertex in $\{v_1, v_4\}$ with a vertex in $\{v_{11}, v_{12}\}$. We may assume v_1v_{11} is an edge. Then, v_5v_{11} is an edge. But then in Observation 2, taking $V' = (V_{11} \setminus \{v_4, v_5\}) \cup \{v\}$, we have $n' = n-10$, $m' \le n-13$ and $|T'| \le (2n-23)/4$. Thus, $T = T' \cup \{v_1, v_6, v_7, v_8, v_{11}\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, neither v_{11} nor v_{12} is adjacent to a vertex in $N(v)$. Thus, by Observation 4, $d(v_{11}) = d(v_{12}) = 3$.

Suppose there is an edge joining a vertex in $\{v_5, v_8\}$ with a vertex in $\{v_{11}, v_{12}\}$. We may assume v_5v_{11} is an edge. By Observation 12, $d(v_5) = 3$. Hence, $d(v_1) = 4$ and v_1 and v_5 have a common neighbor. If v_8v_{12} is not an edge, then in Observation 2, taking $V' = (V_{11} \setminus \{v_4\}) \cup \{v\}$, we have $n' = n-11$, $m' \le n-16$ and $|T'| \le (2n-27)/4$. Thus, $T = T' \cup \{v_1, v_5, v_6, v_7, v_8, v_{11}\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_8v_{12} is an edge. By Observation 12, $d(v_8) = 3$. Hence, $d(v_4) = 4$ and v_4 and v_8 have a common neighbor. In Observation 2, taking $V' = (V_{11} \setminus \{v_1, v_5, v_8\}) \cup \{v\}$, we have $n' = n-9$, $m' \le n-14$ and $|T'| \le (2n-23)/4$. Thus, $T = \{v, v_4, v_6, v_7, v_{11}\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, there is no edge joining $\{v_5, v_8\}$ and $\{v_{11}, v_{12}\}$.

Suppose v_5v_8 is an edge. As in Claim 5, we must have that v_5 and v_8 have a common neighbor. Further, as shown with the v_6 and v_7, at least one of v_5 and v_8 has degree 4. Hence, in Observation 2, taking $V' = V_9 \cup \{v, v_{11}\}$, we have $n' = n-11$, $m' \le n-16$ and $|T'| \le (2n-27)/4$. Thus, $T = T' \cup \{v_1, v_5, v_6, v_7, v_8, v_{11}\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_5v_8 is not an edge.

If v_5 and v_8 have no common neighbor, then in Observation 2, taking $V' = (V_9 \setminus \{v_4\}) \cup \{v, v_{11}\}$, we have $n' = n-10$, $m' \le n-17$ and $|T'| \le (2n-27)/4$. Thus, $T = T' \cup \{v_1, v_5, v_6, v_7, v_8, v_{11}\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_5 and v_8 have a common neighbor. Such a common neighbor is at distance at least 3 from both v_6 and v_7, and so, by Observation 5, has degree 3. Hence, v_5 and v_8 have two common neighbor (of degree 3), say v_{13} and v_{14}. But then in Observation 2, taking $V' = V_8 \cup \{v_{13}, v_{14}\}$, we have $n' = n-11$, $m' \le n-13$ and $|T'| \le (2n-24)/4$. Thus, $T = T' \cup \{v_3, v_5, v_6, v_7, v_{13}\}$ is a transversal of H_G of size at most $(2n-4)/4$, contradicting Observation 1. This completes the proof of Observation 15. $\square$

By Observation 15, $\{v_5, v_6, v_7, v_8\}$ is an independent set.

Observation 16 *If every neighbor of v has degree 3, then $G \in \{F_{11}, F_{12}\}$.*

Proof. By Observation 12, we have that $d(v_i) = 3$ for $i \in \{5, 6, 7, 8\}$. By Observation 4, it follows that v is therefore the only degree-4 vertex in G. Let $N(v_5) = \{v_1, v_9, v_{10}\}$. Then, $v_9v_{10} \in E$.

Suppose first that a vertex in $\{v_5, v_6\}$ and a vertex in $\{v_7, v_8\}$ have a common neighbor. We may assume that v_5 and v_7 have a common neighbor. Thus, $N(v_7) = \{v_3, v_9, v_{10}\}$. Suppose that v_6 and v_8 have no common neighbor. Let $N(v_6) = \{v_2, v_{11}, v_{12}\}$ and $N(v_8) = \{v_4, v_{13}, v_{14}\}$. Then, $v_{11}v_{12} \in E$ and $v_{13}v_{14} \in E$. In Observation 2, taking $V' = (V_{11} \setminus \{v_6, v_8\}) \cup \{v\}$, we have $n' = n - 10$, $m' = n - 13$ and $|T'| \le (2n - 23)/4$. Thus, $T = T' \cup \{v, v_4, v_5, v_9, v_{11}\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, v_6 and v_8 have a common neighbor, and so $G = F_{11}$.

Suppose secondly that v_5 and v_6, or v_7 and v_8, have a common neighbor. We may assume that v_5 and v_6 have a common neighbor; that is, $N(v_6) = \{v_2, v_9, v_{10}\}$. Suppose that v_7 and v_8 have no common neighbor. Let $N(v_7) = \{v_3, v_{11}, v_{12}\}$ and $N(v_8) = \{v_4, v_{13}, v_{14}\}$. Then, $v_{11}v_{12} \in E$ and $v_{13}v_{14} \in E$. In Observation 2, taking $V' = (V_{11} \setminus \{v_7, v_8\}) \cup \{v\}$, we have $n' = n - 10$, $m' = n - 13$ and $|T'| \le (2n - 23)/4$. Thus, $T = T' \cup \{v, v_4, v_5, v_9, v_{11}\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, v_7 and v_8 have a common neighbor, and so $G = F_{12}$.

Hence we may assume that no two vertices in $\{v_5, v_6, v_7, v_8\}$ have a common neighbor, for otherwise $G \in \{F_{11}, F_{12}\}$, as desired. Let $N(v_6) = \{v_2, v_{11}, v_{12}\}$, $N(v_7) = \{v_3, v_{13}, v_{14}\}$, and $N(v_8) = \{v_4, v_{15}, v_{16}\}$. Then, $\{v_9v_{10}, v_{11}v_{12}, v_{13}v_{14}, v_{15}v_{16}\} \subset E$.

Suppose there is an edge joining two triangles each of which contain a vertex from $\{v_5, v_6, v_7, v_8\}$. We may assume that $v_{10}v_{11} \in E$. In Observation 2, taking $V' = (V_6 \setminus \{v_4\}) \cup \{v, v_{10}, v_{11}\}$, we have $n' = n - 8$, $m' = n - 12$ and $|T'| \le (2n - 20)/4$. Thus, $T = T' \cup \{v, v_3, v_{10}, v_{11}\}$ is a transversal of H_G of size at most $(2n - 4)/4$, contradicting Observation 1. Hence there is no edge joining two triangles each of which contain a vertex from $\{v_5, v_6, v_7, v_8\}$.

Suppose there is a vertex, v_{17} say, that is adjacent to two vertices that belong to distinct triangles each of which contain a vertex from $\{v_5, v_6, v_7, v_8\}$. Up to symmetry, there are two cases to consider. Suppose, first, that the vertex v_{17} satisfies $N(v_{17}) = \{v_9, v_{10}, v_{11}\}$. In Observation 2, taking $V' = \{v, v_1, v_2, v_5, v_6, v_9, v_{10}, v_{11}, v_{17}\}$, we have $n' = n - 9$, $m' = n - 12$ and $|T'| \le (2n - 21)/4$. Thus, $T = T' \cup \{v, v_1, v_{11}, v_{17}\}$ is a transversal of H_G of size at most $(2n - 5)/4$, contradicting Observation 1. Suppose, second, that that the vertex v_{17} satisfies $N(v_{17}) = \{v_9, v_{10}, v_{13}\}$. In Observation 2, taking $V' = \{v, v_1, v_3, v_5, v_7, v_9, v_{10}, v_{13}, v_{17}\}$, we have $n' = n - 9$, $m' = n - 12$ and $|T'| \le (2n - 21)/4$. Thus, $T = T' \cup \{v, v_1, v_{13}, v_{17}\}$ is a transversal of H_G of size at most $(2n - 5)/4$, contradicting Observation 1.

Hence there is no vertex that is adjacent to two vertices that belong to distinct triangles each of which contain a vertex from $\{v_5, v_6, v_7, v_8\}$. Thus in Observation 2, taking $V' = V_4 \cup \{v_9, v_{11}, v_{13}\}$, we have $n' = n - 8$, $m' \le n - 15$ and $|T'| \le (2n - 23)/4$. Thus,

$T = T' \cup \{v, v_4, v_9, v_{11}, v_{13}\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. This completes the proof of Observation 16. □

By Observation 16, we may assume that at least one neighbor of v has degree 4. We may assume $d(v_1) = 4$. Let $N(v_1) = \{v, v_2, v_5, v_9\}$. Then, v_5v_9 is an edge. By Observation 15, v_9 is adjacent to no vertex in $\{v_6, v_7, v_8\}$.

Observation 17 *For $i \in \{1, 2, 3, 4\}$, if $d(v_i) = 4$, then the two neighbors of v_i in $V \setminus N[v]$ have no common neighbor other than v_i.*

Proof. For notational convenience, consider the vertex v_1. Suppose that v_5 and v_9 have a common neighbor different from v_1. Then, by Observation 12, $d(v_5) = d(v_9) = 3$. By Observation 15, we may assume that such a common neighbor of v_5 and v_9 is adjacent to no vertex in $\{v_2, v_3, v_4\}$. Let v_{10} be the common neighbor of v_5 and v_9 different from v_1. Since $d(v, v_{10}) = 3$, $d(v_{10}) = 3$ by Observation 4.

If v_6v_{10} is an edge, then in Observation 2, taking $V' = V_2 \cup \{v, v_4, v_5, v_6, v_9, v_{10}\}$, we have $n' = n - 8$, $m' \le n - 11$ and $|T'| \le (2n - 19)/4$. Thus, $T = T' \cup \{v, v_4, v_6, v_{10}\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, v_6v_{10} is not an edge. If v_{10} is adjacent to v_7 or to v_8, say v_7, then in Observation 2, taking $V' = V_3 \cup \{v, v_5, v_7, v_9, v_{10}\}$, we have $n' = n-8$, $m' \le n-11$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v, v_2, v_7, v_{10}\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, v_{10} is not adjacent to any vertex in $\{v_6, v_7, v_8\}$. Let $N(v_{10}) = \{v_5, v_9, v_{11}\}$.

If v_6v_{11} is an edge, then in Observation 2, taking $V' = V_2 \cup \{v, v_5, v_6, v_9, v_{10}, v_{11}\}$, we have $n' = n - 8$, $m' \le n - 11$ and $|T'| \le (2n - 19)/4$. Thus, $T = T' \cup \{v, v_1, v_6, v_{11}\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, v_6v_{11} is not an edge. If v_{11} is adjacent to v_7 or to v_8, say v_7, then in Observation 2, taking $V' = \{v, v_1, v_3, v_5, v_7, v_9, v_{10}, v_{11}\}$, we have $n' = n-8$, $m' \le n-11$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v, v_1, v_7, v_{11}\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, v_{11} is not adjacent to any vertex in $\{v_6, v_7, v_8\}$.

By Observation 4, $d(v_{11}) = 3$. Let $N(v_{11}) = \{v_{10}, v_{12}, v_{13}\}$. Then, $v_{12}v_{13}$ is an edge. By Observation 4, $d(v_{12}) = d(v_{13}) = 3$. In Observation 2, take $V' = \{v, v_1, v_5, v_9, v_{10}, v_{11}, v_{12}\}$, so we have $n' = n-7$, $m' \le n-12$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v, v_1, v_{11}, v_{12}\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. □

By Observation 17, the vertex v_1 is the only common neighbor of v_5 and v_9. Every neighbor of v_3 or v_4 different from v is at distance 3 from v_1 and therefore has degree 3 by Observation 4. In particular, $d(v_7) = d(v_8) = 3$.

Observation 18 *$d(v_3) = 4$ or $d(v_4) = 4$.*

Proof. Suppose that $d(v_3) = d(v_4) = 3$. Let $N(v_7) = \{v_3, v_{10}, v_{11}\}$. Suppose v_7 and v_8 have a common neighbor. Then, $N(v_8) = \{v_4, v_{10}, v_{11}\}$. In Observation 2, taking $V' = \{v, v_1, v_3, v_4, v_7, v_8, v_{10}, v_{11}\}$, we have $n' = n-8$, $m' \le n-11$ and $|T'| \le (2n-19)/4$. Thus, $T = T' \cup \{v, v_1, v_7, v_{10}\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting

Observation 1. Hence, v_7 and v_8 have no common neighbor. Let $N(v_8) = \{v_4, v_{12}, v_{13}\}$. Then, $v_{10}v_{11}$ is an edge and $v_{12}v_{13}$ is an edge.

Suppose there is an edge joining a vertex in $\{v_{10}, v_{11}\}$ and a vertex in $\{v_{12}, v_{13}\}$. We may assume $v_{11}v_{12}$ is an edge. In Observation 2, taking $V' = \{v, v_1, v_3, v_4, v_7, v_8, v_{11}, v_{12}\}$, we have $n' = n - 8$, $m' \leq n - 13$ and $|T'| \leq (2n - 21)/4$. Thus, $T = T' \cup \{v, v_1, v_{11}, v_{12}\}$ is a transversal of H_G of size at most $(2n - 5)/4$, contradicting Observation 1. Hence there is no edge joining a vertex in $\{v_{10}, v_{11}\}$ and a vertex in $\{v_{12}, v_{13}\}$.

Suppose that a vertex in $\{v_{10}, v_{11}\}$ and a vertex in $\{v_{12}, v_{13}\}$ have a common neighbor, say v_{13}. We may assume that $N(v_{13}) = \{v_{10}, v_{11}, v_{12}\}$. Then in Observation 2, taking $V' = \{v, v_3, v_7, v_{10}, v_{11}, v_{13}\}$, we have $n' = n-7$, $m' \leq n-12$ and $|T'| \leq (2n-19)/4$. Thus, $T = T' \cup \{v, v_1, v_{11}, v_{13}\}$ is a transversal of H_G of size at most $(2n - 5)/4$, contradicting Observation 1. Hence a vertex in $\{v_{10}, v_{11}\}$ and a vertex in $\{v_{12}, v_{13}\}$ have no common neighbor.

Suppose that there are two edges joining $\{v_5, v_9\}$ and $\{v_{10}, v_{11}\}$. We may assume that v_5v_{10} and v_9v_{11} are edges. Then in Observation 2, taking $V' = \{v, v_1, v_2, v_3, v_5, v_7, v_9, v_{10}, v_{11}\}$, we have $n' = n-9$, $m' \leq n-11$ and $|T'| \leq (2n-20)/4$. Thus, $T = T' \cup \{v, v_2, v_5, v_{10}\}$ is a transversal of H_G of size at most $(2n-4)/4$, contradicting Observation 1. Hence there is at most one edge joining $\{v_5, v_9\}$ and $\{v_{10}, v_{11}\}$. Similarly, there is at most one edge joining $\{v_5, v_9\}$ and $\{v_{12}, v_{13}\}$. We may therefore assume that there is no edge joining $\{v_5, v_9\}$ and $\{v_{10}, v_{12}\}$. Hence in Observation 2, taking $V' = \{v, v_1, v_3, v_4, v_{10}, v_{12}\}$, we have $n' = n - 6$, $m' \leq n - 13$ and $|T'| \leq (2n - 19)/4$. Thus, $T = T' \cup \{v, v_1, v_{10}, v_{12}\}$ is a transversal of H_G of size at most $(2n - 4)/4$, contradicting Observation 1. $\square$

By Observation 18, we may assume that v_3 or v_4, say v_4, has degree 4. Let $N(v_4) = \{v, v_3, v_8, v_{10}\}$. Then, v_8v_{10} is an edge. Every vertex at distance 2 from v is at distance 3 from either v_1 or v_4 and therefore, by Observation 4, has degree 3. By Observation 17, v_4 is the only common neighbor of v_8 and v_{10}. By the claw-freeness of G, and by Observations 15 and 17, no two vertices at distance 2 from v have a common neighbor in $V \setminus N(v)$.

Observation 19 $d(v_3) = 3$.

Proof. Suppose that $d(v_3) = 4$. Let $N(v_3) = \{v, v_4, v_7, v_{11}\}$. Then, v_7v_{11} is an edge. For $i \in \{7, 8, 10, 11\}$, let v_i' be the neighbor of v_i at distance 3 from v. Hence, $N(v_7) = \{v_3, v_7', v_{11}\}$ and $N(v_{11}) = \{v_3, v_7, v_{11}'\}$, while $N(v_8) = \{v_4, v_8', v_{10}\}$ and $N(v_{10}) = \{v_4, v_8, v_{10}'\}$. Let $W = \{v_7', v_8', v_{10}', v_{11}'\}$.

We show that W is an independent set. Suppose that two vertices in W are adjacent. We may assume that $v_7'v_{11}'$ is an edge or $v_7'v_8'$ is an edge. Suppose $v_7'v_{11}'$ is an edge. Then in Observation 2, taking $V' = \{v, v_1, v_3, v_7, v_7', v_{11}, v_{11}'\}$, we have $n' = n - 7$, $m' \leq n - 12$ and $|T'| \leq (2n-19)/4$. Thus, $T = T' \cup \{v, v_1, v_7, v_7'\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Suppose $v_7'v_8'$ is an edge. Let w be the common neighbor of v_7' and v_8'. Then in Observation 2, taking $V' = \{v, v_3, v_4, v_5, v_6, v_7, v_7', v_8, v_8', w\}$, we have $n' = n-10$, $m' \leq n-17$ and $|T'| \leq (2n-27)/4$. Thus, $T = T' \cup \{v_3, v_4, v_5, v_6, v_7', w\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Thus, W is an independent set.

If no two vertices in W have a common neighbor, then in Observation 2, taking $V' = \{v, v_1, v_3, v_4\} \cup W$, we have $n' = n - 8$, $m' \le n - 19$ and $|T'| \le (2n - 27)/4$. Thus, $T = T' \cup \{v, v_1\} \cup W$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. Hence, two vertices in W have a common neighbor.

Suppose v'_7 and v'_{11} or v'_8 and v'_{10}, say v'_7 and v'_{11}, have a common neighbor. Let $N(v'_7) = \{v_7, v_{12}, v_{13}\}$. Then, $N(v'_{11}) = \{v_{11}, v_{12}, v_{13}\}$. In Observation 2, taking $V' = \{v, v_3, v_7, v'_7, v_{11}, v'_{11}, v_{12}, v_{13}\}$, we have $n' = n - 8$, $m' \le n - 11$ and $|T'| \le (2n - 19)/4$. Thus, $T = T' \cup \{v, v_3, v'_7, v_{12}\}$ is a transversal of H_G of size at most $(2n-3)/4$, contradicting Observation 1. Hence, neither v'_7 and v'_{11} nor v'_8 and v'_{10} have a common neighbor. Hence a vertex in $\{v'_7, v'_{11}\}$ and a vertex in $\{v'_8, v'_{10}\}$ have a common neighbor. We may assume that v'_7 and v'_8 have a common neighbor. Let $N(v'_7) = \{v_7, v_{12} v_{13}\}$. Then, $N(v'_8) = \{v_8, v_{12}, v_{13}\}$. In Observation 2, taking $V' = \{v_3, v_4, v_7, v'_7, v_8, v'_8, v_{12}, v_{13}\}$, we have $n' = n - 8$, $m' \le n - 11$ and $|T'| \le (2n - 19)/4$. Thus, $T = T' \cup \{v_3, v_4, v'_7, v_{12}\}$ is a transversal of H_G of size at most $(2n - 3)/4$, contradicting Observation 1. $\Box$

By Observation 19, $d(v_3) = 3$. An identical argument (interchanging the roles of v_3 and v_4 with v_1 and v_2) shows that $d(v_2) = 3$. Let $N(v_6) = \{v_2, v_{12}, v_{13}\}$ and $N(v_7) = \{v_3, v_{14}, v_{15}\}$. Then, $v_{12}v_{13}$ is an edge and $v_{14}v_{15}$ is an edge. An identical proof as the proof of Observation 18 now produces a contradiction. This completes the proof of Theorem 9. $\Box$

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