# The valuations of the near octagon $\mathbb{I}_4$

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**Abstract**

The maximal and next-to-maximal subspaces of a nonsingular parabolic quadric $Q(2n,2)$, $n \geq 2$, which are not contained in a given hyperbolic quadric $Q^+(2n-1,2) \subset Q(2n,2)$ define a sub near polygon $\mathbb{I}_n$ of the dual polar space $DQ(2n,2)$. It is known that every valuation of $DQ(2n,2)$ induces a valuation of $\mathbb{I}_n$. In this paper, we classify all valuations of the near octagon $\mathbb{I}_4$ and show that they are all induced by a valuation of $DQ(8,2)$. We use this classification to show that there exists up to isomorphism a unique isometric full embedding of $\mathbb{I}_n$ into each of the dual polar spaces $DQ(2n,2)$ and $DH(2n-1,4)$.

## 1 Introduction

### 1.1 Basic Definitions

A *near polygon* is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$, with the property that for every point $x \in \mathcal{P}$ and every line $L \in \mathcal{L}$, there exists a unique point on $L$ nearest to $x$. Here, distances are measured in the *point graph* or *collinearity graph* $\Gamma$ of $\mathcal{S}$. If $d$ is the diameter of $\Gamma$, then the near polygon is called a *near $2d$-gon.* The unique near 0-gon consists of one point (no lines). The near 2-gons are precisely the lines. Near quadrangles are usually called *generalized quadrangles* (Payne and Thas [15]). Near polygons were introduced by Shult and Yanushka [17] because of their connection with the so-called tetrahedrally closed line systems in Euclidean spaces. A detailed treatment of the basic theory of near polygons can be found in the recent book of the author [4].

If $x_1$ and $x_2$ are two points of a near polygon $\mathcal{S}$, then $\mathrm{d}(x_1, x_2)$ denotes the distance between $x_1$ and $x_2$ (in the point graph). If $X_1$ and $X_2$ are two nonempty sets of points, then $\mathrm{d}(X_1, X_2)$ denotes the minimal distance between a point of $X_1$ and a point of $X_2$. If

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$X_1$ is a singleton $\{x_1\}$, then we will also write $\mathrm{d}(x_1, X_2)$ instead of $\mathrm{d}(\{x_1\}, X_2)$. If $X$ is a nonempty set of points and $i \in \mathbb{N}$, then $\Gamma_i(X)$ denotes the set of all points $y$ for which $\mathrm{d}(y, X) = i$. If $X$ is a singleton $\{x\}$, then we will also write $\Gamma_i(x)$ instead of $\Gamma_i(\{x\})$.

A subspace $S$ of a near polygon $\mathcal{S}$ is called *convex* if every point on a shortest path between two points of $S$ is also contained in $S$. The points and lines of a near polygon which are contained in a given convex subspace define a sub(-near-)polygon of $\mathcal{S}$. The maximal distance between two points of a convex subspace $S$ is called the *diameter* of $S$ and is denoted as $\mathrm{diam}(S)$. If $X_i$, $i \in \{1, \dots, k\}$, are nonempty sets of points, then $\langle X_1, \dots, X_k \rangle$ denotes the smallest convex subspace containing $X_1 \cup X_2 \cup \cdots \cup X_k$, i.e., $\langle X_1, \dots, X_k \rangle$ is the intersection of all convex subspaces containing $X_1 \cup X_2 \cup \cdots \cup X_k$.

A near polygon is said to have *order* $(s, t)$ if every line is incident with precisely $s+1$ points and if every point is incident with precisely $t+1$ lines. A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbours. Dense near polygons satisfy several nice properties, see e.g. Chapter 2 of [4]. The most interesting among these properties is without any doubt the following result due to Brouwer and Wilbrink [2]: if $x$ and $y$ are two points of a dense near polygon at distance $\delta$ from each other, then (the point-line geometry induced by) $\langle x, y \rangle$ is a sub-near-$2\delta$-gon. These subpolygons are called *quads* if $\delta = 2$ and *hexes* if $\delta = 3$.

If $x$ is a point and $Q$ is a quad of a dense near polygon such that $\mathrm{d}(x, Q) = \delta$, then precisely one of the following two cases occurs: (i) $Q$ contains a unique point $\pi_Q(x)$ at distance $\delta$ from $x$ and $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_Q(x)) + \mathrm{d}(\pi_Q(x), y)$ for every point $y$ of $Q$; (ii) $\Gamma_\delta(x) \cap Q$ is an ovoid of $Q$. If case (i) occurs, then $x$ is called *classical* with respect to $Q$. If case (ii) occurs, then $x$ is called *ovoidal* with respect to $Q$. If $Q$ is a quad and $\delta \in \mathbb{N}$, then we denote by $\Gamma_{\delta,C}(Q)$, respectively $\Gamma_{\delta,O}(Q)$, the set of points at distance $\delta$ from $Q$ which are classical, respectively ovoidal, with respect to $Q$.

A convex subspace $F$ of a dense near polygon $\mathcal{S}$ is called *classical* in $\mathcal{S}$ if for every point $x$ of $\mathcal{S}$, there exists a unique point $\pi_F(x)$ in $F$ such that $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), y)$ for every point $y$ of $F$. The point $\pi_F(x)$ is called the *projection* of $x$ onto $F$. Classical convex subspaces satisfy the following property:

**Proposition 1.1 (Theorem 2.32 of [4])** *Let $H$ be a convex sub-$2m$-gon of a dense near $2d$-gon $\mathcal{S}$ which is classical in $\mathcal{S}$ and let $x$ be a point of $H$. If $H'$ is a convex sub-$2(d-m+\delta)$-gon through $x$, then* $\mathrm{diam}(H \cap H') \geq \delta$.

An important class of near polygons are the dual polar spaces (Cameron [3]). Suppose $\Pi$ is a nondegenerate polar space of rank $n \geq 2$. Let $\Delta$ be the incidence structure with points, respectively lines, the maximal, respectively next-to- maximal, singular subspaces of $\Pi$, with reverse containment as incidence relation. Then $\Delta$ is a near $2n$-gon, a so-called *dual polar space of rank $n$*. If $\Pi$ is a thick dual polar space, then $\Delta$ is a dense near $2n$-gon. There exists a bijective correspondence between the convex subspaces of $\Delta$ and the singular subspaces of $\Pi$: if $\alpha$ is a singular subspace of $\Pi$, then the set of all maximal singular subspaces containing $\alpha$ is a convex subspace of $\Delta$. Every convex subspace of $\Delta$

is classical in $\Delta$. The dual polar spaces relevant for this paper are the dual polar spaces $DQ(2n, 2)$ and $DH(2n-1, 4)$ related to respectively a nonsingular parabolic quadric $Q(2n, 2)$ in $\mathrm{PG}(2n, 2)$ and a nonsingular hermitian variety $H(2n-1, 4)$ in $\mathrm{PG}(2n-1, 4)$.

Let $Q(2n, 2)$, $n \geq 2$, be a nonsingular parabolic quadric in $\mathrm{PG}(2n, 2)$ and let $\pi$ be a hyperplane of $\mathrm{PG}(2n, 2)$ intersecting $Q(2n, 2)$ in a nonsingular quadric $Q^+(2n-1, 2)$. The generators of $Q(2n, 2)$ define a dual polar space $DQ(2n, 2)$. The generators of $Q(2n, 2)$ not contained in $Q^+(2n-1, 2)$ form a subspace $X$ of $DQ(2n, 2)$. The set $X$ is a hyperplane of $DQ(2n, 2)$, i.e., every line of $DQ(2n, 2)$ is either contained in $X$ or intersects $X$ in a unique point. The point-line incidence structure defined on the set $X$ by the set of lines of $DQ(2n, 2)$ (contained in $X$) is a dense near $2n$-gon which we will denote by $\mathbb{I}_n$. The generalized quadrangle $\mathbb{I}_2$ is isomorphic to the $(3 \times 3)$- grid. For more details on the above construction, we refer to Section 6.4 of [4].

Let $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ be two dense near polygons with respective diameters $d_1$ and $d_2$ and respective distance functions $\mathrm{d}_1(\cdot, \cdot)$ and $\mathrm{d}_2(\cdot, \cdot)$. An *isometric full embedding* $\theta$ of $\mathcal{S}_1$ into $\mathcal{S}_2$ is a map $\theta : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ which satisfies the following properties:

- for all points $x$ and $y$ of $\mathcal{P}_1$, $\mathrm{d}_2(\theta(x), \theta(y)) = \mathrm{d}_1(x, y)$;
- if $L$ is a line of $\mathcal{S}_1$, then $\theta(L) := \{\theta(x) \,|\, x \in L\}$ is a line of $\mathcal{S}_2$.

Two isometric full embeddings $\theta_1$ and $\theta_2$ of $\mathcal{S}_1$ into $\mathcal{S}_2$ are called *isomorphic* if there exists an automorphism $\phi$ of $\mathcal{S}_2$ such that $\theta_2 = \phi \circ \theta_1$. If there exists an isometric full embedding of $\mathcal{S}_1$ into $\mathcal{S}_2$, then obviously $d_2 \geq d_1$. In view of the following proposition, we may restrict the study of isometric full embeddings between dense near polygons to the case in which both dense near polygons have the same diameter.

**Proposition 1.2** *If there exists an isometric full embedding $\theta$ of $\mathcal{S}_1$ into $\mathcal{S}_2$, then there exists a convex subspace $\mathcal{S}_2'$ of diameter $d_1$ in $\mathcal{S}_2$ containing all points $\theta(x)$, $x \in \mathcal{P}_1$.*

**Proof.** Let $x_1$ and $x_2$ be two points of $\mathcal{S}_1$ at maximal distance $d_1$ from each other. Then $\mathrm{d}_2(\theta(x_1), \theta(x_2)) = d_1$ and hence there exists a convex subspace $\mathcal{S}_2'$ of diameter $d_1$ in $\mathcal{S}_2$ containing the points $\theta(x_1)$ and $\theta(x_2)$.

Suppose $x$ is a point of $\mathcal{S}_1$ at distance $d_1$ from $x_1$. Then by Brouwer and Wilbrink [2], there exists a path $x_2 = y_0, y_1, \ldots, y_k = x$ in $\Gamma_{d_1}(x_1)$ connecting the points $x_2$ and $x$. We will prove by induction on $i \in \{0, \ldots, k\}$ that $\theta(y_i)$ is a point of $\mathcal{S}_2'$. Obviously, this holds if $i = 0$. So, suppose $i \in \{1, \ldots, k\}$. The line $y_i y_{i-1}$ contains a point $z_i$ at distance $d_1 - 1$ from $x_1$. Since $\theta$ is an isometric embedding, $\theta(z_i)$ is a point collinear with $\theta(y_{i-1})$ at distance $d_1 - 1$ from $\theta(x_1)$. By the induction hypothesis, $\theta(y_{i-1})$ is a point of $\mathcal{S}_2'$ at distance $d_1$ from $\theta(x_1)$. Hence, also $\theta(z_i)$ is a point of $\mathcal{S}_2'$. It follows that the point $\theta(y_i)$ of the line $\theta(z_i)\theta(y_{i-1})$ belongs to $\mathcal{S}_2'$.

Now, let $x$ denote an arbitrary point of $\mathcal{S}_1$. Then by Brouwer and Wilbrink [2], $x$ is contained in a shortest path connecting $x_1$ with a point $x' \in \Gamma_{d_1}(x_1)$. By the previous paragraph, $\theta(x')$ is a point of $\mathcal{S}_2'$ at distance $d_1$ from $\theta(x_1)$. Since $\theta(x)$ is on a shortest path between the points $\theta(x_1)$ and $\theta(x')$ of $\mathcal{S}_2'$, also $\theta(x)$ belongs to $\mathcal{S}_2'$. This proves the proposition. ■

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a dense near polygon. A function $f$ from $\mathcal{P}$ to $\mathbb{N}$ is called a *valuation* of $\mathcal{S}$ if it satisfies the following properties (we call $f(x)$ the value of $x$):

(V1) there exists at least one point with value 0;

(V2) every line $L$ of $\mathcal{S}$ contains a unique point $x_L$ with smallest value and $f(x) = f(x_L)+1$ for every point $x$ of $L$ different from $x_L$;

(V3) every point $x$ of $\mathcal{S}$ is contained in a convex subspace $F_x$ such that the following properties are satisfied for every $y \in F_x$:

  (i) $f(y) \leq f(x)$;

  (ii) if $z$ is a point collinear with $y$ such that $f(z) = f(y) - 1$, then $z \in F_x$.

One can show, see De Bruyn and Vandecasteele [8, Proposition 2.5], that the convex subspace $F_x$ in property $(V_3)$ is unique. If $f$ is a valuation of $\mathcal{S}$, then we denote by $O_f$ the set of points with value 0. A quad $Q$ of $\mathcal{S}$ is called *special (with respect to $f$)* if it contains two distinct points of $O_f$, or equivalently (see [8]), if it intersects $O_f$ in an ovoid of $Q$. We denote by $G_f$ the partial linear space with points the elements of $O_f$ and with lines the special quads (natural incidence). An important notion is the one of *induced valuation*.

**Proposition 1.3 (Proposition 2.12 of [8])** *Let $\mathcal{S}$ be a dense near polygon and let $F = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be a dense near polygon which is fully and isometrically embedded in $\mathcal{S}$. Let $f$ denote a valuation of $\mathcal{S}$ and put $m := \min\{f(x) \,|\, x \in \mathcal{P}'\}$. Then the map $f_F : \mathcal{P}' \to \mathbb{N}, x \mapsto f(x) - m$ is a valuation of $F$ (a so-called* induced valuation*).*

Valuations of dense near polygons have interesting and important applications in the following areas: (1) the classification of dense near polygons (e.g. [11]); (2) construction of hyperplanes of dense near polygons, in particular of dual polar spaces ([9]); (3) classification of hyperplanes of dual polar spaces ([5]); (4) the study of isometric full embeddings between dual polar spaces ([6]). Valuations will be further discussed in Section 2.

## 1.2 Main results

Valuations are an indispensable tool for classifying dense near polygons (see e.g. [4]). Of particular interest are the dense near polygons of order $(2, t)$ which the authors are trying to classify. At this moment, a complete classification of such dense near polygons is available up to diameter 4 ([15], [1], [11]). In order to obtain new classification results regarding dense near polygons of order $(2, t)$, new classification results regarding valuations seem to be necessary. The classification of the valuations of the dense near hexagons of order $(2, t)$ has been completed by the authors in [10]. The next cases to consider are the near octagons. We start with the near octagon $\mathbb{I}_4$.

The embedding of $\mathbb{I}_n$ in $DQ(2n, 2)$ ($n \geq 2$) described above is an isometric full embedding. So, by Proposition 1.3, every valuation of the dual polar space $DQ(2n, 2)$ induces a valuation of $\mathbb{I}_n$. In [10], the authors classified all valuations of $\mathbb{I}_3$. It turns out that all these valuations are induced by a unique valuation of $DQ(6, 2)$. In the present paper, we prove a similar result for the near octagon $\mathbb{I}_4$:

**Theorem 1.4** *Every valuation $f$ of the near octagon $\mathbb{I}_4$ is induced by a unique valuation $f'$ of $DQ(8, 2)$.*

**Remark.** In [7], it will be shown by one of the authors that also every valuation of $\mathbb{I}_n$, $n \geq 5$, is induced by a unique valuation of $DQ(2n, 2)$. The complete classification of the valuations of $\mathbb{I}_4$ is however necessary to achieve this goal. Paper [7] will also contain a discussion of the structure of the valuations of $\mathbb{I}_n$.

We will see in Corollary 2.8, that there are three types of valuations in $DQ(8, 2)$. We will show in Section 4 that these valuations induce five types of valuations in $\mathbb{I}_4$. More precisely, if $f'$ is a valuation of $DQ(8, 2)$ and if $f$ is the valuation of $\mathbb{I}_4$ induced by $f'$, then one of the following cases occurs (we refer to Sections 2 and 3 for definitions):

($i$) If $f'$ is a classical valuation of $DQ(8, 2)$ such that the unique point with $f'$-value 0 belongs to $\mathbb{I}_4$, then $f$ is a classical valuation of $\mathbb{I}_4$ and $O_f = O_{f'}$.

(ii) If $f'$ is a classical valuation of $DQ(8, 2)$ such that the unique point with $f'$-value 0 does not belong to $\mathbb{I}_4$, then $O_f$ is a so-called projective set.

(iii) Suppose $f'$ is the extension of an ovoidal valuation $f''$ in a quad $Q$ of $DQ(8, 2)$ which is contained in $\mathbb{I}_4$. Then the valuation $f$ of $\mathbb{I}_4$ is also the extension (in $\mathbb{I}_4$) of the ovoidal valuation $f''$ of $Q$. So, $O_f = O_{f'}$.

(iv) Suppose $f'$ is the extension of an ovoidal valuation $f''$ in a quad $Q$ of $DQ(8, 2)$ which is not contained in $\mathbb{I}_4$. Then $O_f \subset O_{f'}$ is an ovoid in the grid-quad $Q \cap \mathbb{I}_4$ of $\mathbb{I}_4$.

(v) Suppose that $f'$ is an SDPS-valuation of $DQ(8, 2)$. Then $|O_f| = 75$ and the linear space $G_f$ is isomorphic to the partial linear space $W'(4)$ obtained from the symplectic generalized quadrangle $W(4)$ by removing two orthogonal hyperbolic lines.

In Section 5, we will use the classification of the valuations of $\mathbb{I}_3$ and $\mathbb{I}_4$ to study isometric full embeddings of $\mathbb{I}_n$ into the dual polar spaces $DQ(2n, 2)$ and $DH(2n-1, 4)$. We will show the following:

**Theorem 1.5** ($i$) *Up to isomorphism, there is a unique isometric full embedding of $\mathbb{I}_n$, $n \geq 2$, into $DQ(2n, 2)$.*

($ii$) *Up to isomorphism, there is a unique isometric full embedding of $\mathbb{I}_n$, $n \geq 2$, into $DH(2n-1, 4)$.*

# 2 Valuations: more advanced notions

## 2.1 Properties of valuations

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a dense near $2n$-gon.

We define four classes of valuations.

(1) For every point $x$ of $\mathcal{S}$, the map $f_x : \mathcal{P} \to \mathbb{N};\ y \mapsto \mathrm{d}(x,y)$ is a valuation of $\mathcal{S}$. We call $f$ a *classical valuation* of $\mathcal{S}$.

(2) Suppose $O$ is an ovoid of $\mathcal{S}$, i.e., a set of points of $\mathcal{S}$ meeting each line in a unique point. For every point $x$ of $\mathcal{S}$, we define $f_O(x) = 0$ if $x \in O$ and $f_O(x) = 1$ otherwise. Then $f_O$ is a valuation of $\mathcal{S}$, which we call an *ovoidal valuation*.

(3) Let $x$ be a point of $\mathcal{S}$ and let $O$ be a set of points of $\mathcal{S}$ at distance $n$ from $x$ such that every line at distance $n-1$ from $x$ has a unique point in common with $O$. For every point $y$ of $\mathcal{S}$, we define $f(y) := \mathrm{d}(x,y)$ if $\mathrm{d}(x,y) \leq n-1$, $f(y) := n-2$ if $y \in O$ and $f(y) := n-1$ otherwise. Then $f$ is a valuation of $\mathcal{S}$, which we call a *semi-classical valuation*.

(4) Suppose $F = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ is a convex subspace of $\mathcal{S}$ which is classical in $\mathcal{S}$, and that $f' : \mathcal{P}' \to \mathbb{N}$ is a valuation of $F$. Then the map $f : \mathcal{P} \to \mathbb{N}; x \mapsto f(x) := \mathrm{d}(x, \pi_F(x)) + f'(\pi_F(x))$ is a valuation of $\mathcal{S}$. We call $f$ the extension of $f'$. If $\mathcal{P}' = \mathcal{P}$, then we say that the extension is *trivial*.

Applying Proposition 1.3 to classical valuations, we obtain:

**Proposition 2.1** *Let $\mathcal{S}$ be a dense near polygon and let $F = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be a dense near polygon which is fully and isometrically embedded in $\mathcal{S}$. For every point $x$ of $\mathcal{S}$ and for every point $y$ of $F$, we define $f_x(y) := d(x,y) - d(x,F)$. Then $f_x$ is a valuation of $F$.*

**Proposition 2.2** *Let $\mathcal{S}$ be a dense near $2n$-gon and let $F = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be a dense near $2n$-gon which is fully and isometrically embedded in $\mathcal{S}$. Let $x$ be a point of $\mathcal{S}$ and let $f_x$ be the valuation of $F$ induced by $x$ (see Proposition 2.1). Then $d(x,F) = n - M$, where $M$ is the maximal value attained by $f_x$.*

**Proof.** We need to show that there is a point in $F$ at distance $n$ from $x$. Let $y$ be a point of $F$ at maximal distance from $x$. Every line of $F$ through $x$ contains a point at distance $\mathrm{d}(x,y) - 1$ from $x$ and hence is contained in the convex subspace $\langle x, y \rangle$ of $\mathcal{S}$. The intersection $\langle x, y \rangle \cap F$ is a convex subspace of $F$ containing all lines of $F$ through $y$. Hence, $\langle x, y \rangle \cap F = F$, i.e., $F \subseteq \langle x, y \rangle$. Since $F$ has diameter $n$, also $\langle x, y \rangle$ must have diameter $n$, i.e. $\mathrm{d}(x,y) = n$. This was what we needed to show. ■

In the following proposition, we collect some properties of valuations. We refer to [8] for proofs.

**Proposition 2.3 ([8])** *The following holds for a valuation $f$ of a dense near $2n$-gon $\mathcal{S}$.*

(*a*) *No two distinct special quads intersect in a line.*

(*b*) *$f(x) = d(x, O_f)$ for every point $x$ of $\mathcal{S}$ with $d(x, O_f) \leq 2$.*

(c) *$f$ is a classical valuation if and only if there exists a point with value $n$.*

(d) *If $x$ is a point such that $f(y) = d(x, y)$ for every point $y$ at distance at most $n-1$ from $x$, then $f$ is either classical or semi-classical.*

(e) *If $\mathcal{S}$ contains lines with $s+1$ points and if $m_i$, $i \in \mathbb{N}$, denotes the total number of points with value $i$, then $\sum_{i=0}^{\infty} m_i(-\frac{1}{s})^i = 0$.*

## 2.2 SDPS-valuations

Let $\Delta$ be a thick dual polar space of rank $2n$, $n \in \mathbb{N}$. (We will take the following convention: a dual polar space of rank 0 is a point and a dual polar space of rank 1 is a line.) A set $X$ of points of $\Delta$ is called an *SDPS-set* of $\Delta$ if it satisfies the following properties.

(1) No two points of $X$ are collinear in $\Delta$.

(2) If $x, y \in X$ such that $\mathrm{d}(x, y) = 2$, then $X \cap \langle x, y \rangle$ is an ovoid of the quad $\langle x, y \rangle$.

(3) The point-line geometry $\widetilde{\Delta}$ whose points are the elements of $X$ and whose lines are the quads of $\Delta$ containing at least two points of $X$ (natural incidence) is a dual polar space of rank $n$.

(4) For all $x, y \in X$, $\mathrm{d}(x, y) = 2 \cdot \delta(x, y)$. Here, $\mathrm{d}(x, y)$ and $\delta(x, y)$ denote the distances between $x$ and $y$ in the respective dual polar spaces $\Delta$ and $\widetilde{\Delta}$.

(5) If $x \in X$ and if $L$ is a line of $\Delta$ through $x$, then $L$ is contained in a quad of $\Delta$ which contains at least two points of $X$.

SDPS-sets were introduced by De Bruyn and Vandecasteele [9], see also [4, Section 5.6.7]. An SDPS-set of a dual polar space of rank 0 consists of the unique point of that dual polar space. An SDPS-set of a thick generalized quadrangle $Q$ is an ovoid of $Q$. For examples of SDPS-sets in thick dual polar spaces of rank $2n \geq 4$, see De Bruyn and Vandecasteele [9, Section 4] or Pralle and Shpectorov [16].

**Proposition 2.4 (Theorem 4 of [9]; Section 5.8 of [4])** *Let $X$ be an SDPS-set of a thick dual polar space $\Delta$ of rank $2n \geq 0$. For every point $x$ of $\Delta$, we define $f(x) := d(x, X)$. Then $f$ is a valuation of $\Delta$ with maximal value equal to $n$.*

Any valuation which can be obtained from an SDPS- set as described in Proposition 2.4 is called an *SDPS- valuation*. In the following two propositions, we characterize SDPS-valuations.

**Proposition 2.5 (Theorem 5 of [9]; Section 5.9 of [4])** *Let $f$ be a valuation of a thick dual polar space $\Delta$ which is the possibly trivial extension of an SDPS-valuation in a convex subspace of $\Delta$, and let $A$ be an arbitrary hex of $\Delta$. Then the valuation induced in $A$ is either classical or the extension of an ovoidal valuation in a quad of $A$.*

**Proposition 2.6 (Theorem 6 of [9]; Section 5.10 of [4])** *Let $f$ be a valuation of a thick dual polar space $\Delta$ such that every induced hex-valuation is either classical or the extension of an ovoidal valuation in a quad, then $f$ is the possibly trivial extension of an SDPS-valuation in a convex subspace of $\Delta$.*

**Proposition 2.7 (Section 6 of [10])** *Every valuation of the dual polar space $DQ(6,2)$ is either classical or the extension of an ovoidal valuation in a quad of $DQ(6,2)$.*

**Corollary 2.8** *If $f$ is a valuation of the dual polar space $DQ(2n,2)$, $n \geq 2$, then $f$ is the possibly trivial extension of an SDPS-valuation in a convex subspace of $DQ(2n,2)$.*

**Proof.** If $f$ is a valuation of $DQ(2n,2)$, $n \geq 2$, then by Proposition 2.7, every induced hex valuation is either classical or the extension of an ovoidal valuation in a quad. By Proposition 2.6, it then follows that $f$ is the possibly trivial extension of an SDPS-valuation in a convex subspace of $DQ(2n,2)$. ■

# 3 Properties of the near $2n$-gon $\mathbb{I}_n$

## 3.1 The convex subspaces of $\mathbb{I}_n$

Consider a nonsingular parabolic quadric $Q(2n,2)$, $n \geq 2$, in $\mathrm{PG}(2n,2)$ and a hyperplane $\pi$ of $\mathrm{PG}(2n,2)$ intersecting $Q(2n,2)$ in a nonsingular hyperbolic quadric $Q^+(2n-1,2)$. Let $DQ(2n,2)$ denote the dual polar space associated with $Q(2n,2)$ and let $\mathbb{I}_n$ be the sub-$2n$-gon of $DQ(2n,2)$ defined on the set of generators of $Q(2n,2)$ not contained in $Q^+(2n-1,2)$.

Let $\alpha$ be a subspace of dimension $n-1-i$, $i \in \{0,\ldots,n\}$, on $Q(2n,2)$ which is not contained in $Q^+(2n-1,2)$ if $i \in \{0,1\}$. If $X_\alpha$ is the set of all maximal subspaces of $Q(2n,2)$ through $\alpha$ not contained in $Q^+(2n-1,2)$, then $X_\alpha$ is a convex subspace of diameter $i$ of $\mathbb{I}_n$. Conversely, every convex subspace of diameter $i$ of $\mathbb{I}_n$ is obtained in this way. Let $\mathcal{A}_\alpha$ denote the point-line geometry on the set $X_\alpha$ induced by the lines of $DQ(2n,2)$. If $i \geq 2$ and if $\alpha$ is not contained in $\pi$, then $\mathcal{A}_\alpha \cong DQ(2i,2)$. If $i \geq 2$ and $\alpha \subseteq \pi$, then $\mathcal{A}_\alpha \cong \mathbb{I}_i$. Every convex subspace of $\mathbb{I}_n$ isomorphic to $DQ(2i,2)$ for some $i \geq 2$ is classical in $\mathbb{I}_n$. If $\alpha_1$ and $\alpha_2$ are two subspaces of $Q(2n,2)$ such that $\alpha_i \not\subseteq \pi$ if $\dim(\alpha_i) \in \{n-2,n-1\}$ ($i \in \{1,2\}$), then $X_{\alpha_1} \subseteq X_{\alpha_2}$ if and only if $\alpha_2 \subseteq \alpha_1$. Using this fact, one can easily see that every line of $\mathbb{I}_n$ is contained in a unique grid- quad. (Recall that $\mathbb{I}_2$ is isomorphic to the $(3\times 3)$-grid.) For more details on the above-mentioned facts, see Section 6.4 of [4].

An important notion is the one of a projective set. Suppose $\alpha$ is a point of $DQ(2n,2)$ not contained in $\mathbb{I}_n$, i.e., $\alpha$ is a generator of $Q^+(2n-1,2)$. Let $V_\alpha$ denote the set of points of $\mathbb{I}_n$ collinear with $\alpha$. Since $\mathbb{I}_n$ is a hyperplane of $DQ(2n,2)$, there is a unique point of $V_\alpha$ on every line of $DQ(2n,2)$ through $\alpha$. The set $V_\alpha$ satisfies the following properties, see Section 8.2 of [10]:

(i) every two points of $V_\alpha$ lie at distance 2 from each other and the unique quad of $\mathbb{I}_n$ containing them is a grid;

(ii) if $x \in V_\alpha$ and if $Q$ is a grid-quad through $x$, then $Q \cap V_\alpha$ is an ovoid of $Q$;

(iii) the incidence structure with points the elements of $V_\alpha$ and with lines the grid-quads of $\mathbb{I}_n$ containing at least two points of $V_\alpha$ is isomorphic to the point-line system of the projective space $\mathrm{PG}(n-1,2)$.

Because of property (iii), the set $V_\alpha$ is called a *projective set*. Projective sets satisfy the following properties, see again Section 8.2 of [10].

(a) Every point is contained in precisely two projective sets.

(b) If $x$ and $y$ are two points at distance 2 from each other such that $\langle x, y\rangle$ is a grid, then there exists a unique projective set containing $x$ and $y$.

## 3.2 The valuations of $\mathbb{I}_3$

We will use the same notations as in Section 3.1 but we suppose that $n = 3$. The valuations of $\mathbb{I}_3$ were classified in Section 8.4 of [10]. The following holds:

**Proposition 3.1** *Every valuation $f$ of $\mathbb{I}_3$ is induced by a unique valuation $f'$ of $DQ(6,2)$.*

There are two types of valuations $f'$ in $DQ(6,2)$ (recall Proposition 2.7) giving rise to four types of valuations $f$ in $\mathbb{I}_3$.

(1) Suppose $f'$ is a classical valuation of $DQ(6,2)$ such that the unique point $x$ with $f'$-value 0 belongs to $\mathbb{I}_3$. Then $f$ is a classical valuation of $\mathbb{I}_3$ and $O_f = \{x\}$.

(2) Suppose $f'$ is a classical valuation of $DQ(6,2)$ such that the unique point with $f'$-value 0 does not belong to $\mathbb{I}_3$. Then $O_f$ is a projective set. We call $f$ a *valuation of Fano-type*: the set of grid-quads of $\mathbb{I}_4$ which are special with respect to $f$ defines a Fano plane on the set $O_f$.

(3) Suppose $f'$ is the extension of an ovoidal valuation in a quad $Q$ of $DQ(6,2)$ which is also a quad of $\mathbb{I}_3$. Then the valuation $f$ of $\mathbb{I}_3$ is the extension of an ovoidal valuation in $Q$. Moreover, $O_f = O_{f'}$. We call $f$ a *valuation of extended type*.

(4) Suppose $f'$ is the extension of an ovoidal valuation in a quad $Q$ of $DQ(6,2)$ which is not a quad of $\mathbb{I}_3$. Then $|O_f| = 3$ and the grid $Q \cap \mathbb{I}_3$ is the unique quad of $\mathbb{I}_3$ which is special with respect to the valuation $f$. We call $f$ a *valuation of grid-type*.

**Lemma 3.2** *Let $f$ be a valuation of $\mathbb{I}_3$ of grid-type and let $G$ denote the unique special grid-quad of $\mathbb{I}_3$ containing $O_f$. Then there are 24 points in $\mathbb{I}_3$ at distance 2 from $G$. From these 24 points, 16 have value 2 and 8 have value 1.*

**Proof.** Let $\overline{G}$ denote the unique $W(2)$-quad of $DQ(6,2)$ containing $G$ and let $\overline{O}$ denote the unique ovoid of $\overline{G}$ containing $O_f$. The points of $\mathbb{I}_3$ at distance 2 from $G$ are precisely the points $x$ of $\mathbb{I}_3$ for which $\pi_{\overline{G}}(x) \notin G$. If $y$ is a point of $\overline{G} \setminus G$, then $y$ is collinear with four points of $\mathbb{I}_3 \setminus G$. If $y \in \overline{O}$, then each of these points has value 1. If $y \notin \overline{O}$, then each of these points has value 2. The lemma now readily follows from the fact that $|\overline{O} \setminus O_f| = 2$ and $|\overline{G} \setminus G| = 6$. ∎

# 4 The valuations of $\mathbb{I}_4$

In this section, we will prove Theorem 1.4. We will regard the near octagon $\mathbb{I}_4$ as a sub-near-polygon of $DQ(8,2)$, see Section 1. Convex subspaces of diameter 2 and 3 of $\mathbb{I}_4$ will be called *quads* and *hexes*, respectively. Convex subspaces of diameter 2 and 3 of $DQ(8,2)$ will be called *QUADS* and *HEXES*, respectively. Every $W(2)$-quad of $\mathbb{I}_4$ is a QUAD of $DQ(8,2)$. A grid-quad of $\mathbb{I}_4$ is not a QUAD of $DQ(8,2)$.

## 4.1 Two lemmas

By Corollary 2.8, every valuation of $DQ(8,2)$ is either a classical valuation, the extension of an ovoidal valuation in a quad of $DQ(8,2)$ or an SDPS-valuation. By Proposition 1.3, each valuation of $DQ(8,2)$ induces a valuation of $\mathbb{I}_4$.

**Lemma 4.1** *Suppose the valuation $f$ of $\mathbb{I}_4$ is induced by a valuation $f'$ of $DQ(8,4)$.*

*$(i)$ If $f'$ is a classical valuation of $DQ(8,2)$ such that $O_{f'} \subseteq \mathbb{I}_4$, then $f$ is a classical valuation of $\mathbb{I}_4$ and $O_f = O_{f'}$.*

*$(ii)$ If $f'$ is a classical valuation of $DQ(8,2)$ such that $O_{f'} \not\subseteq \mathbb{I}_4$, then $O_f$ is a projective set, and every quad of $\mathbb{I}_4$ which is special with respect to $f$ is a grid.*

*$(iii)$ If $f'$ is a valuation of $DQ(8,2)$ which is the extension of an ovoidal valuation in a QUAD $Q$ of $DQ(8,2)$ which is also a quad of $\mathbb{I}_4$, then $f$ is the extension of an ovoidal valuation of $Q$ and $O_f = O_{f'}$.*

*$(iv)$ If $f'$ is a valuation of $DQ(8,2)$ which is the extension of an ovoidal valuation in a QUAD $Q$ of $DQ(8,2)$ which is not a quad of $\mathbb{I}_4$, then $O_f = O_{f'} \cap \mathbb{I}_4$ is a set of 3 points of $Q$.*

*$(v)$ If $f'$ is an SDPS-valuation of $DQ(8,2)$, then $|O_f| \geq 10$ and there exists a $W(2)$-quad in $\mathbb{I}_4$ which is special with respect to $f$.*

**Proof.** Claims (i), (ii), (iii) and (iv) are obvious. We now show claim (v). Let $H_1$ and $H_2$ be two disjoint hexes of $\mathbb{I}_4$ isomorphic to $DQ(6,2)$. Then $H_1$ and $H_2$ are also HEXES of $DQ(8,2)$. By the structure of SDPS-sets, see Lemma 8 of [9], $H_1 \cap O_{f'}$ and $H_2 \cap O_{f'}$ are ovoids in QUADS. Claim (v) follows from the fact that $(H_1 \cap O_{f'}) \cup (H_2 \cap O_{f'}) \subseteq O_f$. ■

**Lemma 4.2** *If $f$ is a valuation of $\mathbb{I}_4$, then $\mathrm{d}(x_1,x_2)$ is even for every two points $x_1$ and $x_2$ of $O_f$.*

**Proof.** By property $(V_2)$ in the definition of valuation, $\mathrm{d}(x_1,x_2) \neq 1$. Suppose $\mathrm{d}(x_1,x_3) = 3$, let $H$ denote the unique hex through $x_1$ and $x_2$ and let $f_H$ denote the valuation of $H$ induced by $f$. Then $x_1, x_2 \in O_{f_H}$. The hex $H$ is isomorphic to either $DQ(6,2)$ or $\mathbb{I}_3$. But neither $DQ(6,2)$ nor $\mathbb{I}_3$ has a valuation for which there exist two points with value 0 at distance 3 from each other. Hence, $\mathrm{d}(x_1,x_2) \in \{0,2,4\}$. ■

If $f$ is a valuation of $\mathbb{I}_4$, then we will consider the following two cases:

(I) any two distinct points of $O_f$ lie at distance 2 from each other;

(II) there exist two points in $O_f$ at distance 4 from each other.

## 4.2 Treatment of Case I

In this subsection, we suppose that $f$ is a valuation of $\mathbb{I}_4$ such that any two distinct points of $O_f$ lie at distance 2 from each other.

**Lemma 4.3** *If $|O_f| = 1$, then the following holds:*
(*i*) *$f$ is a classical valuation;*
(*ii*) *there exists a unique valuation $f'$ in $DQ(8,2)$ inducing $f$;*
(*iii*) *the valuation $f'$ is classical and $O_{f'} = O_f$.*

**Proof.** (i) Put $O_f = \{x\}$. Let $H$ denote an arbitrary hex through $x$ and let $f_H$ denote the valuation induced in $H$. Then $O_{f_H} = \{x\}$. Regardless of whether $H \cong DQ(6,2)$ or $H \cong \mathbb{I}_3$, $f_H$ is classical. It follows that $f(y) = \mathrm{d}(x,y)$ for every point $y$ at distance at most 3 from $x$. By Proposition 2.3 (d), $f$ is classical or semi-classical. Suppose $f$ is semi-classical. Let $y$ be a point at distance 1 from $x$ and let $H$ be a hex through $y$ not containing $x$. Then the valuation induced in $H$ is semi- classical. But this is impossible, because neither $DQ(6,2)$ nor $\mathbb{I}_3$ has semi-classical valuations.

(ii) + (iii) Obviously, $f$ is induced by the classical valuation $f_x$ of $DQ(8,2)$ for which $x$ is the unique point with value 0. By Lemma 4.1, $f_x$ is the unique valuation of $DQ(8,2)$ inducing $f$. ∎

**Lemma 4.4** *Suppose that the maximal distance between two points of $O_f$ is equal to 2 and that there exists a special $W(2)$-quad $Q$. Then:*
(*i*) *$f$ is the extension of an ovoidal valuation in $Q$;*
(*ii*) *there exists a unique valuation $f'$ in $DQ(8,2)$ inducing $f$;*
(*iii*) *the valuation $f'$ is the extension of an ovoidal valuation in $Q$ and $O_{f'} = O_f$.*

**Proof.** (i) We first prove that $Q \cap O_f = O_f$. Suppose the contrary. Then there exists a point $x \in O_f \setminus (O_f \cap Q)$. Since $\mathrm{d}(x,y) = \mathrm{d}(x,\pi_Q(x)) + \mathrm{d}(\pi_Q(x),y) = 2$ for every point $y$ of $O_f \cap Q$, every point of $O_f \cap Q$ has distance at most 1 from $\pi_Q(x)$, a contradiction. Hence $Q \cap O_f = O_f$.

If $x$ is a point of $\mathbb{I}_4$ such that $\mathrm{d}(x,Q) \leq 1$ or ($\mathrm{d}(x,Q) = 2$ and $\pi_Q(x) \in O_f$), then $\mathrm{d}(x,O_f) \leq 2$ and hence $f(x) = \mathrm{d}(x,O_f)$ by Proposition 2.3 (b). Suppose now that $x$ is a point of $\mathbb{I}_4$ such that $\mathrm{d}(x,Q) = 2$ and $\pi_Q(x) \notin O_f$. Let $y$ be a point of $O_f$ collinear with $\pi_Q(x)$, let $H$ be the hex $\langle x,y\rangle$ and let $f_H$ be the valuation of $H$ induced by $f$. Then $O_{f_H} = \{y\}$ since $H \cap O_f = \{y\}$. Hence, $f_H$ is a classical valuation. It follows that $f(x) = 3 = \mathrm{d}(x,O_f)$.

Summarizing, we have $f(x) = \mathrm{d}(x,O_f) = \mathrm{d}(x,\pi_Q(x)) + \mathrm{d}(\pi_Q(x),O_f)$ for every point $x$ of $\mathbb{I}_4$. It follows that $f$ is the extension of the ovoidal valuation of $Q$ determined by the ovoid $O_f$.

(ii) + (iii) If $f'$ is the valuation of $DQ(8,2)$ which is the extension of the ovoidal valuation of the QUAD $Q$ determined by the ovoid $O_f$, then $f'$ induces the valuation $f$ of $\mathbb{I}_4$. By Lemma 4.1, $f'$ is the unique valuation of $DQ(8,2)$ inducing $f$. ∎

**Lemma 4.5** *Suppose that the maximal distance between two points of $O_f$ is equal to* 2 *and that no special $W(2)$-quads exist. Then $O_f$ is either a projective set or an ovoid in a grid-quad.*

**Proof.** Let $x$ denote an arbitrary point of $O_f$. The incidence structure $P$ with points the grid-quads through $x$ and with lines the $\mathbb{I}_3$- hexes through $x$ (natural incidence) is isomorphic to the Fano- plane. Let $X$ denote the set of all special grid-quads through $x$.

**Step 1:** $X$ is a subspace of $P$.
Proof. Let $Q_1$ and $Q_2$ denote two distinct special grid- quads through $x$ and let $Q_3$ denote the third grid-quad through $x$ such that $\{Q_1, Q_2, Q_3\}$ is a line of $P$. Let $H$ denote the unique $\mathbb{I}_3$-hex containing $Q_1$, $Q_2$, $Q_3$, and let $f_H$ be the valuation of $H$ induced by $f$. Since $Q_1$ and $Q_2$ are special quads with respect to $f_H$, $f_H$ is a valuation of Fano- type. So, also $Q_3$ is special with respect to $f_H$ and hence also with respect to $f$.

**Step 2:** $X$ is not a line of $P$.
Proof. Suppose the contrary and let $H$ denote the unique $\mathbb{I}_3$-hex containing all points of $O_f$. Let $H'$ denote a $DQ(6,2)$- hex of $\mathbb{I}_4$ disjoint with $H$ and let $f_{H'}$ denote the valuation of $H'$ induced by $f$. Every point of $\pi_{H'}(O_f)$ has $f$-value 1 and hence belongs to $O_{f_{H'}}$. It follows that $|O_{f_{H'}}| \geq |\pi_{H'}(O_f)| = |O_f| = 7$, a contradiction, because a valuation of $DQ(6,2)$ has at most five points with value 0. This proves the claim.

From Steps 1 and 2, it follows that $X$ is either a point of $P$ or the whole of $P$. In the first case, $O_f$ is an ovoid in a grid-quad. In the second case, $O_f$ must be a projective set since every two points of $O_f$ lie at distance 2 from each other. ∎

**Lemma 4.6** *If $X$ is a projective set of $\mathbb{I}_4$, then there exists a unique valuation $g$ of $\mathbb{I}_4$ such that $X = O_g$. Moreover, $g$ is induced by a unique valuation of $DQ(8,2)$.*

**Proof.** (1) Suppose $g$ is a valuation of $\mathbb{I}_4$ such that $X = O_g$. Let $x$ denote an arbitrary point of $\mathbb{I}_4$ and let $H$ denote a $DQ(6,2)$-hex through $x$. Then $H$ has a unique point in common with $O_g$. Hence, the valuation induced in $H$ by $g$ is classical. It follows that $g(x) = \mathrm{d}(x, X \cap H)$. This proves that there exists at most one valuation $g$ of $\mathbb{I}_4$ such that $O_g = X$.

(2) Let $x$ denote the unique point of $DQ(8,2) \setminus \mathbb{I}_4$ such that $X$ is the set of all points of $\mathbb{I}_4$ collinear with $x$. Let $f_x$ denote the classical valuation of $DQ(8,2)$ such that $f_x(x) = 0$ and let $g$ denote the valuation of $\mathbb{I}_4$ induced by $f_x$. Then $O_g = X$.

(3) By Lemma 4.1, $f_x$ is the unique valuation of $DQ(8,2)$ inducing a valuation $g$ of $\mathbb{I}_4$ with $O_g = X$.

The lemma now follows from (1), (2) and (3). ∎

**Lemma 4.7** *Let $G$ be a grid-quad of the near octagon $\mathbb{I}_4$, let $x$ be a point of $G$ and let $O$ be an ovoid of $G$. Then the following holds:*

($i$) *there are* 128 *points* $u$ *in* $\Gamma_{2,C}(G)$ *for which* $\pi_G(u) = x$*;*

($ii$) *there are* 96 *points* $u$ *in* $\Gamma_{3,O}(G)$ *for which* $\Gamma_3(u) \cap G = O$*.*

**Proof.** Let $\overline{G}$ denote the unique QUAD of $DQ(8,2)$ containing $G$. Let $y$ denote the unique point of $\overline{G} \setminus G$ for which $y^\perp \cap G = O$. The lines and QUADS of $DQ(8,2)$ through any given point of $DQ(8,2)$ define a projective space isomorphic to PG$(3,2)$. So, every point of $\overline{G}$ is contained in precisely 16 QUADS which intersect $\overline{G}$ in only one point.

The 16 QUADS through $x$ intersecting $\overline{G}$ in only the point $x$ are also quads of $\mathbb{I}_4$, since $\overline{G}$ contains the unique line of $DQ(8,2)$ through $x$ not contained in $\mathbb{I}_4$. In each such QUAD, there are 8 points at distance 2 from $x$. If $u$ is one of these 8 points, then $u \in \Gamma_{2,C}(G)$ and $\pi_G(u) = x$. Conversely, if $u$ is a point of $\Gamma_{2,C}(G)$ such that $\pi_G(u) = x$, then $u$ is contained in one of the 16 QUADS through $x$ intersecting $\overline{G}$ in only the point $x$. Hence, there are $16 \cdot 8 = 128$ points $u \in \Gamma_{2,C}(G)$ for which $\pi_G(u) = x$.

Each of the 16 QUADS through $y$ intersecting $\overline{G}$ in only the point $y$ intersects $\mathbb{I}_4$ in a grid. In each such QUAD, there are 6 points of $\mathbb{I}_4$ at distance 2 from $y$. If $u$ is one of these 6 points, then $u \in \Gamma_{3,O}(G)$ and $\Gamma_3(u) \cap G = O$. Conversely, if $u$ is a point of $\Gamma_{3,O}(G)$ such that $\Gamma_3(u) \cap G = O$, then $u$ is contained in one of the 16 QUADS through $y$ intersecting $\overline{G}$ only in the point $y$. This proves that there are $16 \cdot 6 = 96$ points $u \in \Gamma_{3,O}(G)$ for which $\Gamma_3(u) \cap G = O$. ∎

**Lemma 4.8** *If* $X$ *is an ovoid in a grid-quad* $G$ *of* $\mathbb{I}_4$*, then there exists a unique valuation* $g$ *of* $\mathbb{I}_4$ *such that* $X = O_g$*. Moreover,* $g$ *is induced by a unique valuation of* $DQ(8,2)$*.*

**Proof.** (1) Suppose $g$ is a valuation of $\mathbb{I}_4$ such that $X = O_g$. We will count the number of points with a certain $g$-value. Notice that there are no points with $g$-value 4 by Proposition 2.3 (c).

Consider first the points of $G$. In $G$, there are 3 points with value 0 and 6 points with value 1.

Consider the set of points $\Gamma_1(G)$. Since there are 14 lines through each point of $\mathbb{I}_4$, $|\Gamma_1(G)| = 9 \cdot (14-2) \cdot 2 = 216$. If $x \in \Gamma_1(G)$, then $\mathrm{d}(x, O_g) \leq 2$ and hence $g(x) = \mathrm{d}(x, O_g)$ by Proposition 2.3 (b). It follows that there are 72 points in $\Gamma_1(G)$ with value 1 and 144 points in $\Gamma_1(G)$ with value 2.

Now, consider the set $\Gamma_{2,O}(G)$. If $x \in \Gamma_{2,O}(G)$, then $\langle x, G\rangle$ is an $\mathbb{I}_3$-hex. Now, there are three $\mathbb{I}_3$-hexes through $G$ and the valuation induced in each such $\mathbb{I}_3$-hex is of grid-type. By Lemma 3.2, in each $\mathbb{I}_3$-hex through $G$ there are 8 points at distance 2 from $G$ with value 1 and 16 points at distance 2 from $G$ with value 2. Hence, in $\Gamma_{2,O}(G)$ there are 24 points with value 1 and 48 points with value 2.

Now, consider the set $\Gamma_{2,C}(G)$. If $x$ is one of the $128 \cdot 3 = 384$ points of $\Gamma_{2,C}(G)$ such that $\pi_G(x) \in O_g$ (see Lemma 4.7 (i)), then $\mathrm{d}(x, O_g) = 2$ and hence $g(x) = 2$ by Proposition 2.3 (b). Suppose $x$ is one of the $128 \cdot 6 = 768$ points of $\Gamma_{2,C}(G)$ for which $\pi_G(x) \notin O_g$. Let $y$ be a point of $O_g$ collinear with $\pi_G(x)$. Then the valuation induced in the hex $\langle x, y\rangle$ is classical since $O_g \cap \langle x, y\rangle = \{y\}$. It follows that $g(x) = 3$.

Finally, suppose that $x \in \Gamma_3(G)$. Then $\Gamma_3(x) \cap G$ is an ovoid of $G$. By Lemma 4.7 (ii), there are $4 \cdot 96 = 384$ points $x$ in $\Gamma_3(G)$ for which $\Gamma_3(x) \cap G$ is an ovoid meeting $O_g$ and there are $96 \cdot 2 = 192$ points in $\Gamma_3(x) \cap G$ for which $\Gamma_3(x) \cap G$ is an ovoid of $G$ disjoint from $O_g$.

Suppose first that $x \in \Gamma_3(G)$ such that there exists a point $y \in O_g \cap \Gamma_3(x)$. Then the valuation induced in $\langle x, y\rangle$ is classical since $O_g \cap \langle x, y\rangle = \{y\}$. It follows that $g(x) = 3$.

Now, suppose $\Gamma_3(x) \cap O_g = \emptyset$. We show that $x$ cannot have value 1. Suppose the contrary and let $y$ denote a point of $\Gamma_3(x) \cap G$. Then the hex $\langle x, y\rangle$ does not contain points of $O_g$. Since $g(y) = g(x) = 1$, the valuation induced in $\langle x, y\rangle$ has two points with value 0 at distance 3 from each other, which is impossible. Hence, $x$ has value 2 or 3. Suppose that among the 192 points $x$ of $\Gamma_3(G)$ for which $\Gamma_3(x) \cap G \cap O_g = \emptyset$, there are $\alpha$ points with value 3 and $192 - \alpha$ points with value 2.

Now, let $m_i$, $i \in \{0, 1, 2, 3\}$, denote the total number of points with value $i$. Summarizing what has been said before, we can conclude that $m_0 = 3$, $m_1 = 102$, $m_2 = 768 - \alpha$ and $m_3 = 1152 + \alpha$. By Proposition 2.3 (e), $m_0 - \frac{m_1}{2} + \frac{m_2}{4} - \frac{m_3}{8} = -\frac{3\alpha}{8} = 0$. So, $\alpha = 0$ and the valuation is completely determined by the set $X$, i.e., there exists at most one valuation $g$ of $\mathbb{I}_4$ for which $O_g = X$.

(2) Let $\overline{G}$ denote the QUAD of $DQ(8, 2)$ containing $G$, let $O$ denote the unique ovoid of $\overline{G}$ containing $X$ and let $\overline{f}$ denote the valuation of $DQ(8, 2)$ which arises as extension of the ovoidal valuation of $\overline{G}$ with corresponding ovoid $O$. Let $g$ denote the valuation of $\mathbb{I}_4$ induced by $\overline{f}$. Then $O_g = X$.

(3) By Lemma 4.1, it is also clear that $\overline{f}$ is the unique valuation of $DQ(8, 2)$ inducing a valuation $g$ in $\mathbb{I}_4$ such that $O_g = X$.

The lemma now follows from (1), (2) and (3). ∎

## 4.3 Treatment of Case II

In this subsection, we suppose that $f$ is a valuation of $\mathbb{I}_4$ containing two points of $O_f$ at distance 4 from each other.

**Lemma 4.9** *Let $x_1$ and $x_2$ be two points of $O_f$ at distance 4 from each other. Then through $x_1$, there are 4 special $W(2)$-quads and a unique special grid-quad. These special quads partition the set of lines through $x_1$.*

**Proof.** There are 14 hexes through $x_1$ containing a point of $\Gamma_1(x_2)$. Each of the 8 $DQ(6, 2)$-hexes through $x_1$ is classical in $\mathbb{I}_4$ and hence contains a point of $\Gamma_1(x_2)$. So, from the 7 $\mathbb{I}_3$- hexes through $x_1$ there are 6 which contain a point of $\Gamma_1(x_2)$ and a unique other one which does not contain a point of $\Gamma_1(x_2)$.

Let $H$ be an arbitrary $DQ(6, 2)$-hex through $x_1$. Then $f(x_1) = 0$, $f(\pi_H(x_2)) = 1$ and $d(x_1, \pi_H(x_2)) = 3$. It follows that the valuation induced in $H$ is the extension of an ovoidal valuation in a quad of $H$. Hence, $H$ contains a unique special $W(2)$-quad. Conversely, every special $W(2)$-quad is contained in precisely two $DQ(6, 2)$-hexes. As a consequence, there are precisely $\frac{8}{2} = 4$ special $W(2)$-quads through $x_1$. By Proposition 2.3 (a), these four special quads cover twelve lines through $x_1$. Let $L_1$ and $L_2$ denote the remaining two lines of $\mathbb{I}_4$ through $x_1$.

Let $H$ be one of the six $\mathbb{I}_3$ hexes through $x_1$ containing a point collinear with $x_2$. Since $f(x_1) = 0$, $f(\pi_H(x_2)) = 1$ and $\mathrm{d}(x_1, \pi_H(x_2)) = 3$, the valuation $f_H$ induced in $H$ cannot be classical. Suppose $f_H$ is a valuation of Fano-type. Then there exists a special grid-quad through $x_1$ meeting a special $W(2)$- quad in a line, contradicting Proposition 2.3 (a). Hence, $f_H$ is either of extended type or of grid-type.

As every special $W(2)$-quad through $x_1$ is contained in a unique $\mathbb{I}_3$-hex, there are precisely four $\mathbb{I}_3$-hexes $H$ through $x_1$ meeting $\Gamma_1(x_2)$ for which $f_H$ is of extended type. Hence, there exists an $\mathbb{I}_3$-hex $H^*$ through $x_1$ meeting $\Gamma_1(x_2)$ for which $f_{H^*}$ is a valuation of grid-type. Let $Q^*$ denote the unique special quad of $H^*$. By Proposition 2.3 (a), the grid $Q^*$ cannot intersect any of the special $W(2)$-quads through $x_1$ in a line. Hence, $Q^*$ coincides with the quad $\langle L_1, L_2 \rangle$. This proves the lemma. ∎

**Lemma 4.10** *For every point $y_1$ of $O_f$, there exists a point $y_2 \in O_f$ at distance 4 from $y_1$.*

**Proof.** Suppose the contrary. Let $x_1$ and $x_2$ denote two points of $O_f$ at distance 4 from each other. We necessarily have $\mathrm{d}(y_1, x_1) = 2$ and $\mathrm{d}(y_1, x_2) = 2$. Moreover the quads $\langle y_1, x_1 \rangle$ and $\langle y_1, x_2 \rangle$ are different. By Lemma 4.9, there exists a special $W(2)$-quad $Q$ through $x_1$ different from $\langle y_1, x_1 \rangle$. Let $z$ denote a point of $O_f \cap Q$ different from $x_1$. Since the quads $Q$ and $\langle y_1, x_1 \rangle$ do not intersect in a line, the point $z$ has distance 2 from $\langle y_1, x_1 \rangle$. If $z$ is ovoidal with respect to $\langle y_1, x_1 \rangle$, then $\langle y_1, x_1, z \rangle$ is a hex containing a $W(2)$-quad which is not classical in $\langle y_1, x_1, z \rangle$ by Proposition 1.1 ($Q$ and $\langle y_1, x_1 \rangle$ do not intersect in a line). This is impossible since every $W(2)$- quad is classical in $\langle y_1, x_1, z \rangle$. It follows that $z$ is classical with respect to $\langle y_1, x_1 \rangle$ and that $\mathrm{d}(z, y_1) = \mathrm{d}(z, x_1) + \mathrm{d}(x_1, y_1) = 4$. This proves the lemma. ∎

From Lemmas 4.9 and 4.10, it readily follows:

**Corollary 4.11** *Every point of $O_f$ is contained in 4 special $W(2)$-quads and a unique special grid-quad.* ∎

**Lemma 4.12** *If $Q$ is a special $W(2)$-quad and if $x$ is a point of $O_f$ not contained in $Q$, then $d(x, Q) = 2$ and $\pi_Q(x) \in O_f$.*

**Proof.** If $\mathrm{d}(x, Q) = 1$, then $\pi_Q(x) \notin O_f$ and there exists a point $y \in Q \cap O_f$ at distance 2 from $\pi_Q(x)$. Then $\mathrm{d}(x, y) = 3$, contradicting Lemma 4.2. Hence, $\mathrm{d}(x, Q) = 2$. If $\pi_Q(x) \notin O_f$, then there exists a point $y \in O_f \cap Q$ collinear with $\pi_Q(x)$. Then $\mathrm{d}(x, y) = 3$, again contradicting Lemma 4.2. This proves the lemma. ∎

**Lemma 4.13** *If $Q$ is a special grid-quad and if $x$ is a point of $O_f$ not contained in $Q$, then precisely one of the following cases occurs:*

- $(i)$ *there exists a unique point in $Q \cap O_f$ at distance 2 from $x$ and the remaining two points in $Q \cap O_f$ have distance 4 from $x$;*

(*ii*) *all points of* $Q \cap O_f$ *have distance* 4 *from* $x$.

**Proof.** By Lemma 4.2, $\mathrm{d}(x, y) \in \{2, 4\}$ for every point $y$ of $Q \cap O_f$. As in Lemma 4.12 one shows that $\mathrm{d}(x, Q) \neq 1$. So, $\mathrm{d}(x, Q) \geq 2$. Suppose that there exist two points $y_1, y_2 \in Q \cap O_f$ at distance 2 from $x$. Then $\langle x, Q\rangle$ is a hex and the two special quads $\langle x, y_1\rangle$ and $\langle x, y_2\rangle$ are grids since they intersect $Q$ in only points. (Recall Proposition 1.1 and the fact that every $W(2)$-quad is classical in $\langle x, Q\rangle$.) So, the point $x$ is contained in two special grid-quads which contradicts Corollary 4.11. The lemma now follows. ■

**Lemma 4.14** *It holds* $|O_f| = 75$.

**Proof.** Let $Q$ denote a special $W(2)$-quad, let $Q \cap O_f = \{x_1, \ldots, x_5\}$. For every $i \in \{1, 2, 3, 4, 5\}$, let $G_i$ denote the unique special grid-quad through $x_i$ and let $Q_1^{(i)}$, $Q_2^{(i)}$ and $Q_3^{(i)}$ denote the three special $W(2)$-quads through $x_i$ different from $Q$. By Lemma 4.12, every point of $O_f \setminus Q$ is contained in precisely one of the quads $Q_j^{(i)}$, $G_i$ ($i \in \{1, 2, 3, 4, 5\}$, $j \in \{1, 2, 3\}$). It follows that $|O_f| = 5 + 5 \cdot 2 + 15 \cdot 4 = 75$. ■

From Corollary 4.11 and Lemma 4.14, it readily follows:

**Corollary 4.15** *There are* $\frac{75 \cdot 4}{5} = 60$ *special* $W(2)$*-quads and* $\frac{75 \cdot 1}{3} = 25$ *special grid-quads.*

Now, for every grid-quad $G$ of $\mathbb{I}_4$, let $\overline{G}$ denote the unique QUAD of $DQ(8, 2)$ containing $G$. For every special grid-quad $G$, let $O_G$ denote the unique ovoid of the QUAD $\overline{G}$ containing $G \cap O_f$. Now, let $\overline{O_f}$ denote the union of all sets $O_G$, where $G$ is a special grid-quad. Notice that $O_f \subseteq \overline{O_f}$ by Corollary 4.11.

**Lemma 4.16** *If* $x_1$ *and* $x_2$ *are points of* $\overline{O_f}$, *then* $\mathrm{d}(x_1, x_2)$ *is even.*

**Proof.** We distinguish the following cases:

**Case I:** $x_1, x_2 \in O_f$.
The claim has already been shown in Lemma 4.2.

**Case II:** ($x_1 \in O_f, x_2 \in \overline{O_f} \setminus O_f$) or ($x_2 \in O_f, x_1 \in \overline{O_f} \setminus O_f$).
By symmetry, we only need to consider the case $x_1 \in O_f$, $x_2 \in \overline{O_f} \setminus O_f$. Let $G$ denote a special grid-quad such that:

(i) $x_2 \in Q := \overline{G}$;

(ii) for every $i \in \{1, 2, 3\}$, $\mathrm{d}(x_2, a_i) = 2$ where $\{a_1, a_2, a_3\} = G \cap O_f$.

Let $O_G$ denote the unique ovoid of $Q$ containing the points $a_1$, $a_2$ and $a_3$. Then $O_G = \{a_1, a_2, a_3, x_2, x_2'\}$ for a certain point $x_2'$ of $Q$. We distinguish three cases (see Lemma 4.13):

(a) $x_1 \in \{a_1, a_2, a_3\}$. Then $\mathrm{d}(x_1, x_2) = 2$.

(b) $\mathrm{d}(x_1, a_i) = 2$ for a certain $i \in \{1, 2, 3\}$ and $\mathrm{d}(x_1, a_j) = 4$ for all $j \in \{1, 2, 3\} \setminus \{i\}$. Then $\mathrm{d}(x_1, Q) = 2$ and $\pi_Q(x_1) = a_i$. It follows that $\mathrm{d}(x_1, x_2) = 4$.

(c) $\mathrm{d}(x_1, a_i) = 4$ for all $i \in \{1, 2, 3\}$. Then $\mathrm{d}(x_1, Q) = 2$ and $\mathrm{d}(\pi_Q(x_1), a_i) = 2$ for all $i \in \{1, 2, 3\}$. Hence, either $\pi_Q(x_1) = x_2$ or $\pi_Q(x_1) = x_2'$. In the first case, $\mathrm{d}(x_1, x_2) = 2$ and in the latter case $\mathrm{d}(x_1, x_2) = 4$.

**Case III:** $x_1, x_2 \in \overline{O_f} \setminus O_f$.
Let $G_1$ and $G_2$ denote two special grid-quads such that:

(i) $x_i \in Q_i := \overline{G_i}$ for every $i \in \{1, 2\}$;

(ii) for every $i \in \{1, 2, 3\}$, $x_i$ belongs to the ovoid $O_i$ of $Q_i$ containing $G_i \cap O_f = \{a_1^{(i)}, a_2^{(i)}, a_3^{(i)}\}$.

Put $O_i := \{x_i, x_i', a_1^{(i)}, a_2^{(i)}, a_3^{(i)}\}$ for every $i \in \{1, 2\}$.

Suppose $\mathrm{d}(x_2, Q_1) \in \{0, 2\}$. Since $\mathrm{d}(x_2, a_i^{(1)})$ is even for all $i \in \{1, 2, 3\}$ (see Case II), $\mathrm{d}(\pi_{Q_1}(x_2), a_i^{(1)}) \in \{0, 2\}$ for all $i \in \{1, 2, 3\}$. Hence, $\mathrm{d}(\pi_{Q_1}(x_2), x_1) \in \{0, 2\}$ and $\mathrm{d}(x_1, x_2) \in \{0, 2, 4\}$. In a similar way, one shows that $\mathrm{d}(x_1, x_2) \in \{0, 2, 4\}$ if $\mathrm{d}(x_1, Q_2) \in \{0, 2\}$.

So, suppose $\mathrm{d}(x_1, Q_2) = 1$ and $\mathrm{d}(x_2, Q_1) = 1$. If $\mathrm{d}(x_1, x_2) \in \{2, 4\}$, then we are done. Suppose $\mathrm{d}(x_1, x_2) = 1$. Then $\mathrm{d}(x_1, a_i^{(2)}) = \mathrm{d}(x_1, x_2) + \mathrm{d}(x_2, a_i^{(2)})$ is odd for all $i \in \{1, 2, 3\}$, contradicting Case II. So, suppose $\mathrm{d}(x_1, x_2) = 3$.

Suppose the quads $Q_1$ and $Q_2$ intersect in a line $L$. The line $L$ contains at most 1 point of $G_2 \cap O_f$. So, without loss of generality, we may suppose that $a_1^{(2)} \notin L$. By cases (I) and (II), every point of $O_1$ is collinear with $\pi_{Q_1}(a_1^{(2)})$, a contradiction. So, $Q_1$ and $Q_2$ are two disjoint QUADS. Now, $x_1$ and $\pi_{Q_1}(x_2)$ are two points of $Q_1$ at distance 2 from each other collinear with the respective points $\pi_{Q_2}(x_1)$ and $x_2$ of $Q_2$. It follows that $Q_1$ and $Q_2$ are two QUADS at distance 1 from each other.

Now, $\mathrm{d}(a_1^{(1)}, a_i^{(2)})$ is even for all $i \in \{1, 2, 3\}$. It follows that $\pi_{Q_2}(a_1^{(1)})$ is the unique point of $Q_2$ collinear with $a_1^{(2)}$, $a_2^{(2)}$ and $a_3^{(2)}$. Similarly, $\pi_{Q_2}(a_2^{(1)})$ and $\pi_{Q_2}(a_3^{(1)})$ must coincide with the unique point of $Q_2$ collinear with $a_1^{(2)}$, $a_2^{(2)}$ and $a_3^{(2)}$. From $\pi_{Q_2}(a_1^{(1)}) = \pi_{Q_2}(a_2^{(1)}) = \pi_{Q_2}(a_3^{(1)})$, it follows that $a_1^{(1)} = a_2^{(1)} = a_3^{(1)}$, a contradiction. ∎

Let $\Omega$ denote the set of 85 QUADS of $DQ(8, 2)$ consisting of all 60 special $W(2)$-quads and all 25 QUADS $\overline{G}$, where $G$ is some special grid-quad (see Corollary 4.15).

**Lemma 4.17** *If $Q \in \Omega$, then $Q \cap \overline{O_f}$ is an ovoid of $Q$.*

**Proof.** Obviously, this holds if $Q$ is a special $W(2)$-quad. So, suppose that $Q = \overline{G}$ for some special grid-quad $G$. Let $O$ denote the unique ovoid of $Q$ containing $G \cap O_f$. Then obviously, $O \subseteq Q \cap \overline{O_f}$. By Lemma 4.16, $Q \cap \overline{O_f}$ cannot contain points outside $O$. ∎

**Lemma 4.18** *No two QUADS of $\Omega$ intersect in a line.*

**Proof.** Suppose that $Q_1$ and $Q_2$ are two QUADS of $\Omega$ intersecting in a line $L$. Let $y_1$ and $y_2$ denote the two points of $L$ not contained in $\overline{O_f}$. Let $x_i$, $i \in \{1,2\}$, denote a point of $(Q_i \cap \overline{O_f}) \setminus L$ collinear with $y_i$. Then $\mathrm{d}(x_1, x_2) = 3$, contradicting Lemma 4.16. ∎

**Corollary 4.19** *Every point of $\overline{O_f} \setminus O_f$ is contained in at most five QUADS of $\Omega$.*

**Proof.** This follows from Lemma 4.18 and the fact that there are precisely 15 lines through every point of $DQ(8,2)$. ∎

**Lemma 4.20** *Let $Q$ denote a QUAD of $\Omega$ and let $x$ be a point of $\overline{O_f}$ not contained in $Q$. Then $d(x,Q) = 2$ and $\pi_Q(x) \in \overline{O_f}$.*

**Proof.** The proof is similar to the proof of Lemma 4.12. ∎

**Lemma 4.21** *It holds $|\overline{O_f}| = 85$.*

**Proof.** Let $Q$ denote a special $W(2)$-quad. Put $Q \cap O_f = \{x_1, x_2, x_3, x_4, x_5\}$. For every $i \in \{1,2,3,4,5\}$, let $Q_1^{(i)}$, $Q_2^{(i)}$, $Q_3^{(i)}$, $Q_4^{(i)}$ denote the four QUADS of $\Omega$ through $x_i$ different from $Q$ (see Corollary 4.11). By Lemma 4.18, the QUADS $Q$, $Q_1^{(i)}$, $Q_2^{(i)}$, $Q_3^{(i)}$, $Q_4^{(i)}$ partition the set of lines through $x_i$. By Lemma 4.20, the 20 QUADS $Q_j^{(i)}$, $i \in \{1,2,3,4,5\}$ and $j \in \{1,2,3,4\}$, give rise to 80 distinct points of $\overline{O_f}$ not contained in $Q$. Together with the points of $Q \cap \overline{O_f}$ this gives rise to 85 points of $\overline{O_f}$. We will show that these are all the points of $\overline{O_f}$. Suppose that $x$ is a point of $\overline{O_f}$ which we have not yet counted. Without loss of generality, we may suppose that $x_1$ is the unique point of $Q$ at distance 2 from $x$ and that the QUAD $Q_1^{(1)}$ intersects $\langle x, x_1 \rangle$ in a line. It is easily seen that there exists a point in $Q_1^{(1)} \cap \overline{O_f}$ at distance 3 from $x$, which is impossible by Lemma 4.16. So, $|\overline{O_f}| = 85$. ∎

**Lemma 4.22** *Every point $x$ of $\overline{O_f}$ is contained in precisely five QUADS of $\Omega$. These five QUADS partition the set of lines through $x$.*

**Proof.** There are 25 QUADS of $\Omega$ which are of the form $\overline{G}$, where $G$ is a special grid-quad. Each such quad contains two points of $\overline{O_f} \setminus O_f$. On the other hand, $|\overline{O_f} \setminus O_f| = 10$ and each point of $\overline{O_f} \setminus O_f$ is contained in at most 5 QUADS of $\Omega$ by Corollary 4.19. Since $25 \cdot 2 = 10 \cdot 5$, it readily follows that every point of $\overline{O_f} \setminus O_f$ is contained in precisely five QUADS of $\Omega$. By Corollary 4.11, also every point of $O_f$ is contained in five quads of $\Omega$.

By Lemma 4.18, the five QUADS through a point $x$ of $\overline{O_f}$ partition the set of lines through $x$. ∎

**Lemma 4.23** *The incidence structure $\mathcal{Q}$ with point set $\overline{O_f}$ and with line set $\Omega$ is isomorphic to the symplectic generalized quadrangle $W(4)$.*

**Proof.** We will show that $\mathcal{Q}$ is a generalized quadrangle of order 4. The lemma then follows from a well-known result of Payne ([13], [14]) with a gap filled by Tits (see [15]) stating that $W(4)$ is the unique generalized quadrangle of order 4.

By Lemma 4.17, every line of $\mathcal{Q}$ contains five points and by Lemma 4.22, every point of $\mathcal{Q}$ is incident with precisely five lines. From Lemma 4.20, it readily follows that for every line $L$ of $\mathcal{Q}$ and every point $x$ of $\mathcal{Q}$ not incident with $L$, there exists a unique point on $L$ collinear with $x$. So, $\mathcal{Q}$ is a generalized quadrangle of order 4. ■

**Corollary 4.24** *The set $\overline{O_f}$ is an SDPS-set of $DQ(8,2)$.*

Let $\overline{f}$ denote the valuation of $DQ(8,2)$ associated with the SDPS-set $\overline{O_f}$, i.e., $\overline{f}(x) = \mathrm{d}(x, \overline{O_f})$ for every point $x$ of $DQ(8,2)$.

**Lemma 4.25** *For every point $x$ of $\mathbb{I}_4$, $f(x) = \overline{f}(x) = d(x, \overline{O_f}) = d(x, O_f)$.*

**Proof.** Let $H$ denote a $DQ(6,2)$-hex through $x$. By properties of SDPS- sets and SDPS-valuations (see Lemmas 8 and 9 of [9]), we have (i) $\overline{f}(x) = \mathrm{d}(x, \overline{O_f}) = \mathrm{d}(x, \overline{O_f} \cap H)$ and (ii) $H \cap \overline{O_f}$ is an ovoid in a quad of $H$. Since $H \cap O_f = H \cap \overline{O_f}$, $\mathrm{d}(x, O_f) \leq 2$ and hence $f(x) = \mathrm{d}(x, O_f)$ by Proposition 2.3 (b). So, we have

$$f(x) = \mathrm{d}(x, O_f) \leq \mathrm{d}(x, O_f \cap H) = \mathrm{d}(x, \overline{O_f} \cap H) = \overline{f}(x) = \mathrm{d}(x, \overline{O_f}) \leq \mathrm{d}(x, O_f),$$

from which the lemma readily follows. ■

Clearly, the set $\overline{O_f}$ is the unique SDPS-set containing $O_f$, So, by Lemmas 4.1 and 4.25, we have

**Corollary 4.26** *Let $f$ be a valuation of $\mathbb{I}_4$ such that $O_f$ contains two points at distance 4 from each other. Then $f$ is induced by a unique valuation $f'$ of $DQ(8,2)$. Moreover, $f'$ is an SDPS-valuation.*

**Definition.** Let $x$ and $y$ be two points of $W(4)$ at distance 2 from each other. Since the pair $(x,y)$ is regular (see Payne and Thas [15] for definitions), $|\{x,y\}^{\perp}| = |\{x,y\}^{\perp\perp}| = 5$. Here, $\{x,y\}^{\perp}$ denotes the set of points of $W(4)$ collinear with $x$ and $y$, and $\{x,y\}^{\perp\perp}$ denotes the set of points of $W(4)$ collinear with every point of $\{x,y\}^{\perp}$. Let $W'(4)$ denote the incidence structure derived from $W(4)$ by removing all points of $\{x,y\}^{\perp} \cup \{x,y\}^{\perp\perp}$.

**Lemma 4.27** *It holds $G_f \cong W'(4)$.*

**Proof.** The incidence structure $G_f$ is obtained from $G_{\overline{f}} = \mathcal{Q} \cong W(4)$ by removing all points of $\overline{O_f} \setminus O_f$. Let $x_1$ denote an arbitrary point of $\overline{O_f} \setminus O_f$, let $y_1, y_2, y_3, y_4, y_5$ denote the five points of $\overline{O_f} \setminus O_f$ collinear with $x_1$ (in $G_{\overline{f}}$) and let $x_2, x_3, x_4, x_5$ denote the remaining points of $\overline{O_f} \setminus O_f$. Since $G_{\overline{f}}$ is a generalized quadrangle, $y_i \not\sim y_j$ for all $i, j \in \{1, \ldots, 5\}$ with $i \neq j$. Since every $y_j$ is collinear with five points of $\overline{O_f} \setminus O_f$, we have $x_i \sim y_j$ for all $i, j \in \{1, \ldots, 5\}$. As before, we then have $x_i \not\sim x_j$ for all $i, j \in \{1, \ldots, 5\}$ with $i \neq j$. The lemma now readily follows. ■

# 5 Isometric full embeddings of $\mathbb{I}_n$ into $DQ(2n, 2)$ and $DH(2n-1, 4)$

It is well-known that the dual polar space $DQ(2n, 2)$, $n \geq 2$, has an isometric full embedding into the dual polar space $DH(2n-1, 4)$. In De Bruyn [6], it was shown that such an embedding is unique up to isomorphism. In Section 1, we described an isometric full embedding of $\mathbb{I}_n$ into $DQ(2n, 2)$, which we refer to as the *natural embedding* of $\mathbb{I}_n$ in $DQ(2n, 2)$. Composing both isometric embeddings, we obtain an isometric full embedding of $\mathbb{I}_n$ into the dual polar space $DH(2n-1, 4)$. The natural embedding of $\mathbb{I}_n$ in $DQ(2n, 2)$ can be completely described in terms of the near $2n$-gon $\mathbb{I}_n$. For this, we need to give a description (in terms of objects of $\mathbb{I}_n$) of the points and lines of $DQ(2n, 2)$ which are not contained in $\mathbb{I}_n$. This is realized as follows.

(i) The points of $DQ(2n, 2) \setminus \mathbb{I}_n$ are in bijective correspondence with the projective sets of $\mathbb{I}_n$. If $x \in DQ(2n, 2) \setminus \mathbb{I}_n$, then $x^\perp \cap \mathbb{I}_n$ is a projective set of $\mathbb{I}_n$.

(ii) In view of (i), we need to describe the lines of $DQ(2n, 2)$ not contained in $\mathbb{I}_n$ as sets of size 3 whose elements are either points or projective sets of $\mathbb{I}_n$. The set of lines of $DQ(2n, 2)$ not contained in $\mathbb{I}_n$ are in bijective correspondence with the sets $\{x, P_1, P_2\}$, where $x$ is a point of $\mathbb{I}_n$ and where $P_1$ and $P_2$ are the two projective sets through $x$.

We will now prove Theorem 1.5. This theorem is a consequence of the uniqueness of the isometric full embedding of $DQ(2n, 2)$ into $DH(2n-1, 4)$ and the following proposition.

**Proposition 5.1** *Let $\Delta$ be one of the dual polar spaces $DQ(2n, 2)$ or $DH(2n-1, 4)$, $n \geq 2$. If $\theta$ is an isometric full embedding of $\mathbb{I}_n$ into $\Delta$, then $\theta = \theta_2 \circ \theta_1$, where $\theta_1$ is the natural embedding of $\mathbb{I}_n$ into $DQ(2n, 2)$ and $\theta_2$ is an isometric full embedding of $DQ(2n, 2)$ into $\Delta$.*

**Proof.** Obviously, the proposition holds if $n = 2$. So, we will suppose that $n \geq 3$. We will regard $\mathbb{I}_n$ as a subgeometry of $\Delta$, i.e., we will regard $\theta$ as an inclusion map. By Proposition 1.2, every convex subspace $A$ of $\mathbb{I}_n$ is contained in a unique convex subspace $\overline{A}$ of $\Delta$ of the same diameter.

**Claim 1.** *For every projective set $P$ of $\mathbb{I}_n$, there exists a unique point $x_P \in \Delta \setminus \mathbb{I}_n$ such that $x_P^\perp \cap \mathbb{I}_n = P$.*

Proof. Let $x^*$ denote an arbitrary point of $P$ and let $G_1$ and $G_2$ denote two distinct grid-quads of $\mathbb{I}_n$ through $x^*$. Then $|G_1 \cap P| = |G_2 \cap P| = 3$ and there exists a unique $\mathbb{I}_3$-hex $H$ in $\mathbb{I}_n$ containing $G_1$ and $G_2$. Since $\overline{H}$ is a hex of $\Delta$ containing the quads $\overline{G_1}$ and $\overline{G_2}$, $\overline{G_1}$ and $\overline{G_2}$ intersect in a line $L^*$. Let $x_P$ denote the unique point of $L^* \setminus \{x^*\}$ such that $x_P^\perp \cap G_1 = G_1 \cap P$. Then also $x_P^\perp \cap G_2 = G_2 \cap P$, since every point of $(G_2 \cap P) \setminus \{x^*\}$ has distance 2 from every point of $(G_1 \cap P) \setminus \{x^*\}$. Let $G_3$ denote the third grid-quad of $H$ through $x^*$. By Proposition 2.1, the map $f : H \to \mathbb{N}; y \mapsto \mathrm{d}(x_P, y) - 1$ is a valuation of $H$. Since the quads $G_1$ and $G_2$ are special with respect to $f$, the valuation must be of Fano-type. Hence, also $G_3$ is special with respect to $f$. So, $x_P^\perp \cap G_3$ is an ovoid of $G_3$ which necessarily coincides with $G_3 \cap P$, since every point of $(G_3 \cap P) \setminus \{x^*\}$ has distance 2 from any point of $(G_1 \cap P) \setminus \{x^*\}$. Now, suppose $G_4$ is a grid-quad of $\mathbb{I}_n$ through $x^*$ different from $G_1$, $G_2$ and $G_3$. Then there exists a convex suboctagon $F \cong \mathbb{I}_4$ in $\mathbb{I}_n$ containing $G_1$, $G_2$, $G_3$ and $G_4$. By Proposition 2.1, the map $f' : F \to \mathbb{N}; y \mapsto \mathrm{d}(x_P, y) - 1$ is a valuation of $F$ having $G_1$, $G_2$ and $G_3$ as special grid-quads. By the classification of the valuations of $\mathbb{I}_4$, we know that $O_{f'}$ is a projective set of $F$. Hence, $x_P^\perp \cap G_4$ is an ovoid of $G_4$ which necessarily coincides with $G_4 \cap P$, since every point of $(G_4 \cap P) \setminus \{x^*\}$ has distance 2 from every point of $(G_1 \cap P) \setminus \{x^*\}$.

By the above, we know that $P \subseteq x_P^\perp \cap \mathbb{I}_n$. We will now show that $P = x_P^\perp \cap \mathbb{I}_n$. Suppose the contrary. Then there exists a point $u \in (x_P^\perp \cap \mathbb{I}_n) \setminus P$. Since $\mathrm{d}(x^*, u) = 2$, there exists a quad $Q$ in $\mathbb{I}_n$ containing $x^*$ and $u$. Then $x_P^\perp \cap Q$ is an ovoid of $Q$. Now, the grid- quads of $\mathbb{I}_n$ through $x^*$ determine a partition of the lines of $\mathbb{I}_n$ through $x^*$. Hence, there exists a grid-quad $G$ of $\mathbb{I}_n$ through $x^*$ which intersects $Q$ in a line. Since $x_P^\perp \cap G$ is an ovoid of $G$ and $Q \cap x_P^\perp$ is an ovoid of $Q$, it is easily seen that there exists a point $v_1 \in Q \cap x_P^\perp$ and a point $v_2 \in G \cap x_P^\perp$ at distance 3 from each other. But this is impossible, since every two points of $x_P^\perp \cap \mathbb{I}_n$ lie at distance 2 from each other.

Hence, $P = x_P^\perp \cap \mathbb{I}_n$ as claimed.

**Claim 2.** *Let $x$ be a point of $\mathbb{I}_n$ and let $P_1$ and $P_2$ denote the two projective sets through $x$. Then $\{x, x_{P_1}, x_{P_2}\}$ is a line of $\Delta$.*

Proof. Let $y$ denote the point of the line $xx_{P_1}$ different from $x$ and $x_{P_1}$. For every grid-quad $G$ of $\mathbb{I}_n$ through $x$, $x_{P_1}^\perp \cap G$ and $y^\perp \cap G$ are the two ovoids of $G$ through $x$. Since $x_{P_1}^\perp \cap G = P_1 \cap G$, we have $y^\perp \cap G = P_2 \cap G$. Since this holds for every grid-quad $G$ of $\mathbb{I}_n$ through $x$, $x_{P_2} = y$. This proves the claim.

Now, consider the following substructure $\Delta'$ of $\Delta$. The points of $\Delta'$ are of two types: (i) the points of $\mathbb{I}_n$; (ii) the points $x_P$, where $P$ is a projective set of $\mathbb{I}_n$. The lines of $\Delta'$ are of two types: (i) the lines of $\mathbb{I}_n$, (ii) the lines $\{x, x_{P_1}, x_{P_2}\}$, where $x$ is a point of $\mathbb{I}_n$ and where $P_1$ and $P_2$ are the two projective sets of $\mathbb{I}_n$ through $x$.

By the discussion preceding this proposition, the incidence structure $\Delta'$ is isomorphic to $DQ(2n, 2)$. Moreover, the embedding of $\mathbb{I}_n$ in $\Delta'$ is isomorphic to the natural embedding of $\mathbb{I}_n$ in $DQ(2n, 2)$. It remains to show that $\Delta'$ is isometrically embedded in $\Delta$. By the main result of Huang [12], it suffices to show that there exist two opposite points $x_1$ and $x_2$ in $\Delta'$ which are also opposite in the geometry $\Delta$. But this holds as we can take for $x_1$ and $x_2$ two opposite points in $\mathbb{I}_n$. Then $x_1$ and $x_2$ are also opposite points in $\Delta$ as $\mathbb{I}_n$ is isometrically embedded in $\Delta$. ∎

We now take a closer look at the case $n = 4$. Suppose $F_1 := \mathbb{I}_4$ is fully and isometrically embedded in $F_2 := DQ(8, 2)$ which itself is also fully and isometrically embedded in $F_3 := DH(7, 4)$. By Proposition 2.1, every point $x$ of $F_3$ induces a valuation $f_x : F_1 \to \mathbb{N}; y \mapsto \mathrm{d}(x, y) - \mathrm{d}(x, F_1)$ of $F_1$. So, we can distinguish the points of $F_3$ by means of the type of valuation of $F_1$ they induce. From this point of view, there are five possible types of points $x \in F_3$:

(I) $O_{f_x}$ is a singleton, or equivalently, $f_x$ is a classical valuation;

(II) $O_{f_x}$ is a projective set of $F_1$;

(III) $O_{f_x}$ is an ovoid in a $W(2)$-quad of $F_1$;

(IV) $O_{f_x}$ is an ovoid in a grid-quad of $F_1$;

(V) $O_{f_x}$ is a set of 75 points.

We can make the following conclusions (recall Proposition 2.2 and Lemma 4.1):

The points of type (I) are precisely the points of $F_1$. There are precisely 2025 such points.

The points of type (II) are the points of $F_2 \setminus F_1$. There are precisely 270 such points.

The points of type (III) belong to $\Gamma_1(F_2) \cap \Gamma_1(F_1)$. If $x$ is a point of type (III), then $\Gamma_1(x) \cap F_2 = \Gamma_1(x) \cap F_1$ is an ovoid in a $W(2)$-quad of $F_2$ which is also a quad of $F_1$. There are 45360 points of type (III).

The points of type (IV) belong to $\Gamma_1(F_2) \cap \Gamma_1(F_1)$. If $x$ is a point of type (IV), then $\Gamma_1(x) \cap F_2$ is an ovoid in a $W(2)$-quad of $F_2$ and $\Gamma_1(x) \cap F_1$ is an ovoid in a grid-quad of $F_1$. There are 18900 points of type (IV).

The points of type (V) belong to $\Gamma_2(F_2) \cap \Gamma_2(F_1)$. If $x$ is a point of type (V), then $\Gamma_2(x) \cap F_2$ is a set of 85 points which carries the structure of a generalized quadrangle of order 4, and $\Gamma_2(x) \cap F_1$ is a set of 75 points which carries the structure of a generalized quadrangle of order 4 in which two orthogonal hyperbolic lines have been removed. There are 48384 points of type (V).

# References


[1] A. E. Brouwer, A. M. Cohen, J. I. Hall and H. A. Wilbrink. Near polygons and Fischer spaces. *Geom. Dedicata* 49 (1994), 349–368.

[2] A. E. Brouwer and H. A. Wilbrink. The structure of near polygons with quads. *Geom. Dedicata* 14 (1983), 145–176.

[3] P. J. Cameron. Dual polar spaces. *Geom. Dedicata* 12 (1982), 75–86.

[4] B. De Bruyn. *Near Polygons.* Frontiers in Mathematics 6, Birkhäuser, 2006.

[5] B. De Bruyn. A characterization of the SDPS- hyperplanes of dual polar spaces. *European J. Combin.*, to appear.

[6] B. De Bruyn. Isometric full embeddings of $DW(2n-1,q)$ into $DH(2n-1,q^2)$. preprint.

[7] B. De Bruyn. The valuations of the near $2n$-gon $\mathbb{I}_n$. preprint.

[8] B. De Bruyn and P. Vandecasteele. Valuations of near polygons. *Glasg. Math. J.* 47 (2005), 347–361.

[9] B. De Bruyn and P. Vandecasteele. Valuations and hyperplanes of dual polar spaces. *J. Combin. Theory Ser. A* 112 (2005), 194–211.

[10] B. De Bruyn and P. Vandecasteele. The distance-2- sets of the slim dense near hexagons. *Ann. Combin.*, to appear.

[11] B. De Bruyn and P. Vandecasteele. The classification of the slim dense near octagons. *European J. Combin.*, to appear.

[12] W.-l. Huang. Adjacency preserving mappings between point- line geometries. *Innovations in Incidence Geometry*, to appear.

[13] S. E. Payne. Generalized quadrangles of order 4, I. *J. Combin. Theory* 22 (1977), 267–279.

[14] S. E. Payne. Generalized quadrangles of order 4, II. *J. Combin. Theory* 22 (1977), 280–288.

[15] S. E. Payne and J. A. Thas. *Finite Generalized Quadrangles.* Research Notes in Mathematics 110. Pitman, 1984.

[16] H. Pralle and S. V. Shpectorov. The ovoidal hyperplanes of a dual polar space of rank 4. *Adv. Geom*, to appear.

[17] E. E. Shult and A. Yanushka. Near $n$-gons and line systems. *Geom. Dedicata* 9 (1980), 1–72.