Neighbour-distinguishing edge colourings of random regular graphs

Catherine Greenhill *
School of Mathematics and Statistics
University of New South Wales
Sydney NSW 2052, Australia
csg@unsw.edu.au

Andrzej Ruciński
Department of Discrete Mathematics
Adam Mickiewicz University
Poznań, Poland
rucinski@amu.edu.pl

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Abstract

A proper edge colouring of a graph is *neighbour-distinguishing* if for all pairs of adjacent vertices v, w the set of colours appearing on the edges incident with v is not equal to the set of colours appearing on the edges incident with w. Let $\mathrm{ndi}(G)$ be the least number of colours required for a proper neighbour-distinguishing edge colouring of G. We prove that for $d \geq 4$, a random d-regular graph G on n vertices asymptotically almost surely satisfies $\mathrm{ndi}(G) \leq \lceil 3d/2 \rceil$. This verifies a conjecture of Zhang, Liu and Wang for almost all 4-regular graphs.

1 Introduction

Suppose that $G = (V, E)$ is a graph and $h : E \to [k]$ is a proper edge colouring of G. All edge colourings considered in this paper are proper and from now on we will not explicitly mention this. For each vertex $v \in V$, let $S(v) = \{h(e) : v \in e\}$ be the set of colours on the neighbourhood of v. An edge colouring h is said to be *neighbour-distinguishing* if $S(v) \neq S(w)$ for all $\{v, w\} \in E$. A neighbour-distinguishing edge colouring of G exists if G has no isolated edges. Let the *neighbour-distinguishing index* of G, denoted by $\mathrm{ndi}(G)$, be the least number of colours needed in a neighbour-distinguishing edge colouring of G (where $\mathrm{ndi}(G) = \infty$ if G contains an isolated edge). We sometimes abbreviate "neighbour-distinguishing edge colouring" to "nd-colouring". This notion was introduced by Zhang, Liu and Wang in [12]. (Note that nd-colourings are also called *strong edge colourings* [12] or *adjacent vertex distinguishing colourings* [2]. Our terminology and notation follows [4].)

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As an example which will be important in our proof, the cycle C_n of length $n \geq 3$ has

$$\mathrm{ndi}(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 3, \\ 4 & \text{if } n \neq 5 \text{ and } n \not\equiv 0 \pmod 3, \\ 5 & \text{if } n = 5. \end{cases}$$

Let $\Delta(G) = \Delta$ be the maximum degree of the graph G. Clearly $\mathrm{ndi}(G) \geq \Delta$, and if there are adjacent vertices of maximum degree in G then $\mathrm{ndi}(G) \geq \Delta + 1$. Zhang, Liu and Wang [12] conjectured that

$$\mathrm{ndi}(G) \leq \Delta + 2$$

whenever G is a connected graph with at least three vertices which is not C_5. Balister et al. [2] proved the conjecture for all graphs with $\Delta = 3$, as well as for all bipartite graphs. They also showed that the bound is tight.

Only much weaker bounds are known for general graphs without isolated edges. Akbari et al. [1] obtained the bound

$$\mathrm{ndi}(G) \leq 3\Delta$$

for all graphs G without isolated edges. For very large Δ, Hatami [7] improved that to

$$\mathrm{ndi}(G) \leq \Delta + 300$$

(if $\Delta \geq 10^{20}$), and Ghandehari and Hatami [6] proved that

$$\mathrm{ndi}(G) \leq \Delta + 27\sqrt{\Delta \log \Delta}$$

(if $\Delta \geq 10^6$). For k-chromatic graphs G, Balister et al. [2] proved the bound

$$\mathrm{ndi}(G) = \Delta + O(\log k) = \Delta + O(\log \Delta), \tag{1}$$

with an implicit constant in the $O(\cdot)$ term (see Remark 1 below).

In related work, Baril, Kheddouci and Togni [3] proved that $\mathrm{ndi}(G) = \Delta + 1$ whenever G is a multidimensional mesh or a hypercube, and Edwards, Hornak and Wozniak [4] showed that $\mathrm{ndi}(G) \leq \Delta + 1$ if G is a planar bipartite graph with $\Delta \geq 12$.

The main goal of this note is to verify the above conjecture for *almost all* 4-regular graphs, and to establish bounds on $\mathrm{ndi}(G)$ for *almost all* d-regular graphs G, where $d \geq 4$ is constant. Let $\mathcal{G}_{n,d}$ be the uniform probability space of all d-regular graphs on vertex set $[n] = \{1, 2, \ldots, n\}$, where nd is even. Here d is a fixed constant and our asymptotics are as n tends to infinity. Following [9], we will identify the probability space $\mathcal{G}_{n,d}$ with a random graph sampled from it. The phrase *asymptotically almost surely* (a.a.s.) means "with probability which tends to 1 as n tends to infinity".

Theorem 1 *Let $d \geq 4$. Then a.a.s.* $\mathrm{ndi}(\mathcal{G}_{n,d}) \leq \lceil 3d/2 \rceil$.

We prove this theorem with the aid of contiguity. Section 2 contains background on contiguity of random regular graphs. The main proof is presented in Section 3, while two crucial probabilistic claims used in that proof are deferred to Section 4.

Remark 1 Clearly, for very large d, our bound is superceded by the above mentioned results from [2], [7] and [6]. (Note that, by Brooks' theorem, every connected, d-regular, n-vertex graph is d-chromatic for $d \geq 3$ and $n \geq d+2$.) But our bound beats the bound from [2] for $d \leq 56$. Indeed, it follows from the proof given in [2] that

$$d - 1 + 5\lceil \log_2 d \rceil$$

is a lower bound on the upper bound in (1), and it is easy to check that the inequality

$$\lceil 3d/2 \rceil < d - 1 + 5\lceil \log_2 d \rceil$$

holds for $d \leq 56$.

2 Contiguity background

Two sequences of probability spaces $\mathcal{A}_n$ and $\mathcal{B}_n$, with the same underlying set Ω_n, are said to be *contiguous* (written $\mathcal{A}_n \approx \mathcal{B}_n$) if for any sequence of events $(\mathcal{E}_n)$ with $\mathcal{E}_n \subseteq \Omega_n$ for $n \geq 1$, we have

$$\mathbb{P}_{\mathcal{A}_n}(\mathcal{E}_n) \to 1 \text{ if and only if } \mathbb{P}_{\mathcal{B}_n}(\mathcal{E}_n) \to 1.$$

That is, the same (sequences of) events hold a.a.s. in both sequences of probability spaces. See [8], [11] or [9, Chapter 9] for more information about contiguity.

2.1 Random regular multigraphs

Let $\mathcal{G}'_{n,d}$ be the (non-uniform) probability space of all d-regular multigraphs on vertex set $[n]$ with no loops, which arise from the pairings model (see [9, Chapter 9] or [11]). If G is a d-regular multigraph on $[n]$ with no loops and with r_k edges of multiplicity k, for $k \geq 1$, then the probability of G in this model is proportional to $\prod_{k \geq 1}(k!)^{-r_k}$. In particular, the probability space obtained from $\mathcal{G}'_{n,d}$ by conditioning on no multiple edges is exactly the space $\mathcal{G}_{n,d}$ of uniformly random d-regular (simple) graphs on $[n]$. (For readers unfamiliar with the pairings model, it does not hurt much to instead think of uniformly random d-regular multigraphs with no loops, since this model is contiguous with $\mathcal{G}'_{n,d}$ (see [8, Theorem 12]).)

The definition of neighbour-distinguishing colourings and the neighbour-distinguishing index $\mathrm{ndi}(G)$ extend naturally to all multigraphs with no connected component of order two. (Define $\mathrm{ndi}(G) = \infty$ if G has a connected component of order two.) For technical reasons, we will prove an analogue of Theorem 1 for the multigraph model $\mathcal{G}'_{n,d}$. That is, we will prove the following.

Theorem 2 *Let $d \geq 4$. Then a.a.s.* $\mathrm{ndi}(\mathcal{G}'_{n,d}) \leq \lceil 3d/2 \rceil$.

Since the probability that $\mathcal{G}'_{n,d}$ has no multiple edges is bounded away from 0 (see for example [8, Remark 13]), Theorem 1 follows immediately from Theorem 2. Indeed, for every event $\mathcal{E}_n$ which holds a.a.s. in $\mathcal{G}'_{n,d}$, we have

$$\mathbb{P}_{\mathcal{G}_{n,d}}(\neg\mathcal{E}_n) = \mathbb{P}_{\mathcal{G}'_{n,d}}(\neg\mathcal{E}_n \mid \mathcal{G}'_{n,d} \text{ has no multiple edges}) = o(1).$$

2.2 Contiguity arithmetic

For given multigraphs A and B on the vertex set $[n]$, the *sum* of A and B, written $A+B$, is the multigraph on $[n]$ with edges given by the multiset union of the edges of A and B. Define the sum of more than two multigraphs in the same way.

If $\mathcal{A}_n$ and $\mathcal{B}_n$ are both probability spaces on the set Ω_n of all multigraphs on the vertex set $[n]$, then their *sum* $\mathcal{A}_n + \mathcal{B}_n$ is the probability space obtained by choosing $A \in \mathcal{A}_n$ and $B \in \mathcal{B}_n$ independently and forming the multigraph $A+B$. Denote the sum of k copies of $\mathcal{A}_n$ by $k\mathcal{A}_n$.

Now we list all contiguity instances which are relevant to our proof. Let $\mathcal{H}_n$ be the uniform probability space on the set of all Hamilton cycles on the vertex set $[n]$. Frieze et al. [5] proved that for $d \geq 3$,

$$\mathcal{G}'_{n,d} \approx \mathcal{G}'_{n,d-2} + \mathcal{H}_n \tag{2}$$

while Kim and Wormald [10] proved that

$$\mathcal{G}'_{n,4} \approx 2\mathcal{H}_n. \tag{3}$$

The contiguous decompositions (2) and (3) give rise to an inductive proof of Theorem 2, described in the next section.

3 Proof of Theorem 2

In this section we give the proof of Theorem 2, which was already shown to yield our main result, Theorem 1. We begin with an outline of the proof.

3.1 Outline of the proof

We will prove Theorem 2 by induction on d, with increments of two (separately for d odd and even), and with the inductive step based on the contiguous decomposition (2) and the following deterministic lemma.

Lemma 1 *Fix $d \geq 5$ and $n \geq 6$. Let G be a $(d-2)$-regular multigraph G on the vertex set $[n]$ and let $H = C_n$ be a Hamilton cycle on the same vertex set $[n]$. Then*

$$\mathrm{ndi}(G+H) \leq \mathrm{ndi}(G) + 3.$$

There are two base cases, namely $d = 3$ and $d = 4$. A result from [2] implies that $\mathrm{ndi}(G) \leq 5$ for *all* multigraphs G with maximum degree 3 and no connected component of size 2.

The following lemma provides the second base case.

Lemma 2 *A.a.s.* $\mathrm{ndi}(2\mathcal{H}_n) \leq 6$.

Let us see now how these two lemmas yield the proof of Theorem 2.

Proof of Theorem 2. Note that when $d = 3$ the contiguity result (2) implies that $\mathcal{G}'_{n,3}$ is a.a.s. Hamiltonian, and hence connected. In particular, a.a.s. $\mathcal{G}'_{n,3}$ has no connected component of order two. Using this fact the theorem holds when $d = 3$, by [2]. By Lemma 2 and (3), the theorem holds when $d = 4$. Since

$$\lceil 3(d-2)/2 \rceil + 3 = \lceil 3d/2 \rceil$$

the result follows by induction for all $d \geq 3$, using Lemma 1 and (2). □

As an aside, note that working with graphs rather than multigraphs and substituting the deterministic upper bound of 8 for the asymptotically almost sure upper bound of 6 in Lemma 2 gives the following deterministic result.

Lemma 3 *Let G be a d-regular graph on the vertex set $[n]$.*

(i) *If d is odd and the edge set of G can be partitioned into the edge sets of $(d-3)/2$ disjoint Hamilton cycles and one cubic graph then* $\mathrm{ndi}(G) \leq \lceil 3d/2 \rceil$.

(ii) *If d is even and the edge set of G can be partitioned into the edge sets of $d/2$ disjoint Hamilton cycles then* $\mathrm{ndi}(G) \leq \lceil 3d/2 \rceil + 2$.

Now we continue with the proof of Theorem 2. It remains to prove Lemma 1 and Lemma 2. Both lemmas are quite trivial for $n \equiv 0 \pmod 3$ while some difficulties arise in the other cases. We handle each value of $n \pmod 3$ separately.

In what follows, we say that vertices v and w are *distinguishable* under a given edge colouring if $S(v) \neq S(w)$. (Here v and w need not be neighbours.) Vertices which are not distinguishable will be called *indistinguishable*.

The following fact, though obvious, is quite useful in the proofs.

Fact 1 *Let G_1 and G_2 be multigraphs on the same vertex set. Then*

$$\mathrm{ndi}(G_1 + G_2) \leq \mathrm{ndi}(G_1) + \mathrm{ndi}(G_2).$$

Proof. The inequality holds trivially if either G_1 or G_2 has a component of size two. Suppose then that G_i has an nd-colouring h_i with the set of colours C_i for $i = 1, 2$, where $C_1 \cap C_2 = \emptyset$. We define an edge colouring h of $G_1 + G_2$ using the colours in $C_1 \cup C_2$ by letting $h(e) = h_i(e)$ if $e \in G_i$, $i = 1, 2$. It is easy to check that h is an nd-colouring of $G_1 + G_2$. □

Note that for $n \not\equiv 0 \pmod 3$ and $n \geq 6$ we have $\mathrm{ndi}(C_n) = 4$. Thus Lemma 1 can be viewed as a sharpening (by 1) of Fact 1 when $G_2 = C_n$. Moreover, Lemma 2 shows that in the special case when also $G_1 = C_n$ we gain 2 a.a.s. if G_2 is drawn randomly from $\mathcal{H}_n$. The idea behind these improvements is to allow some pairs of vertices to be indistinguishable in the colouring of G_2, provided that they are distinguishable in the colouring of G_1.

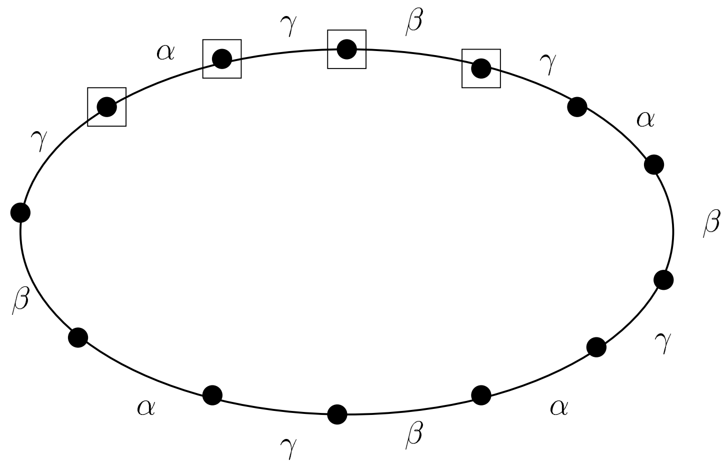

Figure 1: The colouring of H used in the second case of the proof of Lemma 1 when $n \equiv 1 \pmod 3$

3.2 Proof of Lemma 1

Fix $d \geq 5$, $n \geq 6$, and let G be a $(d-2)$-regular multigraph on the vertex set $[n]$. If G has a connected component of size two then the lemma holds trivially, so we may assume that G has no such component. If $n \equiv 0 \,(\mathrm{mod}\, 3)$ then $\mathrm{ndi}(C_n) = 3$ and Lemma 1 holds (deterministically) by Fact 1. Otherwise, fix an optimal nd-colouring h of G and suppose that h uses the colour set $[r]$. Let $H = C_n$ be a Hamilton cycle on the same vertex set $[n]$.

Case $n \equiv 1 \,(\mathrm{mod}\, 3)$**:**

Suppose that there exists an edge uv of H such that some colour $\delta \in [r]$ is missing at both u and v. Then we may colour uv with the colour δ in H, and colour the rest of H with three new colours to give an nd-colouring of H. This gives an nd-colouring of $G+H$ using $r+3$ colours.

On the other hand, if no such edge exists in H then for every edge uv of H we have $|S(u) \cup S(v)| = r$. Since G is $(d-2)$-regular we know that $r \geq d-1$, which implies that $|S(u) \cap S(v)| \leq d-3$. Thus there is at least one colour in $S(u) - S(v)$, which implies that u and v are distinguishable under h. As this holds for any edge of H, consider four consecutive vertices $u_1, \ldots, u_4$ of H. We may colour H with three new colours in such a way that all vertices are distinguishable from their H-neighbours except for the pairs u_1, u_2 and u_3, u_4. (See Figure 1 for an example, where the vertices $u_1, \ldots, u_4$ have boxes drawn around them.) This gives an nd-colouring of $G+H$ using $r+3$ colours.

Case $n \equiv 2 \,(\mathrm{mod}\, 3)$**:**

Let $V_1, \ldots, V_k$ be the partition of $[n]$ given by the colour classes of the (proper) vertex colouring of G induced by h. That is, vertices v and w belong to the same part of the partition if and only if $S(v) = S(w)$ under h. (Here k is the number of distinct sets $S(v)$ under h, which could be as large as $\binom{r}{d}$.)

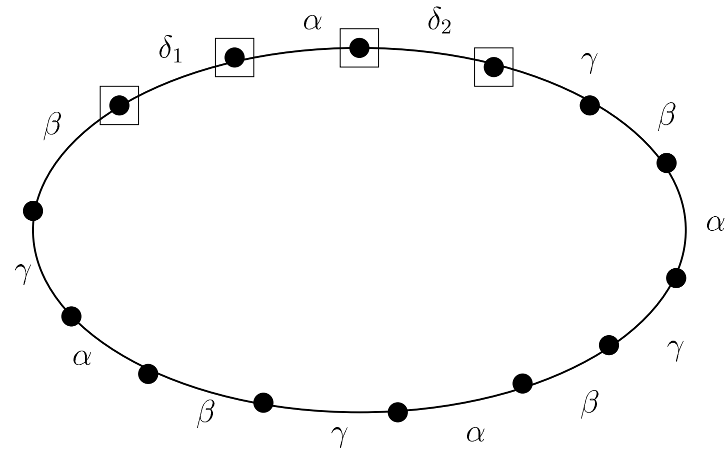

Figure 2: The colouring of H used in the second case of the proof of Lemma 1 when $n \equiv 2 \pmod 3$

First suppose that there is a 2-path uvw on H such that u and v are distinguishable under h and v and w are distinguishable under h. Then we may colour the edges of H using three new colours in such a way that every vertex is distinguishable from its H-neighbours except for the pairs u, v and v, w. This gives an nd-colouring of $G + H$ using $r + 3$ colours.

Next, suppose that there is no such 2-path on H. Then whenever H enters a set V_i, it stays in V_i for at least one more vertex (that is, $H[V_i]$ has no isolated vertices). Choose an edge u_2v_1 of H with $u_2 \in V_i$ and $v_1 \in V_j$ for some $i \neq j$. Then we have a 3-path $u_1u_2v_1v_2$ in H such that $u_1, u_2 \in V_i$ and $v_1, v_2 \in V_j$. Hence there exists distinct colours $\delta_1, \delta_2 \in [r]$ such that δ_1 is missing at u_1 and at u_2 and δ_2 is missing at v_1 and at v_2. We may now construct an nd-colouring of H using δ_1 for the edge u_1u_2, δ_2 for the edge v_1v_2, and using three new colours for all other edges of H. (See Figure 2 for an example, where the vertices u_1, u_2, v_1, v_2 have boxes around them.) This produces an nd-colouring of $G + H$ using $r + 3$ colours, as required, completing the proof of Lemma 1.

3.3 Proof of Lemma 2

Again, if $n \equiv 0 \pmod 3$ then $\mathrm{ndi}(C_n) = 3$ and Lemma 2 holds (deterministically) using Fact 1. Otherwise, write $G = H_1 + H_2$, where H_1 and H_2 are two Hamilton cycles on $[n]$. Assume that H_1 is fixed and that H_2 is a random element of $\mathcal{H}_n$.

Case $n \equiv 1 \pmod 3$**:**

We will show in Claim 1 below (see Section 4) that when $n \equiv 1 \pmod 3$, a.a.s. there is an edge vw of H_2 such that the distance from v to w in H_1 is congruent to $2 \pmod 3$ (in which case both paths from v to w in H_1 have lengths congruent to $2 \pmod 3$).

Colour the edge vw with the colour γ, and colour the rest of H_2 with colours δ, ϵ, ζ to give an nd-colouring of H_2. Next, colour the edges of H_1 with colours α, β, γ in such a way that v is adjacent to edges coloured α, β and so is w, and all adjacent vertices of H_2

are distinguishable except that v and w are not distinguishable from their neighbours. To achieve this, use the colouring

$$\alpha,\ \beta,\ \gamma,\ \alpha,\ \beta,\ \gamma,\ldots,\alpha,\ \beta$$

from v to w around one side of H_1, and use the colouring

$$\beta,\ \alpha,\ \gamma,\ \beta,\ \alpha,\ \gamma,\ldots,\beta,\ \alpha$$

from v to w around the other side (see Figure 3). In the induced edge-colouring of

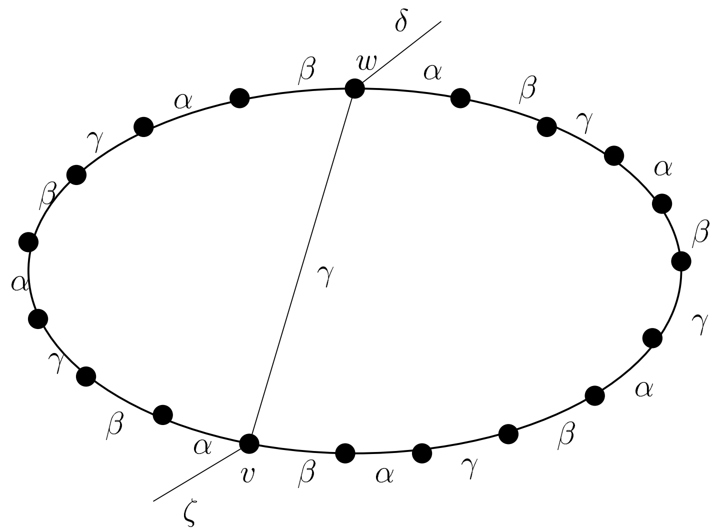

Figure 3: The colouring of H_1 used in the proof of Lemma 2 when $n \equiv 1\,(\mathrm{mod}\,3)$

$H_1 + H_2$, vertices v and w are incident with three edges coloured with colours $\{\alpha, \beta, \gamma\}$, and they are the only two vertices in the multigraph with this property, which makes them distinguishable from their H_1-neighbours. So this is an nd-colouring of $H_1 + H_2$.

Case $n \equiv 2\,(\mathrm{mod}\,3)$**:**

We will show in Claim 2 below (see Section 4) that when $n \equiv 2\,(\mathrm{mod}\,3)$, a.a.s. there exist edges v_1w_1 and v_2w_2 of H_2 which cut H_2 into two paths of positive lengths divisible by 3, and such that the vertices v_1, w_1, v_2, w_2 cut H_1 into four paths, $P_1, \ldots, P_4$, of lengths congruent to $2\,(\mathrm{mod}\,3)$.

Colour v_1w_1 and v_2w_2 with colour γ and colour the rest of H_2 by δ, ϵ, ζ, so that all pairs of adjacent vertices are distinguishable. Finally, colour H_1 with colours α, β, γ so that each path P_i begins and ends with the colour sequence α, β and all pairs of adjacent vertices on H_1 are distinguishable except that v_1, w_1, v_2, w_2 are not distinguishable from their neighbours on H_1. It follows similarly to the case when $n \equiv 1 \pmod 3$ that all pairs of adjacent vertices of $H_1 + H_2$ are distinguishable. $\square$

4 Adding a random Hamilton cycle

It remains to prove the two final claims, both about the effect of adding a random Hamilton cycle to a given graph.

To choose a uniformly random Hamilton cycle H on the set $[n]$, it will be convenient to consider the following random process. Take an arbitrary start-vertex u_1 and proceed randomly around $[n]$ creating H vertex by vertex. Specifically, suppose that $u_1 u_2 \cdots u_j$ have already been chosen. Then u_{j+1} is selected uniformly at random from the remaining $n-j$ vertices, for $j = 1, \ldots, n-1$ (and the edge $u_n u_1$ is added at the end to complete the cycle). Every Hamilton cycle will have two chances to appear, one for each direction, each with probability $1/(n-1)!$ (and thus with global probability $2/(n-1)!$, as it should be). In this process, let $e_i = u_i u_{i+1}$, $i = 1, \ldots, n$ be the ith random edge of H. (The edge e_n is not really random, since $u_{n+1} = u_1$.) Then, for each i the sequence $(e_1, \ldots, e_i)$ will be called *the history of H until time i*. We refer to this process and the notation described above throughout this section.

Throughout this section we will write n/c instead of $\lfloor n/c \rfloor$ in a few places, where c is a constant. Since n tends to infinity the error in doing this is negligible.

Below, H_1 is a fixed Hamilton cycle on $[n]$, while H_2 is an element of $\mathcal{H}_n$ selected uniformly at random.

Claim 1 *Suppose that $n \equiv 1 \pmod 3$. Then a.a.s. H_2 contains an edge vw such that the distance from v to w in H_1 is congruent to $2 \pmod 3$.*

Proof. Choose H_2 vertex by vertex, as described above. Call the ith edge $e_i = u_i u_{i+1}$ of H_2 *bad* if the distance from u_i to u_{i+1} in H_1 is not equal to $2 \pmod 3$ (in some direction). Let E_i be the event that e_i is bad. Then

$$\mathbb{P}\left(\bigcap_{i=1}^{n} E_i\right) \le \mathbb{P}\left(\bigcap_{i=1}^{n/12} E_i\right) = \prod_{i=1}^{n/12} \mathbb{P}\left(E_i \;\middle|\; \bigcap_{j=1}^{i-1} E_j\right).$$

In order to estimate $\mathbb{P}\left(E_i \;\middle|\; \bigcap_{j=1}^{i-1} E_j\right)$, we first estimate $\mathbb{P}\left(E_i \;\middle|\; e_1, \ldots, e_{i-1}\right)$; that is, the probability of the event E_i conditioned on the history of the process up to time i. Given u_i there are at most $2n/3$ vertices which are not on H_2 yet, and which make a bad pair with u_i. Since we choose u_{i+1} out of at least $n - n/12 = 11n/12$ vertices, we have

$$\mathbb{P}\left(E_i \;\middle|\; e_1, \ldots, e_{i-1}\right) \le 8/11.$$

Summing over all possible histories $e_1, \ldots, e_{i-1}$ such that $E_1, \ldots, E_{i-1}$ all hold, we obtain

$$\mathbb{P}\left(E_i \;\middle|\; \bigcap_{j=1}^{i-1} E_j\right) \le 8/11.$$

Therefore
$$\mathbb{P}\left(\bigcap_{i=1}^{n} E_i\right) \leq (8/11)^{n/12} = o(1)$$
as required. ∎

Claim 2 *Suppose that $n \equiv 2 \pmod 3$. Then a.a.s. H_2 contains edges v_1w_1 and v_2w_2 which cut H_2 into two paths of positive lengths divisible by 3 and such that the vertices v_1, w_1, v_2, w_2 cut H_1 into four paths, $P_1, \ldots, P_4$, of lengths congruent to $2 \pmod 3$.*

Proof. Call the edge $e_i = u_iu_{i+1}$ of H_2 *good* if the distance from u_i to u_{i+1} in H_1 is at most $n/4$ and is congruent to $2 \pmod 3$. We modify the proof of Claim 1 to show that a.a.s. there exists a good edge e_i with $1 \leq i \leq n/12$. Let E_i be the event that edge e_i is bad. Given the history up until step i, there are at most $n/2$ choices for u_{i+1} which (do not yet lie on H_2 and) are too far away from u_i and at most $n/3$ choices which (do not yet lie on H_2 and) are close enough to u_i but with the wrong modulus. At least $11n/12$ vertices do not yet lie on H_2, so arguing as in Claim 1,
$$\mathbb{P}\left(E_i \;\middle|\; \bigcap_{j=1}^{i-1} E_j\right) \leq \frac{n/2 + n/3}{11n/12} = 10/11.$$
Therefore
$$\mathbb{P}\left(\bigcap_{i=1}^{n/12} E_i\right) \leq (10/11)^{n/12} = o(1).$$
This says that a.a.s. there exists a good edge e_i with $1 \leq i \leq n/12$. This edge e_i is the edge v_1w_1. Call this Phase 1.

Assume for the rest of the proof that Phase 1 is successful (that is, a good edge was found in the first $n/12$ steps). The vertices v_1, w_1 split H_1 into a short path (of length at most $n/4$) and a long path. Call the vertices of the long path *active*, and call the vertices of the short path *inactive*. In Phase 2, we say that the edge $e_j = u_ju_{j+1}$ is *good* if

(i) u_j is active,

(ii) the distance from u_j to the closer of v_1, w_1 in H_1 is at most $n/4$ and is congruent to $2 \pmod 3$,

(iii) u_{j+1} is active,

(iv) the path from u_j to u_{j+1} in H_1 which does not contain v_1, w_1 has length congruent to $2 \pmod 3$,

(v) if P is the path in H_1 of length at most $n/4$ between u_j and the closer of v_1, w_1, then u_{j+1} does not lie on P.

We say that Phase 2 is successful if there exists a good edge e_j such that $j = i + 1 + 3\ell$ where $1 \leq \ell \leq n/72$. We will show that a.a.s. Phase 2 is successful, conditioned on Phase 1 being successful. If Phase 2 is sucessful then the edge e_j is the edge $v_2 w_2$.

For example, consider Figure 4. The edge $v_1 w_1$ is shown, together with the possible choices for u_j which satisfy (i) and (ii). Then for a particular choice of u_j, Figure 5 shows

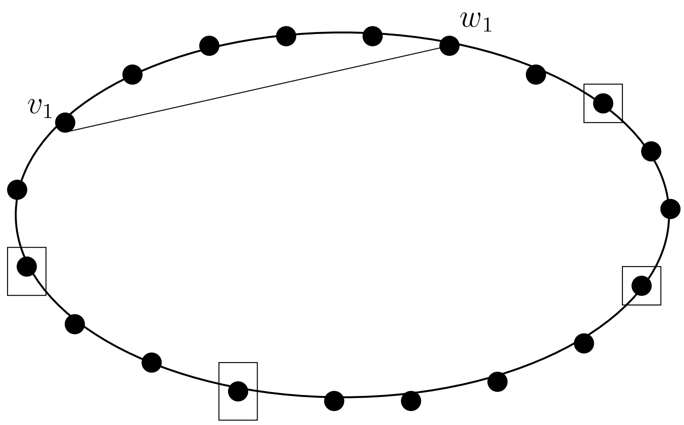

Figure 4: Choices for u_j in Phase 2

the possible choices for u_{j+1} which satisfy (iii)–(v).

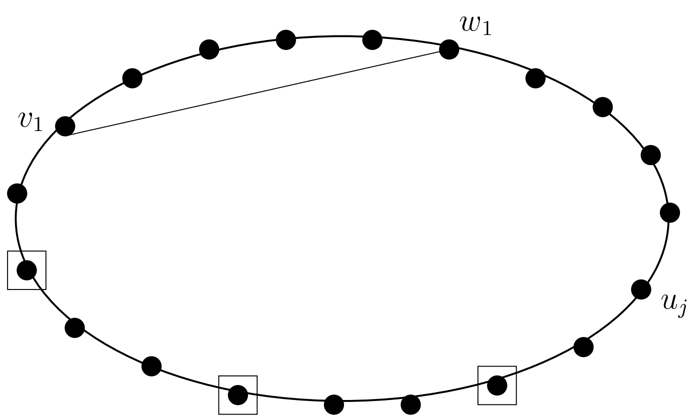

Figure 5: Choices for u_{j+1} in Phase 2

Let F_j be the event that e_j is bad, where $j = i + 1 + 3\ell$ and $1 \leq \ell \leq n/72$. Let $e_1, \ldots, e_{j-2}$ be the history up until step $j - 1$, and assume that Phase 1 succeeds for this history. We next choose u_j, and this choice succeeds if (i) and (ii) hold. There are $n/2$ active vertices which are close enough to v_1 or w_1, and 1/3 of these have distance which is the correct modulus. Of these, at most $n/8$ already lie on H_2. Therefore the probability that u_j satisfies (i) and (ii), conditioned on the history up until step $j-1$, is at least 1/24. If u_j satisfies (i) and (ii) then the probability that u_{j+1} satisfies (iii) - (v) is also at least 1/24, since there are at least $n/2$ active vertices which do not lie in P, of which 1/3 of these have distance which is the correct modulus (in (iv)), and only at most $n/8$ of these already lie on H_2. It follows that

$$\mathbb{P}(F_j \mid e_1, \ldots, e_{j-2}) \leq \frac{575}{576}$$

and by the usual arguments, the probability that Phase 2 fails, conditioned on Phase 1 succeeding, is at most $(575/576)^{n/72} = o(1)$. Hence a.a.s. Phases 1 and 2 both succeed, as required. □

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