Plethysm for wreath products and homology of sub-posets of Dowling lattices

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Abstract

We prove analogues for sub-posets of the Dowling lattices of the results of Calderbank, Hanlon, and Robinson on homology of sub-posets of the partition lattices. The technical tool used is the wreath product analogue of the tensor species of Joyal.

Introduction

For any positive integer n and finite group G, the Dowling lattice $Q_n(G)$ is a poset with an action of the wreath product group $G \wr S_n$. If G is trivial, $Q_n(\{1\})$ can be identified with the partition lattice Π_{n+1} (on which S_n acts as a subgroup of S_{n+1}). If G is the cyclic group of order r for $r \geq 2$, $Q_n(G)$ can be identified with the lattice of intersections of reflecting hyperplanes in the reflection representation of $G \wr S_n$. For general G, the underlying set of $Q_n(G)$ can be thought of as the set of all pairs (I, π) where $I \subseteq \{1, \cdots, n\}$ and π is a set partition of $G \times (\{1, \cdots, n\} \setminus I)$ whose parts G permutes freely; see Definition 1.1 below for the partial order.

In Section 1 we will define various sub-posets P of $Q_n(G)$, containing the minimum element $\hat{0}$ and the maximum element $\hat{1}$, which are stable under the action of $G \wr S_n$. For completeness' sake we include the cases of $Q_n(G)$ itself and two other sub-posets which have been studied before, but the main interest lies in two new families of sub-posets, defined using a fixed integer $d \geq 2$: $Q_n^{1 \bmod d}(G)$, given by the congruence conditions $|I| \equiv 0 \bmod d$ and $|K| \equiv 1 \bmod d$ for all parts K of π, and $Q_n^{0 \bmod d}(G)$, given by the condition $|K| \equiv 0 \bmod d$ for all parts K of π. These definitions are modelled on those of the sub-posets $\Pi_n^{(1,d)}$ and $\Pi_n^{(0,d)}$ of the partition lattice studied by Calderbank, Hanlon, and Robinson in [4]. We will prove that all our sub-posets P are pure (i.e. graded) and

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Cohen-Macaulay, so the only non-vanishing reduced homology group of $P \setminus \{\hat{0}, \hat{1}\}$ is the top homology $\widetilde{H}_{l(P)-2}(P \setminus \{\hat{0}, \hat{1}\}; \mathbb{Q})$. We take rational coefficients so that we can regard this homology as a representation of $G \wr S_n$ over $\mathbb{Q}$.

The main aim of this paper is to find in each case a formula for the character of this representation, analogous to the formulae proved in [4]. The last paragraph of that paper hoped specifically for a Dowling lattice analogue of [4, Theorem 6.5], but our Theorem 2.7 casts doubt on its existence. (In the cases of the previously-studied posets, we recover Hanlon's formula from [7] and other results which were more or less known.)

In [14], Rains applied [4, Theorem 4.7] to compute the character of S_n on the cohomology of the manifold $\overline{M_{0,n}}(\mathbb{R})$ (the real points of the moduli space of stable genus 0 curves with n marked points). In subsequent work he has generalized this, giving a description of the cohomology of any real De Concini-Procesi model in terms of the Whitney homology of an associated poset; the posets which arise in types B and D are closely related to our $Q_n^{1 \bmod 2}(\{\pm 1\})$. This application to algebraic topology, which was the original motivation for studying such sub-posets of Dowling lattices, will be explained in a forthcoming joint paper; for a sample, see (5.11) below.

In Section 2 we recall the combinatorial framework used by Macdonald to write down characters of representations of wreath products, and state our main results. In Section 3 we introduce the functorial concept of a $(G \wr \mathbb{S})$-module, a generalization of Joyal's notion of tensor species; this concept comes from [8], and we recall the connection proved there with generalizations of plethysm. In Section 4 we use this technology, and the 'Whitney homology method' of Sundaram, to prove our results. In Section 5 we extend the results to the setting of Whitney homology, thus computing the 'equivariant characteristic polynomials' of our posets.

1 Some Cohen-Macaulay sub-posets of Dowling lattices

In this section we define the Dowling lattices and the sub-posets of interest to us, and prove that they are Cohen-Macaulay. A convenient reference for the basic definitions and techniques of Cohen-Macaulay posets is [16]; the key result for us is the Björner-Wachs criterion, [16, Theorem 4.2.2] (proved in [2]), that a pure bounded poset with a recursive atom ordering is Cohen-Macaulay. For any nonnegative integer n, write $[n]$ for $\{1, \cdots, n\}$ (so $[0]$ is the empty set), and S_n for the symmetric group of permutations of $[n]$.

For any finite set I, let $\Pi(I)$ denote the poset of partitions of the set I, where a **partition** π of I is a set of nonempty disjoint subsets of I whose union is I. These subsets $K \in \pi$ are referred to as the **parts** of π. The partial order on $\Pi(I)$ is by refinement; $\Pi(I)$ is a geometric lattice, isomorphic to $\Pi_n = \Pi([n])$ where $n = |I|$. (We use the convention that the empty set has a single partition, which as a set is itself empty. Therefore $\Pi(\emptyset) = \Pi_0$ is a one-element poset, like Π_1.)

Fix a finite group G, and view the wreath product $G \wr S_n$ as the group of permutations of $G \times [n]$ which commute with the action of G (by left multiplication on the first factor).

Our definition of the corresponding Dowling lattice is as follows.

Definition 1.1. For $n \geq 1$, let $Q_n(G)$ be the poset of pairs (J, π) where J is a G-stable subset of $G \times [n]$ and $\pi \in \Pi((G \times [n]) \setminus J)$ is such that G permutes its parts freely, i.e. for all $1 \neq g \in G$ and $K \in \pi$, $K \neq g.K \in \pi$. The partial order on these pairs is defined so that $(J, \pi) \leq (J', \pi')$ is equivalent to the following two conditions:

1. $J \subseteq J'$, and
2. for all parts $K \in \pi$, either $K \subseteq J'$ or K is contained in a single part of π'.

We have an obvious action of $G \wr S_n$ on the poset $Q_n(G)$. Of course, J must be of the form $G \times I$ for some subset $I \subseteq [n]$, so we could just as well have used I in the definition, as in the introduction; once one has taken into account this and other such variations, it should be clear that $Q_n(G)$ is isomorphic, as a $(G \wr S_n)$-poset, to Dowling's original lattice in [5] and to the various alternative definitions given in [7], [6], and [9]. (The justification for adding yet another definition to the list will come when we adopt a functorial point of view.) The minimum element $\hat{0}$ is the pair $(\emptyset, \{\{(g, m)\} \,|\, g \in G, m \in [n]\})$, and the maximum element $\hat{1}$ is the pair $(G \times [n], \emptyset)$. Dowling proved in [5] that $Q_n(G)$ is a geometric lattice, and hence it is Cohen-Macaulay; its rank function is

$$\mathrm{rk}(J, \pi) = n - \frac{|\pi|}{|G|}, \tag{1.1}$$

so the length of the lattice as a whole is n.

Special cases of this lattice are more familiar. Clearly $Q_n(\{1\}) \cong \Pi(\{0, 1, \cdots, n\})$ via the map which sends (J, π) to $\{J \cup \{0\}\} \cup \pi$; and $Q_n(\{\pm 1\})$ is the signed partition lattice, also known as the poset of (conjugate) parabolic subsystems of a root system of type B_n. More generally, when G is cyclic of order $r \geq 2$, $Q_n(G)$ can be identified with the lattice of intersections of reflecting hyperplanes in the reflection representation of $G \wr S_n$, i.e. the lattice denoted $L(\mathcal{A}_n(r))$ in [13, §6.4].

Before we define the sub-posets we are mainly interested in, let us also consider two sub-posets given by a condition on J:

Definition 1.2. For $n \geq 1$, let $R_n(G)$ be the sub-poset of $Q_n(G)$ consisting of pairs (J, π) where either $J = \emptyset$ or $J = G \times [n]$. For $n \geq 2$ and assuming that $G \neq \{1\}$, let $Q_n^{\sim}(G)$ be the sub-poset of $Q_n(G)$ consisting of pairs (J, π) where $\frac{|J|}{|G|} \neq 1$.

Clearly the minimum and maximum elements of $Q_n(G)$ are in $R_n(G)$ (indeed, we allow $J = G \times [n]$ merely in order to include $\hat{1}$); likewise for $Q_n^{\sim}(G)$, given that $n \geq 2$. It is easy to see that both $R_n(G)$ and $Q_n^{\sim}(G)$ are pure of length n, with rank function again given by (1.1). It is also easy to see that $R_n(G) \setminus \{\hat{1}\}$ is a geometric semilattice in the sense of Wachs and Walker (see [16, Definition 4.2.6]), so $R_n(G)$ is Cohen-Macaulay by [16, Theorem 4.2.7]. An alternative proof of this is provided by [9, Corollary 3.12], where $R_n(G) \setminus \{\hat{0}, \hat{1}\}$ is called $\underline{\Pi}_n^G$. Note that $R_n(\{1\}) \setminus \{\hat{1}\} \cong \Pi_n$, so $R_n(\{1\})$ is Π_n with an extra maximum element adjoined. One can also interpret $R_n(\{\pm 1\}) \setminus \{\hat{1}\}$ as the poset of (conjugate) parabolic subsystems of a root system of type B_n all of whose components are of type A. As for $Q_n^{\sim}(G)$, note that it is closed under the join operation of $Q_n(G)$, and two elements of $Q_n^{\sim}(G)$ have a meet in $Q_n^{\sim}(G)$ which is $\leq$ their meet in $Q_n(G)$. The assumption that $G \neq \{1\}$ ensures that every element of $Q_n^{\sim}(G)$ is a join of atoms, so $Q_n^{\sim}(G)$ is another geometric lattice, and hence it is Cohen-Macaulay. When G is cyclic of order $r \geq 2$, $Q_n^{\sim}(G)$ can be identified with the lattice denoted $L(\mathcal{A}_n^0(r))$ in [13, Section 6.4]; for instance, $Q_n^{\sim}(\{\pm 1\})$ is the poset of (conjugate) parabolic subsystems of a root system of type D_n.

Now we turn to the analogues of the sub-posets of the partition lattices considered by Calderbank, Hanlon, and Robinson.

Definition 1.3. For $n \geq 1$ and $d \geq 2$, let $Q_n^{1 \bmod d}(G)$ be the sub-poset of $Q_n(G)$ consisting of pairs (J, π) satisfying the following conditions:

1. for all $K \in \pi$, $|K| \equiv 1 \bmod d$; and

2. either $\frac{|J|}{|G|} \equiv 0 \bmod d$ or $J = G \times [n]$.

Note that when $n \equiv 0 \bmod d$, there is no need to explicitly allow $J = G \times [n]$; otherwise, allowing this has the effect of ensuring that the maximum element $\hat{1}$ is included. Clearly the minimum element $\hat{0}$ of $Q_n(G)$ also belongs to $Q_n^{1 \bmod d}(G)$. To explain the congruence condition in (2), note that under the isomorphism $Q_n(\{1\}) \cong \Pi(\{0, 1, \cdots, n\}) \cong \Pi_{n+1}$, $Q_n^{1 \bmod d}(\{1\})$ corresponds to the '1 mod d partition lattice' $\Pi_{n+1}^{(1,d)}$ considered in [4]. Also, one can interpret $Q_n^{1 \bmod d}(\{\pm 1\}) \setminus \{\hat{1}\}$ as the poset of proper (conjugate) parabolic subsystems of a root system of type B_n all of whose components have rank divisible by d; the following result was proved by Rains in that case.

Proposition 1.4. *$Q_n^{1 \bmod d}(G)$ is a totally semimodular pure poset with rank function*

$$\mathrm{rk}(J, \pi) = \begin{cases} \lceil \frac{n}{d} \rceil, & \text{if } J = G \times n, \\ \frac{n}{d} - \frac{|\pi|}{d|G|}, & \text{otherwise.} \end{cases}$$

Its length is $\lceil \frac{n}{d} \rceil$.

Proof. If (J', π') covers (J, π) in $Q_n^{1 \bmod d}(G)$, then there are two possibilities:

1. $J' = J$, in which case π' must be obtained from π by merging $(d+1)$ G-orbits of parts into a single G-orbit of parts, **or**

2. $J' \supset J$, in which case J' must be the union of J together with d G-orbits of parts of π (or $n - d\lfloor \frac{n}{d} \rfloor$ G-orbits, if $J' = G \times [n]$ and $n \not\equiv 0 \bmod d$).

In either case one sees immediately that the purported rank of (J', π') is one more than that of (J, π), so this is indeed the rank function, and $Q_n^{1 \bmod d}(G)$ is pure. To show that $Q_n^{1 \bmod d}(G)$ is totally semimodular (see [16, 4.2]), it suffices to check the condition at $\hat{0}$, since for every $(J, \pi) \in Q_n^{1 \bmod d}(G)$, the principal upper order ideal $[(J, \pi), \hat{1}]$ is isomorphic to $Q^{1 \bmod d}_{|\pi|/|G|}(G)$. That is, we need only prove the following: if a and b are distinct atoms of $Q^{1 \bmod d}_n(G)$, $a \vee b$ is their join in the lattice $Q_n(G)$, and $c \in Q^{1 \bmod d}_n(G)$ satisfies $c \geq a \vee b$ and is minimal with this property in $Q^{1 \bmod d}_n(G)$, then $\mathrm{rk}(c) = 2$ for the rank function we have just found. Since there are two types of atoms corresponding to the two kinds of covering relation, we have several cases to consider.

Case 1: $a = (\emptyset, \pi_a)$ and $b = (\emptyset, \pi_b)$. Let A be the union of the non-singleton parts of π_a, which all have size $d+1$ and form a single G-orbit. Define B similarly for π_b. Let $\pi_a \vee \pi_b$ denote the join in $\Pi(G \times [n])$.

Subcase 1a: $\frac{|A \cap B|}{|G|} = 0$ or 1. Then $a \vee b = (\emptyset, \pi_a \vee \pi_b)$ is itself in $Q^{1 \bmod d}_n(G)$, so $c = a \vee b$ and $\mathrm{rk}(c) = 2$.

Subcase 1b: $\frac{|A \cap B|}{|G|} \geq 2$, and $a \vee b = (\emptyset, \pi_a \vee \pi_b)$. (This means that the mergings of A and of B are 'compatible' on the overlap.) Then $\pi_a \vee \pi_b$ has a unique G-orbit of non-singleton parts, whose union is $A \cup B$. Since $A \neq B$, we have $d + 2 \leq \frac{|A \cup B|}{|G|} = 2d + 2 - \frac{|A \cap B|}{|G|} \leq 2d$. Then either $c = (J, \hat{0}_{\Pi((G \times [n]) \setminus J)})$ where $J \supseteq A \cup B$ has size $\min\{2d|G|, n|G|\}$, or $c = (\emptyset, \pi_c)$ where π_c has a unique G-orbit of non-singleton parts, all of size $2d + 1$, whose union contains $A \cup B$. In either case $\mathrm{rk}(c) = 2$.

Subcase 1c: $\frac{|A \cap B|}{|G|} \geq 2$, and $a \vee b = (A \cup B, \hat{0}_{\Pi((G \times [n]) \setminus (A \cup B))})$. (This means that the mergings of A and of B are 'not compatible' on the overlap, as can happen when G is non-trivial). We have $d + 1 \leq \frac{|A \cup B|}{|G|} = 2d + 2 - \frac{|A \cap B|}{|G|} \leq 2d$, so c must be of the form $(J, \hat{0}_{\Pi((G \times [n]) \setminus J)})$ where $J \supseteq A \cup B$ has size $\min\{2d|G|, n|G|\}$. Thus $\mathrm{rk}(c) = 2$.

Case 2: the atoms a and b are of different types. Without loss of generality, assume $a = (A, \hat{0}_{\Pi((G \times [n]) \setminus A)})$ where $|A| = d|G|$, and $b = (\emptyset, \pi_b)$ for B as above.

Subcase 2a: $A \cap B = \emptyset$. Then $a \vee b = (A, \pi_b|_{(G \times [n]) \setminus A})$ is itself in $Q^{1 \bmod d}_n(G)$, so $c = a \vee b$ and $\mathrm{rk}(c) = 2$.

Subcase 2b: $A \cap B \neq \emptyset$. Then $a \vee b = (A \cup B, \hat{0}_{\Pi((G \times [n]) \setminus (A \cup B))})$, and $d + 1 \leq \frac{|A \cup B|}{|G|} = 2d + 1 - \frac{|A \cap B|}{|G|} \leq 2d$, so c must be as in Subcase 1c.

Case 3: $a = (A, \hat{0}_{\Pi((G \times [n]) \setminus A)})$, $b = (B, \hat{0}_{\Pi((G \times [n]) \setminus B)})$ where $|A| = |B| = d|G|$. Then $a \vee b = (A \cup B, \hat{0}_{\Pi((G \times [n]) \setminus (A \cup B))})$. Since $A \neq B$, we have $d + 1 \leq \frac{|A \cup B|}{|G|} = 2d - \frac{|A \cap B|}{|G|} \leq 2d$, so c must be as in Subcase 1c. $\square$

We deduce via [16, Theorem 4.2.3] that $Q^{1 \bmod d}_n(G)$ is Cohen-Macaulay.

Finally, we consider the analogue of the 'd-divisible partition lattice'.

Definition 1.5. For $n \geq 1$ and $d \geq 2$, let $Q^{0 \bmod d}_n(G)$ be the sub-poset of $Q_n(G)$ consisting of pairs (J, π) such that either

1. (J, π) is the minimum element of $Q_n(G)$, i.e. $J = \emptyset$ and $|K| = 1$ for all $K \in \pi$, or
2. $|K| \equiv 0 \bmod d$ for all $K \in \pi$.

Note that the maximum element of $Q_n(G)$ vacuously satisfies condition (2), so this poset is certainly bounded. If $n \equiv -1 \bmod d$, then under the isomorphism $Q_n(\{1\}) \cong \Pi(\{0, 1, \cdots, n\}) \cong \Pi_{n+1}$, $Q^{0 \bmod d}_n(\{1\})$ corresponds to the poset $\Pi^{(0,d)}_{n+1}$ considered in [4]. If $n \not\equiv -1 \bmod d$, then $Q^{0 \bmod d}_n(\{1\})$ does not correspond to anything in [4].

Proposition 1.6. *$Q_n^{0 \bmod d}(G)$ is a pure lattice with a recursive atom ordering. Its rank function is*

$$\mathrm{rk}(J,\pi) = \begin{cases} 0, & \text{if } (J,\pi) = \hat{0}, \\ \lfloor \frac{n}{d} \rfloor + 1 - \frac{|\pi|}{|G|}, & \text{otherwise.} \end{cases}$$

Its length is $\lfloor \frac{n}{d} \rfloor + 1$.

Proof. It is obvious that $Q_n^{0 \bmod d}(G) \setminus \{\hat{0}\}$ is an upper order ideal of $Q_n(G)$, so $Q_n^{0 \bmod d}(G)$ is a lattice. The atoms of $Q_n^{0 \bmod d}(G)$ are those (J,π) where $|K| = d$ for all $K \in \pi$ and $\frac{|\pi|}{|G|} = \lfloor \frac{n}{d} \rfloor$; these all have rank $(n - \lfloor \frac{n}{d} \rfloor)$ as elements of $Q_n(G)$, so $Q_n^{0 \bmod d}(G)$ is pure and has the claimed rank function. To find a recursive atom ordering (see [16, Definition 4.2.1]), note that for any non-mimimum element (J,π) of $Q_n^{0 \bmod d}(G)$, the principal upper order ideal $[(J,\pi),\hat{1}]$ is totally semimodular, being isomorphic to $Q_{|\pi|/|G|}(G)$. Thus we need only check that the atoms of $Q_n^{0 \bmod d}(G)$ can be ordered $a_1, \cdots, a_t$ so that:

$$a_i, a_j < y, i < j \Rightarrow \exists z \leq y, z \text{ covers } a_j \text{ and } a_k, \exists k < j. \tag{1.2}$$

Such an ordering (inspired by [16, Exercise 4.3.6(a)]) can be defined as follows. For each d-element subset $I \subset [n]$, let $\Psi(I)$ be the set of partitions of $G \times I$ on whose parts G acts freely and transitively (there are $|G|^{d-1}$ elements in this set). An atom (J,π) of $Q_n^{0 \bmod d}(G)$ is uniquely determined by the following data:

1. a $(n - d\lfloor \frac{n}{d} \rfloor)$-element subset I_0 of $[n]$, such that $J = G \times I_0$; and
2. a partition of $[n] \setminus I_0$ into d-element subsets $I_1, \cdots, I_{\lfloor \frac{n}{d} \rfloor}$, each I_s equipped with a partition $\psi_s \in \Psi(I_s)$, such that $\pi = \bigcup_s \psi_s$.

From these data, construct a word by concatenating the elements of I_0 (in increasing order) followed by the elements of I_1 (in increasing order), I_2 (in increasing order), and so on up to $I_{\lfloor \frac{n}{d} \rfloor}$, where the ordering of $I_1, \cdots, I_{\lfloor \frac{n}{d} \rfloor}$ themselves is determined by the order of their smallest elements. Then order the atoms by lexicographic order of these words; within atoms with the same word, use the order given by some arbitrarily chosen orderings of the sets $\Psi(I)$ for all d-element subsets I (applied lexicographically, so the ordering of $\Psi(I_1)$ is applied first, then in case of equality of ψ_1 the ordering of $\Psi(I_2)$ is applied, etc.).

We now prove that this ordering satisfies the condition (1.2). Let $a_j = (J,\pi)$ have associated I_s and ψ_s as above, let $y = (J',\pi') \in Q_n^{0 \bmod d}(G)$ be such that $(J',\pi') > (J,\pi)$, and suppose that (J',π') is not greater than any common cover of a_j and an earlier atom. We must deduce from this that (J,π) is the earliest atom which is $< (J',\pi')$. Firstly, let K be any part of π', and let $s_1 < \cdots < s_t$ be such that $\bigcup_{g \in G} g.K = G \times (I_{s_1} \cup \cdots \cup I_{s_t})$. Suppose that for some i, $a = \max(I_{s_i}) > \min(I_{s_{i+1}}) = b$. Let $g_a, g_b \in G$ be such that $(g_a, a), (g_b, b) \in K$. There is an element $w \in G \wr S_n$ defined by

$$w.(g,c) = \begin{cases} (g g_a^{-1} g_b, b), & \text{if } c = a, \\ (g g_b^{-1} g_a, a), & \text{if } c = b, \\ (g,c), & \text{if } c \neq a, b. \end{cases}$$

It is clear that $w.(J,\pi)$ is an earlier atom than (J,π), and their join is a common cover which is $\leq (J',\pi')$, contrary to assumption. Hence we must have $\max(I_{s_i}) < \min(I_{s_{i+1}})$ for all i. Thus the parts of π contained in K are simply those which one obtains by ordering the elements of K by their second component, and chopping that list into d-element sublists. By similar arguments (details omitted), one can show that I_0 must consist of the $(n - d\lfloor \frac{n}{d} \rfloor)$ smallest numbers occurring in the second components of elements of J', and that the parts of π contained in J' are those obtained by listing the remaining such numbers in increasing order, chopping that list into d-element sublists, and choosing for each resulting I_s the smallest element of $\Psi(I_s)$ (for the fixed order on this set). It is clear from this construction of (J,π) that it is the earliest atom which is $< (J',\pi')$. $\square$

We deduce via [16, Theorem 4.2.2] that $Q_n^{0 \bmod d}(G)$ is Cohen-Macaulay.

2 Statement of the main results

In this section, after introducing some necessary notation, we state our results on the character of $G \wr S_n$ on $\widetilde{H}_{l(P)-2}(\overline{P};\mathbb{Q})$ for each of the sub-posets P of $Q_n(G)$ defined in the previous section; here $\overline{P}$ denotes the 'proper part' $P \setminus \{\hat{0},\hat{1}\}$. Since the posets are Cohen-Macaulay, this is the only reduced homology group of $\overline{P}$ which can be nonzero. Hence $\dim \widetilde{H}_{l(P)-2}(\overline{P};\mathbb{Q}) = (-1)^{l(P)}\mu(P)$. (We follow the usual convention that $\widetilde{H}_{-1}(\emptyset;\mathbb{Q})$ is one-dimensional.)

Let G_* denote the set of conjugacy classes of G. Following [12, Chapter I, Appendix B], we introduce the polynomial ring $\Lambda_G := \mathbb{Q}[p_i(c)]$ in indeterminates $p_i(c)$, one for each positive integer i and conjugacy class $c \in G_*$. This ring is $\mathbb{N}$-graded by setting $\deg(p_i(c)) = i$. The character of a representation M of $G \wr S_n$ over $\mathbb{Q}$ is encapsulated in its **Frobenius characteristic**

$$\mathrm{ch}_{G\wr S_n}(M) := \frac{1}{|G|^n n!} \sum_{x \in G\wr S_n} \mathrm{tr}(x,M)\Psi(x), \tag{2.1}$$

which is a homogeneous element of Λ_G of degree n. The definition of the **cycle index** $\Psi(x)$ is $\prod_{i\geq 1, c\in G_*} p_i(c)^{a_i(c)}$ if x lies in the conjugacy class of elements with $a_i(c)$ cycles of length i and type c (see [loc. cit.]). For any $f \in \Lambda_G$, we write $f^\natural$ for its 'non-equivariant specialization', the element of $\mathbb{Q}[x]$ obtained from f by setting $p_1(\{1\})$ to x and all other $p_i(c)$ to 0. Clearly

$$\mathrm{ch}_{G\wr S_n}(M)^\natural = (\dim M)\frac{x^n}{|G|^n n!}. \tag{2.2}$$

When $G = \{1\}$ we write p_i for $p_i(\{1\})$, as usual in the theory of symmetric groups and symmetric functions.

Now it is well known that $\Lambda_{\{1\}}$ has an associative operation called **plethysm**, for which p_1 is an identity. Less well known is that Λ_G has a pair of 'plethystic actions' of $\Lambda_{\{1\}}$, one on the left and one on the right; in the terminology of [3], $\Lambda_{\{1\}}$ is a plethory, and Λ_G is a $\Lambda_{\{1\}}$–$\Lambda_{\{1\}}$–biring. The left plethystic action is an operation $\circ : \Lambda_{\{1\}} \times \Lambda_G \to \Lambda_G$, which is uniquely defined by:

1. for all $g \in \Lambda_G$, the map $\Lambda_{\{1\}} \to \Lambda_G : f \mapsto f \circ g$ is a homomorphism of $\mathbb{Q}$-algebras;

2. for any $i \geq 1$, the map $\Lambda_G \to \Lambda_G : g \mapsto p_i \circ g$ is a homomorphism of $\mathbb{Q}$-algebras;

3. $p_i \circ p_j(c) = p_{ij}(c)$.

This action implicitly appears in [11]. The more interesting right plethystic action, made explicit for the first time in [8, Section 5], is an operation $\circ : \Lambda_G \times \Lambda_{\{1\}} \to \Lambda_G$, which is uniquely defined by:

1. for all $g \in \Lambda_{\{1\}}$, the map $\Lambda_G \to \Lambda_G : f \mapsto f \circ g$ is a homomorphism of $\mathbb{Q}$-algebras;

2. for any $i \geq 1$, $c \in G_*$, the map $\Lambda_{\{1\}} \to \Lambda_G : g \mapsto p_i(c) \circ g$ is a homomorphism of $\mathbb{Q}$-algebras;

3. $p_i(c) \circ p_j = p_{ij}(c^j)$, where c^j denotes the conjugacy class of jth powers of elements of c.

If $G = \{1\}$ both these actions become the usual operation of plethysm. We have $(f \circ g) \circ h = f \circ (g \circ h)$ whenever f, g, h live in the right combination of $\Lambda_{\{1\}}$ and/or Λ_G for both sides to be defined; moreover, $p_1 \circ f = f \circ p_1 = f$ for all $f \in \Lambda_G$. Note that under the non-equivariant specialization, all cases of $\circ$ become simply the substitution of one polynomial in $\mathbb{Q}[x]$ into another. For the 'meaning' of these plethystic actions, see [8, Section 5].

Since our formulae use generating functions which combine $(G \wr S_n)$-modules for infinitely many n, we need to enlarge Λ_G to the formal power series ring $\mathbb{A}_G := \mathbb{Q}[\![p_i(c)]\!]$, which we give its usual topology (coming from the $\mathbb{N}$-filtration). We extend the non-equivariant specialization in the obvious way (that is, by continuity): for $f \in \mathbb{A}_G$, $f^\natural$ is an element of the formal power series ring $\mathbb{Q}[\![x]\!]$. Just as one cannot substitute a formal power series with nonzero constant term into another formal power series, the extensions of $\circ$ to this context require a slight restriction. Let $\mathbb{A}_{G,+}$ be the ideal of $\mathbb{A}_G$ consisting of elements whose degree-0 term vanishes. Then the left plethystic action extends to an operation $\circ : \mathbb{A}_{\{1\}} \times \mathbb{A}_{G,+} \to \mathbb{A}_G$, and the right plethystic action extends to an operation $\circ : \mathbb{A}_G \times \mathbb{A}_{\{1\},+} \to \mathbb{A}_G$. The associativity and identity properties continue to hold.

An important element of $\mathbb{A}_G$ is the sum of the characteristics of the trivial representations:

$$\mathrm{Exp}_G := \sum_{n \geq 0} \mathrm{ch}_{G \wr S_n}(\mathbf{1}) = \exp\Big(\sum_{\substack{i \geq 1 \\ c \in G_*}} \frac{|c| p_i(c)}{|G| i}\Big).$$

Clearly $\mathrm{Exp}_G^\natural = \exp(\frac{x}{|G|})$. We write $\mathrm{Exp}_{\{1\}} = \exp(\sum_{i \geq 1} \frac{p_i}{i})$ simply as Exp. It is well known that the plethystic inverse of $\mathrm{Exp} - 1$ in $\mathbb{A}_{\{1\},+}$ is

$$L := \sum_{d \geq 1} \frac{\mu(d)}{d} \log(1 + p_d). \tag{2.3}$$

In other words, $L \circ (\mathrm{Exp} - 1) = (\mathrm{Exp} - 1) \circ L = p_1$. With this notation, the famous result of Stanley that $\widetilde{H}_{n-3}(\overline{\Pi_n}; \mathbb{Q}) \cong \varepsilon_n \otimes \mathrm{Ind}_{\mu_n}^{S_n}(\psi)$, where ψ is a faithful character of the cyclic group μ_n generated by an n-cycle, can be rephrased as

$$p_1 + \sum_{n \geq 2} (-1)^{n-1} \mathrm{ch}_{S_n}(\widetilde{H}_{n-3}(\overline{\Pi_n}; \mathbb{Q})) = L. \tag{2.4}$$

(The proof of this fact will be recalled in Section 4.) We can rephrase [7, Corollary 2.2] in a similar way:

Theorem 2.1. (Hanlon) *In $\mathbb{A}_G$ we have the equation*

$$1 + \sum_{n \geq 1} (-1)^n \mathrm{ch}_{G \wr S_n}(\widetilde{H}_{n-2}(\overline{Q_n(G)}; \mathbb{Q})) = (\mathrm{Exp}_G \circ L)^{-1}.$$

We will give a new proof of Theorem 2.1 in Section 4. To see that this is equivalent to Hanlon's statement, note that

$$\begin{aligned}
(\mathrm{Exp}_G \circ L)^{-1} &= \exp(-\sum_{\substack{i \geq 1 \\ c \in G_*}} \frac{|c| p_i(c)}{|G| i}) \circ (\sum_{d \geq 1} \frac{\mu(d)}{d} \log(1 + p_d)) \\
&= \exp(-\sum_{\substack{i \geq 1 \\ d \geq 1 \\ c \in G_*}} \frac{|c| \mu(d)}{|G| i d} \log(1 + p_{id}(c^d))) \\
&= \prod_{\substack{l \geq 1 \\ c \in G_*}} (1 + p_l(c))^{F(l,c)},
\end{aligned}$$

where

$$F(l, c) := -\frac{1}{|G| l} \sum_{d | l} \mu(d) |\{g \in G \mid g^d \in c\}|, \tag{2.5}$$

which is easily equated with Hanlon's $F(l, c, 1)$. Applying $\natural$ to Theorem 2.1, we derive the non-equivariant version:

$$1 + \sum_{n \geq 1} (-1)^n \dim \widetilde{H}_{n-2}(\overline{Q_n(G)}; \mathbb{Q}) \frac{x^n}{|G|^n n!} = (1 + x)^{-1/|G|}, \tag{2.6}$$

which is equivalent to the well-known fact that

$$\dim \widetilde{H}_{n-2}(\overline{Q_n(G)}; \mathbb{Q}) = (|G| + 1)(2|G| + 1) \cdots ((n-1)|G| + 1). \tag{2.7}$$

Note that the $G = \{1\}$ special case of Theorem 2.1 is

$$1 + \sum_{n \geq 1} (-1)^n \mathrm{ch}_{S_n}(\widetilde{H}_{n-2}(\overline{Q_n(\{1\})}; \mathbb{Q})) = (1 + p_1)^{-1},$$

which can also be obtained by applying $\frac{\partial}{\partial p_1}$ to both sides of (2.4).

For the sub-poset $R_n(G)$, we have the following result, to be proved in Section 4.

Theorem 2.2. *In $\mathbb{A}_G$ we have the equation*

$$\sum_{n\geq 1}(-1)^n \mathrm{ch}_{G\wr S_n}(\widetilde{H}_{n-2}(\overline{R_n(G)};\mathbb{Q})) = 1 - \mathrm{Exp}_G \circ L.$$

The non-equivariant version is

$$\sum_{n\geq 1}(-1)^n \dim \widetilde{H}_{n-2}(\overline{R_n(G)};\mathbb{Q})\frac{x^n}{|G|^n n!} = 1-(1+x)^{1/|G|}, \tag{2.8}$$

which is equivalent to the result of Hultman ([9, Corollary 3.12]):

$$\dim \widetilde{H}_{n-2}(\overline{R_n(G)};\mathbb{Q}) = (|G|-1)(2|G|-1)\cdots((n-1)|G|-1). \tag{2.9}$$

Note also that the $G = \{1\}$ case of Theorem 2.2 is

$$\sum_{n\geq 1}(-1)^n \mathrm{ch}_{S_n}(\widetilde{H}_{n-2}(\overline{R_n(\{1\})};\mathbb{Q})) = -p_1.$$

This reflects the fact that for $n \geq 2$, $R_n(\{1\}) \setminus \{\hat{0},\hat{1}\} \cong \Pi_n \setminus \{\hat{0}\}$ is contractible. A more interesting consequence is:

Corollary 2.3. *For $n \geq 1$, $\widetilde{H}_{n-2}(\overline{Q_n(G)};\mathbb{Q})$ is isomorphic to*

$$\bigoplus_{\substack{m\geq 1\\ n_1,\cdots,n_m\geq 1\\ n_1+\cdots+n_m=n}} \mathrm{Ind}^{G\wr S_n}_{(G\wr S_{n_1})\times\cdots\times(G\wr S_{n_m})}(\widetilde{H}_{n_1-2}(\overline{R_{n_1}(G)};\mathbb{Q}) \boxtimes \cdots \boxtimes \widetilde{H}_{n_m-2}(\overline{R_{n_m}(G)};\mathbb{Q}))$$

as a representation of $G \wr S_n$.

Proof. From Theorems 2.1 and 2.2 we deduce that

$$\begin{aligned}
1 + &\sum_{n\geq 1}(-1)^n \mathrm{ch}_{G\wr S_n}(\widetilde{H}_{n-2}(\overline{Q_n(G)};\mathbb{Q})) \\
&= (1 - \sum_{n\geq 1}(-1)^n \mathrm{ch}_{G\wr S_n}(\widetilde{H}_{n-2}(\overline{R_n(G)};\mathbb{Q})))^{-1} \\
&= \sum_{m\geq 0}(\sum_{n\geq 1}(-1)^n \mathrm{ch}_{G\wr S_n}(\widetilde{H}_{n-2}(\overline{R_n(G)};\mathbb{Q})))^m \\
&= \sum_{\substack{m\geq 0\\ n_1,\cdots,n_m\geq 1}} (-1)^{n_1+\cdots+n_m} \prod_{i=1}^m \mathrm{ch}_{G\wr S_{n_i}}(\widetilde{H}_{n_i-2}(\overline{R_{n_i}(G)};\mathbb{Q})).
\end{aligned}$$

Since multiplication of Frobenius characteristics corresponds to induction product of representations ([12, Chapter I, Appendix B, (6.3)]), this gives the result. ☐

Corollary 2.3 can also be proved by applying a poset fibre theorem of Björner, Wachs, and Welker to the 'forgetful' poset map $Q_n(G) \to \Pi(\{0,1,\cdots,n\})$; see [16, (5.3.5)].

For $Q_n^{\sim}(G)$, assuming that $G \neq \{1\}$, we will prove the following result in Section 4:

Theorem 2.4. *In $\mathbb{A}_G$ we have the equation*

$$1+\sum_{n\geq 2}(-1)^n \mathrm{ch}_{G\wr S_n}(\widetilde{H}_{n-2}(\overline{Q_n^{\sim}(G)};\mathbb{Q})) = (1+\sum_{c\in G_*}\frac{|c|}{|G|}p_1(c))(\mathrm{Exp}_G \circ L)^{-1}.$$

The non-equivariant version is

$$1+\sum_{n\geq 2}(-1)^n \dim \widetilde{H}_{n-2}(\overline{Q_n^{\sim}(G)};\mathbb{Q})\frac{x^n}{|G|^n n!} = (1+\frac{x}{|G|})(1+x)^{-1/|G|}, \tag{2.10}$$

equivalent to the result which is well known at least for cyclic G (see [13, Corollary 6.86]):

$$\dim \widetilde{H}_{n-2}(\overline{Q_n^{\sim}(G)};\mathbb{Q}) = (n-1)(|G|-1)(|G|+1)(2|G|+1)\cdots((n-2)|G|+1). \tag{2.11}$$

A further consequence of Theorem 2.4 is:

Corollary 2.5. *For $n \geq 2$, $\widetilde{H}_{n-2}(\overline{Q_n(G)};\mathbb{Q})$ is isomorphic to*

$$\mathrm{Ind}_{G^n}^{G\wr S_n}(\mathbf{1}) \oplus \bigoplus_{k=0}^{n-2} \mathrm{Ind}_{G^k\times(G\wr S_{n-k})}^{G\wr S_n}(\mathbf{1}\boxtimes \widetilde{H}_{n-k-2}(\overline{Q_{n-k}^{\sim}(G)};\mathbb{Q}))$$

as a representation of $G\wr S_n$.

Proof. From Theorems 2.1 and 2.4 we deduce that

$$\begin{aligned}
1+&\sum_{n\geq 1}(-1)^n \mathrm{ch}_{G\wr S_n}(\widetilde{H}_{n-2}(\overline{Q_n(G)};\mathbb{Q}))\\
&= (1+\sum_{c\in G_*}\frac{|c|}{|G|}p_1(c))^{-1}(1+\sum_{m\geq 2}(-1)^m \mathrm{ch}_{G\wr S_m}(\widetilde{H}_{m-2}(\overline{Q_m^{\sim}(G)};\mathbb{Q}))\\
&= 1+\sum_{n\geq 1}(-1)^n \mathrm{ch}_{G\wr S_1}(\mathbf{1})^n + \sum_{\substack{n\geq 2\\ 0\leq k\leq n-2}} (-1)^n \mathrm{ch}_{G\wr S_1}(\mathbf{1})^k \mathrm{ch}_{G\wr S_{n-k}}(\widetilde{H}_{n-k-2}(\overline{Q_{n-k}^{\sim}(G)};\mathbb{Q})),
\end{aligned}$$

which implies the claim. □

Perhaps this Corollary too follows from a suitable poset fibre theorem.

To state the results for the Calderbank-Hanlon-Robinson-style sub-posets we need a bit more notation. For any $d \geq 2$, $\mathrm{Exp}_G^{0 \bmod d}$ denotes the sum of all terms of Exp_G whose degree is $\equiv 0 \bmod d$, and $\mathrm{Exp}_G^{\neq 0 \bmod d}$ denotes the sum of the other terms; we use similar notations with $0 \bmod d$ replaced by $1 \bmod d$, and with subscripts omitted when $G=\{1\}$. Since $\mathrm{Exp}^{1 \bmod d}$ is an element of $\mathbb{A}_{\{1\},+}$ whose degree-1 term is p_1, it has a unique two-sided plethystic inverse in $\mathbb{A}_{\{1\},+}$, which we write as $(\mathrm{Exp}^{1 \bmod d})^{[-1]}$.

In Section 4 we will prove the following.

Theorem 2.6. *In $\mathbb{A}_G$ we have the equation*

$$1+\sum_{n\geq 1}(-1)^{\lceil \frac{n}{d}\rceil}\mathrm{ch}_{G\wr S_n}(\widetilde{H}_{\lceil \frac{n}{d}\rceil-2}(\overline{Q_n^{1 \bmod d}(G)};\mathbb{Q})) \\ =[(1-\mathrm{Exp}_G^{\neq 0 \bmod d})(\mathrm{Exp}_G^{0 \bmod d})^{-1}]\circ(\mathrm{Exp}^{1 \bmod d})^{[-1]}.$$

In the $d=2$ case, the right-hand side could be written more suggestively as $(\mathrm{Sech}_G-\mathrm{Tanh}_G)\circ\mathrm{Arcsinh}$. The non-equivariant version of this special case is:

$$1+\sum_{n\geq 1}(-1)^{\lceil \frac{n}{2}\rceil}\dim\widetilde{H}_{\lceil \frac{n}{2}\rceil-2}(\overline{Q_n^{1 \bmod 2}(G)};\mathbb{Q})\frac{x^n}{|G|^n n!} \\ =\mathrm{sech}(\frac{1}{|G|}\mathrm{arcsinh}(x))-\tanh(\frac{1}{|G|}\mathrm{arcsinh}(x)). \tag{2.12}$$

The $G=\{\pm 1\}$ special case of this is equivalent to:

$$\dim\widetilde{H}_{\lceil \frac{n}{2}\rceil-2}(\overline{Q_n^{1 \bmod 2}(\{\pm 1\})};\mathbb{Q})=\begin{cases}\frac{(2n)!}{2^n(n+1)!}, & \text{when } n \text{ is even,}\\ \frac{n!(n-1)!}{(\frac{n+1}{2})!(\frac{n-1}{2})!}, & \text{when } n \text{ is odd.}\end{cases} \tag{2.13}$$

Note that the $G=\{1\}$ special case of Theorem 2.6 can also be obtained by applying $\frac{\partial}{\partial p_1}$ to both sides of [4, Theorem 4.7], which in our notation is

$$p_1+\sum_{n\geq 2}(-1)^{\lceil \frac{n-1}{d}\rceil}\mathrm{ch}_{S_n}(\widetilde{H}_{\lceil \frac{n-1}{d}\rceil-2}(\overline{\Pi_n^{(1,d)}};\mathbb{Q}))=(1+p_1-\mathrm{Exp}^{\neq 1 \bmod d})\circ(\mathrm{Exp}^{1 \bmod d})^{[-1]}. \tag{2.14}$$

Finally, we have the result for the 'd-divisible' sub-poset of the Dowling lattice, also to be proved in Section 4.

Theorem 2.7. *In $\mathbb{A}_G$ we have the equation*

$$\sum_{n\geq 1}(-1)^{\lfloor \frac{n}{d}\rfloor+1}\mathrm{ch}_{G\wr S_n}(\widetilde{H}_{\lfloor \frac{n}{d}\rfloor-1}(\overline{Q_n^{0 \bmod d}(G)};\mathbb{Q}))=1-\mathrm{Exp}_G\cdot(\mathrm{Exp}_G\circ L\circ(\mathrm{Exp}^{0 \bmod d}-1))^{-1}.$$

The non-equivariant version of the $d=2$ case is:

$$\sum_{n\geq 1}(-1)^{\lfloor \frac{n}{2}\rfloor+1}\dim\widetilde{H}_{\lfloor \frac{n}{2}\rfloor-1}(\overline{Q_n^{0 \bmod 2}(G)};\mathbb{Q})\frac{x^n}{|G|^n n!}=1-(1+\tanh(x))^{1/|G|}. \tag{2.15}$$

Note that in the $G=\{1\}$ case of Theorem 2.7, the expression $\mathrm{Exp}\circ L$ collapses to $1+p_1$, so we have

$$\sum_{n\geq 1}(-1)^{\lfloor \frac{n}{d}\rfloor+1}\mathrm{ch}_{S_n}(\widetilde{H}_{\lfloor \frac{n}{d}\rfloor-1}(\overline{Q_n^{0 \bmod d}(\{1\})};\mathbb{Q}))=1-\frac{\mathrm{Exp}}{\mathrm{Exp}^{0 \bmod d}}.$$

Taking only the terms of degree $\equiv -1 \bmod d$ in this formula, we get

$$\sum_{\substack{n\geq d-1\\ n\equiv -1 \bmod d}}(-1)^{\frac{n+1}{d}}\mathrm{ch}_{S_n}(\widetilde{H}_{\frac{n+1}{d}-2}(\overline{Q_n^{0 \bmod d}(\{1\})};\mathbb{Q}))=-\frac{\mathrm{Exp}^{-1 \bmod d}}{\mathrm{Exp}^{0 \bmod d}}.$$

This can also be obtained by applying $\frac{\partial}{\partial p_1}$ to both sides of [4, Corollary 4.7], which in our notation is

$$\sum_{\substack{n\geq d\\ n\equiv 0 \bmod d}} (-1)^{\frac{n}{d}} \mathrm{ch}_{S_n}(\widetilde{H}_{\frac{n}{d}-2}(\overline{\Pi_n^{(0,d)}};\mathbb{Q})) = -L\circ(\mathrm{Exp}^{0 \bmod d}-1). \tag{2.16}$$

The fact that Theorem 2.7 for nontrivial G does not simplify to the same extent perhaps implies that no Dowling lattice analogue of [4, Theorem 6.5] can be found.

3 $(G \wr \mathbb{S})$-modules

The technical tool we will use to prove the Theorems stated in the previous section is an extension of Joyal's theory of tensor species (for which see [10] or the textbook [1]) to the case of wreath products $G \wr S_n$. This was introduced in [8, Section 5] (as observed at the end of that section, the assumption that G is cyclic is unnecessary). We recall the main points here. All vector spaces and representations are over $\mathbb{Q}$.

Define the category $\mathbf{B}_G$ whose objects are finite sets equipped with a **free** action of G, and whose morphisms are G-equivariant **bijections**. Thus every object in $\mathbf{B}_G$ is isomorphic to $G\times[n]$ for a unique $n\in\mathbb{N}$. The automorphism group of $G\times[n]$ in $\mathbf{B}_G$ is the wreath product $G\wr S_n$. A **tensor species**, also known as an **$\mathbb{S}$-module**, is a functor from $\mathbf{B}_{\{1\}}$ to vector spaces; the natural generalization is as follows.

Definition 3.1. A $(G\wr\mathbb{S})$**-module** (over $\mathbb{Q}$) is a functor U from $\mathbf{B}_G$ to the category of finite-dimensional vector spaces over $\mathbb{Q}$. That is, to each finite set I with a free G-action it associates a finite-dimensional vector space $U(I)$, and to each G-equivariant bijection $f: I \xrightarrow{\sim} J$ between such sets it associates an isomorphism $U(f): U(I)\xrightarrow{\sim} U(J)$.

(In [8], where G was cyclic of order r, I called this a $\mathbf{B}_r$-module.)

Clearly any $(G\wr\mathbb{S})$-module U gives rise to a sequence $(U(G\times[n]))_{n\geq 0}$ of representations of the various wreath products $G\wr S_n$; moreover, U is determined up to isomorphism (in the usual sense of isomorphism for functors) by this sequence of representations. This means that U is determined up to isomorphism by its **character**

$$\mathrm{ch}(U) := \sum_{n\geq 0}\mathrm{ch}_{G\wr S_n}(U(G\times[n]))\in\mathbb{A}_G. \tag{3.1}$$

The convenience of defining U as a functor rather than just a sequence of representations will become clear shortly.

An important example is the trivial $(G\wr\mathbb{S})$-module $\mathbf{1}_G$, which is defined by $\mathbf{1}_G(I)=\mathbb{Q}$ for all objects I of $\mathbf{B}_G$, and $\mathbf{1}_G(f)=\mathrm{id}$ for all morphisms f of $\mathbf{B}_G$. Clearly $\mathbf{1}_G(G\times[n])$ is the trivial representation of $G\wr S_n$, so $\mathrm{ch}(\mathbf{1}_G)=\mathrm{Exp}_G$.

For any $(G\wr\mathbb{S})$-module U we can define various sub-$(G\wr\mathbb{S})$-modules by imposing a restriction on degree, which we will write as a superscript. For instance, $U^{1 \bmod d}$ is the $(G\wr\mathbb{S})$-module defined by $U^{1 \bmod d}(I)=U(I)$ when $\frac{|I|}{|G|}\equiv 1 \bmod d$, and $U^{1 \bmod d}(I)=0$

when $\frac{|I|}{|G|} \not\equiv 1 \bmod d$; the definition of $U^{1 \bmod d}$ on morphisms of $\mathbf{B}_G$ is the same as that of U when the morphisms are of the form $f : I \xrightarrow{\sim} J$ for $\frac{|I|}{|G|} = \frac{|J|}{|G|} \equiv 1 \bmod d$, and zero otherwise. Clearly $\mathrm{ch}(U^{1 \bmod d})$ is the sum of all terms of $\mathrm{ch}(U)$ whose degree is $\equiv 1 \bmod d$. Similarly we define $U^{0 \bmod d}$, $U^{\neq 1 \bmod d}$, $U^{\neq 0 \bmod d}$, and $U^{\geq 1}$.

Now the main point of Joyal's theory is that certain natural operations of tensor species, called sum, product, and substitution (or partitional composition), correspond to the analogous operations on their characters; the analogue of substitution is plethysm. These operations can be extended to our context as follows. We define the sum and product of two $(G \wr \mathbb{S})$-modules U and V by

$$\begin{aligned}(U+V)(I) &= U(I) \oplus V(I),\\ (U \cdot V)(I) &= \bigoplus_{\substack{J \subseteq I\\ G\text{-stable}}} U(J) \otimes V(I \setminus J),\end{aligned} \tag{3.2}$$

for any object I of $\mathbf{B}_G$; the definition on morphisms is the obvious one. (It is clear that if J is a G-stable subset of an object of $\mathbf{B}_G$, then both J and $I \setminus J$ are objects of $\mathbf{B}_G$.)

If U is an $\mathbb{S}$-module and V is a $(G \wr \mathbb{S})$-module such that $V(\emptyset) = 0$, we define the substitution $U \circ V$, a $(G \wr \mathbb{S})$-module, by

$$(U \circ V)(I) = \bigoplus_{\substack{\pi \in \Pi(I)\\ \text{partwise } G\text{-stable}}} \left(U(\pi) \otimes \bigotimes_{J \in \pi} V(J) \right) \tag{3.3}$$

for any object I of $\mathbf{B}_G$. Here $\pi \in \Pi(I)$ is 'partwise G-stable' if $g.J = J$ for all $g \in G$ and all parts $J \in \pi$. The definition on morphisms is the obvious one.

If U is a $(G \wr \mathbb{S})$-module and V is an $\mathbb{S}$-module such that $V(\emptyset) = 0$, we define the substitution $U \circ V$, a $(G \wr \mathbb{S})$-module, by

$$(U \circ V)(I) = \bigoplus_{\substack{\pi \in \Pi(I)\\ \pi \in \mathbf{B}_G}} \left(U(\pi) \otimes \bigotimes_{\mathcal{O} \in G \backslash \pi} V(\mathcal{O}) \right) \tag{3.4}$$

for any object I of $\mathbf{B}_G$. Here the condition $\pi \in \mathbf{B}_G$ means just that G acts freely on the set of parts of the partition, i.e. for all $1 \neq g \in G$ and $J \in \pi$, $J \neq g.J \in \pi$. If $\mathcal{O}$ is a G-orbit on the set of parts, $V(\mathcal{O})$ should be thought of as $V(J)$ for some $J \in \mathcal{O}$, the choice making no difference up to isomorphism; but in order to be able to repeat the mantra that "the definition on morphisms is the obvious one", we must make the more canonical definition that

$$V(\mathcal{O}) := \{(v_J) \in \prod_{J \in \mathcal{O}} V(J) \mid v_{g.J} = V(g|_J)(v_J),\ \forall J \in \mathcal{O}, g \in G\}.$$

By functoriality and freeness, the choice of one v_J uniquely determines the whole $\mathcal{O}$-tuple.

We have the following generalization of Joyal's result (which is the case $G = \{1\}$).

Theorem 3.2. *1. If U and V are $(G \wr \mathbb{S})$-modules,*

$$\mathrm{ch}(U+V) = \mathrm{ch}(U) + \mathrm{ch}(V), \ \mathrm{ch}(U \cdot V) = \mathrm{ch}(U)\mathrm{ch}(V).$$

2. If U is an $\mathbb{S}$-module and V a $(G \wr \mathbb{S})$-module such that $V(\emptyset) = 0$,

$$\mathrm{ch}(U \circ V) = \mathrm{ch}(U) \circ \mathrm{ch}(V).$$

3. If U is a $(G \wr \mathbb{S})$-module and V an $\mathbb{S}$-module such that $V(\emptyset) = 0$,

$$\mathrm{ch}(U \circ V) = \mathrm{ch}(U) \circ \mathrm{ch}(V).$$

Proof. See [8, Proposition 5.1, Theorems 5.6 and 5.9] for a proof using analytic functors (the assumption there that G was cyclic was never actually used, except in the trivial sense that conjugacy classes were replaced by elements in the notation). $\square$

Since the representations we want to apply this to are homology groups of posets, we will need to have a 'super' version of the above. So we define a **super-$(G \wr \mathbb{S})$-module** to be a functor from $\mathbf{B}_G$ to the category of finite-dimensional $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces over $\mathbb{Q}$. If U is a super-$(G \wr \mathbb{S})$-module, define its super-character

$$\mathrm{sch}(U) := \sum_{n \geq 0} \mathrm{ch}_{G \wr S_n}(U(G \times [n])_{\bar{0}}) - \mathrm{ch}_{G \wr S_n}(U(G \times [n])_{\bar{1}}). \tag{3.5}$$

Any $(G \wr \mathbb{S})$-module U may be viewed as a super-$(G \wr \mathbb{S})$-module which is purely even, so that $\mathrm{sch}(U) = \mathrm{ch}(U)$. The above definitions of sum, product, and substitution can be carried over to the super context, using the usual sign-commutativity convention for tensor products.

Theorem 3.3. *1. If U and V are super-$(G \wr \mathbb{S})$-modules,*

$$\mathrm{sch}(U+V) = \mathrm{sch}(U) + \mathrm{sch}(V), \ \mathrm{sch}(U \cdot V) = \mathrm{sch}(U)\mathrm{sch}(V).$$

2. If U is a super-$\mathbb{S}$-module and V a super-$(G \wr \mathbb{S})$-module such that $V(\emptyset) = 0$,

$$\mathrm{sch}(U \circ V) = \mathrm{sch}(U) \circ \mathrm{sch}(V).$$

3. If U is a super-$(G \wr \mathbb{S})$-module and V a super-$\mathbb{S}$-module such that $V(\emptyset) = 0$,

$$\mathrm{sch}(U \circ V) = \mathrm{sch}(U) \circ \mathrm{sch}(V).$$

Proof. A more complicated version of this theorem was proved in [8, Section 7]. To deduce the present version, set $q \to 1$ in [8, Corollaries 7.3 and 7.6] to obtain (1) and (3); the analogue of (2) was not stated there but follows by the same method. $\square$

4 Proof of the main results

In this section we prove Theorems 2.1, 2.2, 2.4, 2.6 and 2.7. To make our arguments more legible, we need a notational convention: for any Cohen-Macaulay poset P and elements $x < y$, we write $\widetilde{H}_P(x,y)$ for the top-degree reduced homology $\widetilde{H}_{\ell(x,y)-2}((x,y);\mathbb{Q})$ of the open interval $(x,y) \subset P$. We view this as a super vector space of parity equal to that of $\ell(x,y)$. Since the top degree is the only one in which the reduced homology could be nonzero, we have

$$\operatorname{sdim} \widetilde{H}_P(x,y) = (-1)^{\ell(x,y)} \dim \widetilde{H}_P(x,y) = \mu_P(x,y). \tag{4.1}$$

We also define $\widetilde{H}_P(x,x)$ to be a one-dimensional even super vector space, so that (4.1) holds when $x = y$. The key fact we use, an application of the Euler-Poincaré principle observed by Sundaram (see [15, Section 1] and [16, Theorem 4.4.1]), is as follows:

Theorem 4.1. *If P is a Cohen-Macaulay pure bounded poset where $\hat{0} \neq \hat{1}$ (i.e. P has more than one element), then $\bigoplus_{x\in P} \widetilde{H}_P(\hat{0},x)$ is balanced; that is, its odd and even parts are isomorphic, as vector spaces and as representations of any group that acts on P. The same is true for $\bigoplus_{x\in P} \widetilde{H}_P(x,\hat{1})$.*

Of course the second statement is just the first applied to the dual poset. (Taking sdim, we recover the familiar recursive properties of the Möbius function.)

In order to apply the theory of the previous section, we need to define our posets functorially. We illustrate by rewriting Sundaram's proof of Stanley's theorem (2.4) on the partition lattices (see [16, Theorem 4.4.7]). We have already defined $\Pi(I)$ as the lattice of set partitions of the set I; with an obvious definition on morphisms, this constitutes a functor Π from $\mathbf{B}_{\{1\}}$ to the category of posets. We then define three related super-$\mathbb{S}$-modules, $\widetilde{H}_\Pi$, WH_Π, and WH_Π^*. The definitions on objects of $\mathbf{B}_{\{1\}}$ are:

$$\begin{aligned} \widetilde{H}_\Pi(I) &= \widetilde{H}_{\Pi(I)}(\hat{0},\hat{1}), \\ WH_\Pi(I) &= \bigoplus_{\pi\in\Pi(I)} \widetilde{H}_{\Pi(I)}(\hat{0},\pi), \text{ and} \\ WH_\Pi^*(I) &= \bigoplus_{\pi\in\Pi(I)} \widetilde{H}_{\Pi(I)}(\pi,\hat{1}), \end{aligned}$$

and the definitions on morphisms are the obvious ones. (By our convention that $\Pi(\emptyset)$ has one element, $\widetilde{H}_\Pi(\emptyset)$, $WH_\Pi(\emptyset)$, and $WH_\Pi^*(\emptyset)$ are one-dimensional and of even parity.) Sundaram's argument rests on the recursive property of partition lattices, namely that for any $\pi \in \Pi_n$, the principal upper order ideal $[\pi,\hat{1}]$ is isomorphic to $\Pi_{|\pi|}$. With functorial language we can state this more precisely: for any $\pi \in \Pi(I)$, $[\pi,\hat{1}]$ is **canonically** isomorphic to $\Pi(\pi)$. Recalling the definition of substitution from the previous section, we see that we have an isomorphism of super-$\mathbb{S}$-modules:

$$WH_\Pi^* \cong \widetilde{H}_\Pi \circ \mathbf{1}_{\{1\}}^{\geq 1}. \tag{4.2}$$

Taking sch and applying the $G=\{1\}$ case of Theorem 3.3, we get

$$\operatorname{sch}(WH_\Pi^*)=\operatorname{sch}(\widetilde{H}_\Pi)\circ(\operatorname{Exp}-1).$$

Now by Theorem 4.1, $\operatorname{sch}(WH_\Pi^*)=1+p_1$ (only the one-element posets contribute). We deduce that $\operatorname{sch}(\widetilde{H}_\Pi)=1+L$, which is exactly (2.4). A slightly more complicated argument uses lower order ideals: for any $\pi\in\Pi(I)$, $[\hat{0},\pi]$ is canonically isomorphic to $\prod_{K\in\pi}\Pi(K)$. Applying the Künneth formula [16, second statement of Theorem 5.1.5], we see that we have an isomorphism of super-$\mathbb{S}$-modules:

$$WH_\Pi\cong\mathbf{1}_{\{1\}}\circ\widetilde{H}_\Pi^{\geq 1},\tag{4.3}$$

and the result follows as before. Notice how the sign convention in the definition of substitution of super-$\mathbb{S}$-modules takes into account the sign-commutativity of the Künneth formula.

With these arguments as models, we turn to our sub-posets of the Dowling lattices. First we have to reinterpret them as functors Q, R, $Q^{\sim}$, $Q^{1\,\mathrm{mod}\,d}$, $Q^{0\,\mathrm{mod}\,d}$ from $\mathbf{B}_G$ to the category of posets, so that $Q(G\times[n])=Q_n(G)$, $R(G\times[n])=R_n(G)$, and so on. The definitions in Section 1 were deliberately written so that this is simply a matter of replacing $G\times[n]$ with a general object I of $\mathbf{B}_G$. (Following usual conventions, this will result in $Q(\emptyset)$, $R(\emptyset)$, $Q^{\sim}(\emptyset)$, $Q^{1\,\mathrm{mod}\,d}(\emptyset)$, and $Q^{0\,\mathrm{mod}\,d}(\emptyset)$ all being one-element posets; we also stipulate that when $\frac{|J|}{|G|}=1$, $Q^{\sim}(J)$ is a one-element poset.) We then define three super-$(G\wr\mathbb{S})$-modules attached to each functor, as with Π: $\widetilde{H}_Q$, WH_Q, WH_Q^*, $\widetilde{H}_R$, WH_R, and WH_R^*, and so forth.

Proof. **(Theorem 2.1)** For any $(J,\pi)\in Q(I)$, $[(J,\pi),\hat{1}]$ is isomorphic to $Q(\pi)$, so

$$WH_Q^*(I)\cong\bigoplus_{\substack{J\subseteq I\\ G\text{-stable}}}\ \bigoplus_{\substack{\pi\in\Pi(I\setminus J)\\ \pi\in\mathbf{B}_G}}\widetilde{H}_Q(\pi).$$

Clearly this amounts to an isomorphism of super-$(G\wr\mathbb{S})$-modules:

$$WH_Q^*\cong\mathbf{1}_G\cdot(\widetilde{H}_Q\circ\mathbf{1}_{\{1\}}^{\geq 1}).\tag{4.4}$$

Taking sch and applying parts (1) and (3) of Theorem 3.3, we get

$$\operatorname{sch}(WH_Q^*)=\operatorname{Exp}_G\cdot(\operatorname{sch}(\widetilde{H}_Q)\circ(\operatorname{Exp}-1)).$$

Now by Theorem 4.1, $\operatorname{sch}(WH_Q^*)=1$ (only the one-element poset $Q(\emptyset)$ contributes). Hence

$$\operatorname{sch}(\widetilde{H}_Q)=\operatorname{Exp}_G^{-1}\circ L=(\operatorname{Exp}_G\circ L)^{-1},\tag{4.5}$$

which is exactly the statement. For reference, we also give the alternative proof using WH_Q. For any $(J,\pi)\in Q(I)$, $[\hat{0},(J,\pi)]$ is isomorphic to $Q(J)\times\prod_{\mathcal{O}\in G\setminus\pi}\Pi(K_{\mathcal{O}})$ where $K_{\mathcal{O}}$ denotes a representative of the orbit $\mathcal{O}$, so by the same Künneth formula as above,

$$WH_Q(I)\cong\bigoplus_{\substack{J\subseteq I\\ G\text{-stable}}}\widetilde{H}_Q(J)\otimes\Big(\bigoplus_{\substack{\pi\in\Pi(I\setminus J)\\ \pi\in\mathbf{B}_G}}\ \bigotimes_{\mathcal{O}\in G\setminus\pi}\widetilde{H}_\Pi(K_{\mathcal{O}})\Big).$$

Clearly this amounts to an isomorphism of super-$(G \wr \mathbb{S})$-modules:

$$WH_Q \cong \widetilde{H}_Q \cdot (\mathbf{1}_G \circ \widetilde{H}_\Pi^{\geq 1}). \tag{4.6}$$

Using $\mathrm{sch}(WH_Q) = 1$ and $\mathrm{sch}(\widetilde{H}_\Pi^{\geq 1}) = L$, we reach the result again. $\square$

Proof. **(Theorem 2.2)** For any $(\emptyset, \pi) \in R(I)$, $[(\emptyset, \pi), \hat{1}]$ is isomorphic to $R(\pi)$. Hence we have an isomorphism of super-$(G \wr \mathbb{S})$-modules:

$$WH_R^* \cong \mathbf{1}_G^{\geq 1} + \widetilde{H}_R \circ \mathbf{1}_{\{1\}}^{\geq 1}, \tag{4.7}$$

where the first term comes from the $\widetilde{H}_{R(I)}(\hat{1}, \hat{1})$ terms. Taking sch and applying parts (1) and (3) of Theorem 3.3, we get

$$\mathrm{sch}(WH_R^*) = \mathrm{Exp}_G - 1 + \mathrm{sch}(\widetilde{H}_R) \circ (\mathrm{Exp} - 1).$$

Now by Theorem 4.1, $\mathrm{sch}(WH_R^*) = 1$ (only the one-element poset $R(\emptyset)$ contributes). Hence

$$\mathrm{sch}(\widetilde{H}_R) = (2 - \mathrm{Exp}_G) \circ L = 2 - \mathrm{Exp}_G \circ L, \tag{4.8}$$

which (subtracting 1 from both sides) is exactly the statement. The alternative proof would use the isomorphism

$$WH_R \cong \widetilde{H}_R^{\geq 1} + \mathbf{1}_G \circ \widetilde{H}_\Pi^{\geq 1} \tag{4.9}$$

of super-$(G \wr \mathbb{S})$-modules. $\square$

Proof. **(Theorem 2.4)** Here the proof via $WH_{Q^\sim}$ is more convenient. For any $(J, \pi) \in Q^\sim(I)$, $[\hat{0}, (J, \pi)] \cong Q^\sim(J) \times \prod_{\mathcal{O} \in G \backslash \pi} \Pi(K_\mathcal{O})$, so the analogue of (4.6) is

$$WH_{Q^\sim} \cong \widetilde{H}_{Q^\sim}^{\neq 1} \cdot (\mathbf{1}_G \circ \widetilde{H}_\Pi^{\geq 1}). \tag{4.10}$$

Taking sch and applying parts (1) and (3) of Theorem 3.3, we get

$$\mathrm{sch}(WH_{Q^\sim}) = \mathrm{sch}(\widetilde{H}_{Q^\sim})^{\neq 1} \cdot (\mathrm{Exp}_G \circ L).$$

Now by Theorem 4.1, $\mathrm{sch}(WH_{Q^\sim}) = 1 + \sum_{c \in G_*} \frac{|c|}{|G|} p_1(c)$ (only the one-element posets $Q^\sim(\emptyset)$ and $Q^\sim(G \times [1])$ contribute). Hence

$$\mathrm{sch}(\widetilde{H}_{Q^\sim})^{\neq 1} = (1 + \sum_{c \in G_*} \frac{|c|}{|G|} p_1(c))(\mathrm{Exp}_G \circ L)^{-1}, \tag{4.11}$$

which is exactly the statement. $\square$

Proof. **(Theorem 2.6)** For any $(J,\pi) \in Q^{1 \bmod d}(I)$, $[(J,\pi),\hat{1}]$ is isomorphic to $Q^{1 \bmod d}(\pi)$. So we have an isomorphism of super-$(G \wr \mathbb{S})$-modules:

$$WH^*_{Q^{1 \bmod d}} \cong \mathbf{1}_G^{\not\equiv 0 \bmod d} + \mathbf{1}_G^{0 \bmod d} \cdot (\widetilde{H}_{Q^{1 \bmod d}} \circ \mathbf{1}^{1 \bmod d}), \tag{4.12}$$

where the two terms cover the cases of $\frac{|J|}{|G|} \not\equiv 0 \bmod d$ (forcing $J = I$) and $\frac{|J|}{|G|} \equiv 0 \bmod d$ respectively. Taking sch as usual, we get

$$\mathrm{sch}(WH^*_{Q^{1 \bmod d}}) = \mathrm{Exp}_G^{\not\equiv 0 \bmod d} + \mathrm{Exp}_G^{0 \bmod d}(\mathrm{sch}(\widetilde{H}_{Q^{1 \bmod d}}) \circ \mathrm{Exp}^{1 \bmod d}).$$

Now by Theorem 4.1, $\mathrm{sch}(WH^*_{Q^{1 \bmod d}}) = 1$. Hence

$$\mathrm{sch}(\widetilde{H}_{Q^{1 \bmod d}}) = [(1 - \mathrm{Exp}_G^{\not\equiv 0 \bmod d})(\mathrm{Exp}_G^{0 \bmod d})^{-1}] \circ (\mathrm{Exp}^{1 \bmod d})^{[-1]}, \tag{4.13}$$

which is exactly the statement. The alternative proof would use the isomorphism

$$WH_{Q^{1 \bmod d}} \cong \widetilde{H}^{\not\equiv 0 \bmod d}_{Q^{1 \bmod d}} + \widetilde{H}^{0 \bmod d}_{Q^{1 \bmod d}} \cdot (\mathbf{1}_G \circ \widetilde{H}^{1 \bmod d}_{\Pi^{1 \bmod d}}), \tag{4.14}$$

and the Calderbank-Hanlon-Robinson result (2.14), which implies that $\mathrm{sch}(\widetilde{H}^{1 \bmod d}_{\Pi^{1 \bmod d}}) = (\mathrm{Exp}^{1 \bmod d})^{[-1]}$. (Here $\Pi^{1 \bmod d}$ denotes the functor from $\mathbf{B}_{\{1\}}$ to the category of posets such that $\Pi^{1 \bmod d}([n])$ is the poset $\Pi_n^{(1,d)}$ studied in [4].) $\square$

Proof. **(Theorem 2.7)** For any $(J,\pi) \in Q^{0 \bmod d}(I)$ except $\hat{0}$, $[(J,\pi),\hat{1}]$ is isomorphic to $Q(\pi)$. So we have an isomorphism of super-$(G \wr \mathbb{S})$-modules:

$$WH^*_{Q^{0 \bmod d}} \cong \widetilde{H}^{\geq 1}_{Q^{0 \bmod d}} + \mathbf{1}_G \cdot (\widetilde{H}_Q \circ \mathbf{1}^{0 \bmod d, \geq d}), \tag{4.15}$$

where the two terms cover respectively the cases where $(J,\pi) = \hat{0}$, $I \neq \emptyset$, and where $|K| \equiv 0 \bmod d$ for all $K \in \pi$. Taking sch and using Theorem 2.1, we get

$$\mathrm{sch}(WH^*_{Q^{0 \bmod d}}) = \mathrm{sch}(\widetilde{H}_{Q^{0 \bmod d}}) - 1 + \mathrm{Exp}_G \cdot ((\mathrm{Exp}_G \circ L)^{-1} \circ (\mathrm{Exp}^{0 \bmod d} - 1)).$$

Now by Theorem 4.1, $\mathrm{sch}(WH^*_{Q^{0 \bmod d}}) = 1$. Hence

$$\mathrm{sch}(\widetilde{H}_{Q^{0 \bmod d}}) = 2 - \mathrm{Exp}_G \cdot (\mathrm{Exp}_G \circ L \circ (\mathrm{Exp}^{0 \bmod d} - 1))^{-1}, \tag{4.16}$$

which (subtracting 1 from both sides) is exactly the statement. The alternative proof via $WH_{Q^{0 \bmod d}}$ requires a bit of care, since for $\hat{0} \neq (J,\pi) \in Q^{0 \bmod d}(I)$, it is not the closed interval $[\hat{0},(J,\pi)]$ but rather the semi-closed interval $(\hat{0},(J,\pi)]$ which is naturally a product of smaller posets. Hence one must use the 'once-suspended' Künneth formula [16, first statement of Theorem 5.1.5]. The upshot is the following isomorphism of super-$(G \wr \mathbb{S})$-modules, where $U[1]$ denotes U with parities interchanged:

$$WH_{Q^{0 \bmod d}}[1] \cong \mathbf{1}_G[1] + \mathbf{1}_G^{\geq 1} \circ (\widetilde{H}^{0 \bmod d, \geq d}_{\Pi^{0 \bmod d}}[1]) + \widetilde{H}^{\geq 1}_{Q^{0 \bmod d}}[1] \cdot (\mathbf{1}_G \circ \widetilde{H}^{0 \bmod d, \geq d}_{\Pi^{0 \bmod d}}[1]). \tag{4.17}$$

Here the first term corresponds to the minimum element $\hat{0}$, the second to the non-minimum elements of the form $(\emptyset,\pi)$, and the third to the elements of the form (J,π) with $J\neq\emptyset$; $\Pi^{0 \bmod d}$ denotes the functor from $\mathbf{B}_{\{1\}}$ to the category of posets such that $\Pi^{0 \bmod d}([n])$ is the poset $\Pi_n^{(0,d)}$ studied in [4]. Now the Calderbank-Hanlon-Robinson result (2.16) implies $\mathrm{sch}(\widetilde{H}^{0 \bmod d,\geq d}_{\Pi^{0 \bmod d}}[1]) = L\circ(\mathrm{Exp}^{0 \bmod d}-1)$, so we obtain

$$-1 = -\mathrm{Exp}_G + (\mathrm{Exp}_G - 1)\circ L\circ(\mathrm{Exp}^{0 \bmod d}-1) + (1-\mathrm{sch}(\widetilde{H}_{Q^{0 \bmod d}}))(\mathrm{Exp}_G\circ L\circ(\mathrm{Exp}^{0 \bmod d}-1)),$$

which simplifies to (4.16) again. □

5 Whitney homology

The name WH for the super-$(G\wr\mathbb{S})$-modules used in the previous section of course stands for 'Whitney homology'; but properly speaking, the Whitney homology of a poset has a $\mathbb{Z}$-grading, not a $(\mathbb{Z}/2\mathbb{Z})$-grading. Recall that if P is a Cohen-Macaulay poset with minimum element $\hat{0}$, its (rational) Whitney homology groups are defined by

$$WH_i(P) := \bigoplus_{\substack{x\in P\\ \mathrm{rk}(x)=i}} \widetilde{H}_P(\hat{0},x), \text{ for all } i\in\mathbb{Z}.$$

If a group Γ acts on the poset P, it also acts on each $WH_i(P)$, and the characters of these Whitney homology representations encapsulate the 'equivariant characteristic polynomials':

$$\sum_{i\in\mathbb{Z}} \mathrm{tr}(\gamma, WH_i(P))\,(-t)^i = \sum_{x\in P^\gamma} \mu_{P^\gamma}(\hat{0},x)\,t^{\mathrm{rk}_P(x)}, \text{ for all } \gamma\in\Gamma. \tag{5.1}$$

It is an easy matter to find formulae analogous to those in §2 for these characters in the case of of our posets, since we have effectively already worked out the relationship between the Whitney homology and its highest-degree part.

We need to introduce the concept of a **graded $(G\wr\mathbb{S})$-module**, which is merely a functor from $\mathbf{B}_G$ to the category of finite-dimensional $\mathbb{Z}$-graded vector spaces over $\mathbb{Q}$. Any $(G\wr\mathbb{S})$-module may be regarded as a graded $(G\wr\mathbb{S})$-module concentrated in degree 0. If U is a graded $(G\wr\mathbb{S})$-module, we define its graded character to be

$$\mathrm{ch}_t(U) := \sum_{n\geq 0}\sum_{i\in\mathbb{Z}} \mathrm{ch}_{G\wr S_n}(U(G\times[n])_i)\,(-t)^i,$$

an element of $\mathbb{A}_G\otimes_{\mathbb{Q}}\mathbb{Q}[t,t^{-1}]$. Note that setting $t\to 1$ recovers the super-character of U regarded as a super-$(G\wr\mathbb{S})$-module. The definitions of sum, product, and substitution carry over to this graded context in the usual way, incorporating the sign-commutativity in tensor products. We can also extend the definitions of our plethystic actions to $\mathbb{A}_G\otimes_{\mathbb{Q}}\mathbb{Q}[t,t^{-1}]$ by adding the rules that $p_i\circ t = t^i$, $p_i(c)\circ t = t^i$.

Theorem 5.1. *1. If U and V are graded $(G \wr \mathbb{S})$-modules,*

$$\mathrm{ch}_t(U+V) = \mathrm{ch}_t(U) + \mathrm{ch}_t(V), \;\; \mathrm{ch}_t(U \cdot V) = \mathrm{ch}_t(U)\mathrm{ch}_t(V).$$

2. If U is a graded $\mathbb{S}$-module and V a graded $(G \wr \mathbb{S})$-module such that $V(\emptyset) = 0$,

$$\mathrm{ch}_t(U \circ V) = \mathrm{ch}_t(U) \circ \mathrm{ch}_t(V).$$

3. If U is a graded $(G \wr \mathbb{S})$-module and V a graded $\mathbb{S}$-module such that $V(\emptyset) = 0$,

$$\mathrm{ch}_t(U \circ V) = \mathrm{ch}_t(U) \circ \mathrm{ch}_t(V).$$

Proof. Again, a more complicated version was proved in [8, Section 7]. For (1), set $q \to 1$ in [8, Proposition 7.2]; for (3), set $q \to 1$ in [8, Theorem 7.5]; part (2) follows by the same method. ☐

To apply this, we must reinterpret $\widetilde{H}$, WH, and WH^* as graded $(G\wr\mathbb{S})$-modules for each of our poset functors, by viewing each $\widetilde{H}_P(x,y)$ as a graded vector space concentrated in degree $\ell(x,y)$. (Recall that the actual homological degree, when $x<y$, is $\ell(x,y)-2$; the 'double suspension' is required by the Künneth formula [16, second statement of Theorem 5.15].) It is easy to check that every isomorphism of super-$(G \wr \mathbb{S})$-modules stated in the previous section remains true verbatim as an isomorphism of graded $(G \wr \mathbb{S})$-modules, to which we can apply ch_t and use the appropriate parts of Theorem 5.1. (The notation [1] in (4.17) now denotes a shift of grading, such that $U[1]_i = U_{i+1}$.)

The flow of information is reversed from that in the previous section: initially we knew $\mathrm{sch}(WH)$ by Theorem 4.1 and deduced $\mathrm{sch}(\widetilde{H})$; now we can easily obtain $\mathrm{ch}_t(\widetilde{H})$ from this, and deduce $\mathrm{ch}_t(WH)$. To illustrate the procedure on the partition lattice once more, the first step is to complete the second equation of the following analogy:

$$\begin{aligned}\mathrm{sch}(\widetilde{H}_\Pi) &= 1 + \sum_{n\geq 1} (-1)^{n-1} \mathrm{ch}_{S_n}(\widetilde{H}_{\Pi_n}(\hat{0},\hat{1})) = 1 + L,\\ \mathrm{ch}_t(\widetilde{H}_\Pi) &= 1 + \sum_{n\geq 1} \mathrm{ch}_{S_n}(\widetilde{H}_{\Pi_n}(\hat{0},\hat{1}))\,(-t)^{n-1} = \;???\end{aligned}$$

Since the only difference from first equation to second is that the degree-n term for each $n \geq 1$ is multiplied by t^{n-1}, the answer is clearly that

$$\mathrm{ch}_t(\widetilde{H}_\Pi) = 1 + t^{-1} L \circ tp_1. \tag{5.2}$$

Hence (4.2) and (4.3), viewed as isomorphisms of graded $\mathbb{S}$-modules, imply that

$$\begin{aligned}\mathrm{ch}_t(WH_\Pi) &= \mathrm{Exp} \circ t^{-1} L \circ tp_1,\\ \mathrm{ch}_t(WH^*_\Pi) &= 1 + t^{-1} L \circ t(\mathrm{Exp} - 1).\end{aligned}$$

Unravelling the definitions to rewrite the left-hand sides, these become the known facts

$$1 + \sum_{n\geq 1} \sum_{i\in\mathbb{Z}} \mathrm{ch}_{S_n}(WH_i(\Pi_n))\,(-t)^i = \mathrm{Exp} \circ t^{-1}L \circ tp_1,$$

$$1 + \sum_{n\geq 1} \sum_{i\in\mathbb{Z}} \mathrm{ch}_{S_n}(WH_i(\Pi_n^*))\,(-t)^i = 1 + t^{-1}L \circ t(\mathrm{Exp} - 1),$$

which are equivariant versions of the familiar generating functions for the characteristic polynomials of the partition lattices and their duals.

In the remainder of the section we give the results of applying this same procedure to our wreath product posets. (We omit the dual forms, which seem less interesting.)

Theorem 5.2. (Hanlon) *In $\mathbb{A}_G \otimes_{\mathbb{Q}} \mathbb{Q}[t, t^{-1}]$ we have the equation*

$$1 + \sum_{n\geq 1} \sum_{i\in\mathbb{Z}} \mathrm{ch}_{G\wr S_n}(WH_i(Q_n(G)))\,(-t)^i = \mathrm{Exp}_G \circ (t^{-1} - 1)L \circ tp_1.$$

Proof. From Theorem 2.1 we deduce that $\mathrm{ch}_t(\widetilde{H}_Q) = (\mathrm{Exp}_G \circ L \circ tp_1)^{-1}$. Substituting this fact and (5.2) into (4.6) gives

$$\mathrm{ch}_t(WH_Q) = (\mathrm{Exp}_G \circ L \circ tp_1)^{-1}(\mathrm{Exp}_G \circ t^{-1}L \circ tp_1),$$

which is the statement. □

Note that this Theorem is just a rephrasing of [7, Corollary 2.3]. Its non-equivariant version is

$$1 + \sum_{n\geq 1} \sum_{i\in\mathbb{Z}} \dim WH_i(Q_n(G))\,(-t)^i \frac{x^n}{|G|^n n!} = (1 + tx)^{\frac{1}{|G|}(t^{-1}-1)}, \tag{5.3}$$

which is equivalent to the well-known formula

$$\sum_{i\in\mathbb{Z}} \dim WH_i(Q_n(G))\,(-t)^i = (1 - t)(1 - (|G| + 1)t) \cdots (1 - ((n - 1)|G| + 1)t). \tag{5.4}$$

A similar proof using (4.9) gives:

Theorem 5.3. *In $\mathbb{A}_G \otimes_{\mathbb{Q}} \mathbb{Q}[t, t^{-1}]$ we have the equation*

$$\sum_{n\geq 1} \sum_{i\in\mathbb{Z}} \mathrm{ch}_{G\wr S_n}(WH_i(R_n(G)))\,(-t)^i = \mathrm{Exp}_G \circ t^{-1}L \circ tp_1 - \mathrm{Exp}_G \circ L \circ tp_1.$$

The non-equivariant version is

$$\sum_{n\geq 1} \sum_{i\in\mathbb{Z}} \dim WH_i(R_n(G))\,(-t)^i \frac{x^n}{|G|^n n!} = (1 + tx)^{\frac{1}{|G|}t^{-1}} - (1 + tx)^{\frac{1}{|G|}}, \tag{5.5}$$

which is equivalent to

$$\begin{aligned}\sum_{i\in\mathbb{Z}} \dim WH_i(R_n(G))\,(-t)^i &= (1 - |G|t)(1 - 2|G|t) \cdots (1 - (n - 1)|G|t) \\ &\quad + (|G| - 1)(2|G| - 1) \cdots ((n - 1)|G| - 1)(-t)^n.\end{aligned} \tag{5.6}$$

Another similar proof using (4.10) gives (assuming $G \neq \{1\}$):

Theorem 5.4. *In $\mathbb{A}_G \otimes_{\mathbb{Q}} \mathbb{Q}[t, t^{-1}]$ we have the equation*

$$1 + \sum_{n\geq 2} \sum_{i\in\mathbb{Z}} \mathrm{ch}_{G\wr S_n}(WH_i(Q_n^{\sim}(G)))\,(-t)^i = (1 + \sum_{c\in G_*} \frac{|c|}{|G|} tp_1(c))(\mathrm{Exp}_G \circ (t^{-1} - 1)L \circ tp_1).$$

The non-equivariant version is

$$1 + \sum_{n\geq 2} \sum_{i\in\mathbb{Z}} \dim WH_i(Q_n^{\sim}(G))\,(-t)^i \frac{x^n}{|G|^n n!} = (1 + \frac{1}{|G|} tx)(1 + tx)^{\frac{1}{|G|}(t^{-1}-1)}, \tag{5.7}$$

equivalent to the formula

$$\sum_{i\in\mathbb{Z}} \dim WH_i(Q_n^{\sim}(G))\,(-t)^i = (1-t)(1-(|G|+1)t) \times \cdots$$
$$\cdots \times (1-((n-2)|G|+1)t)(1-(n-1)(|G|-1)t), \tag{5.8}$$

which for cyclic G is a consequence of [13, Corollary 6.86].

To state the analogous results for $Q^{1 \bmod d}$ and $Q^{0 \bmod d}$, we need to abuse notation slightly. For instance, to apply ch_t to (4.14) we need the graded version of the equation $\mathrm{sch}(\widetilde{H}^{1 \bmod d}_{\Pi^{1 \bmod d}}) = (\mathrm{Exp}^{1 \bmod d})^{[-1]}$, namely

$$\mathrm{ch}_t(\widetilde{H}^{1 \bmod d}_{\Pi^{1 \bmod d}}) = t^{-1/d}(\mathrm{Exp}^{1 \bmod d})^{[-1]} \circ t^{1/d} p_1. \tag{5.9}$$

The right-hand side makes sense because every term of $(\mathrm{Exp}^{1 \bmod d})^{[-1]}$ has degree $\equiv 1 \bmod d$, so the exponents of t all come out as integers. (Explicitly, the degree-n term is multiplied by $t^{\frac{n-1}{d}}$, as required since $l(\Pi_n^{(1,d)}) = \frac{n-1}{d}$ when $n \equiv 1 \bmod d$). Similar remarks apply to the right-hand side of the following result, deduced from (4.14).

Theorem 5.5. *In $\mathbb{A}_G \otimes_{\mathbb{Q}} \mathbb{Q}[t, t^{-1}]$ we have the equation*

$$\begin{aligned}
1+&\sum_{n\geq 1} \sum_{i\in\mathbb{Z}} \mathrm{ch}_{G\wr S_n}(WH_i(Q_n^{1 \bmod d}(G)))\,(-t)^i \\
&= -\sum_{j=1}^{d-1} t^{\frac{d-j}{d}} (\mathrm{Exp}_G^{j \bmod d}(\mathrm{Exp}_G^{0 \bmod d})^{-1}) \circ (\mathrm{Exp}^{1 \bmod d})^{[-1]} \circ t^{1/d} p_1 \\
&\quad + (\mathrm{Exp}_G^{0 \bmod d} \circ (\mathrm{Exp}^{1 \bmod d})^{[-1]} \circ t^{1/d} p_1)^{-1} (\mathrm{Exp}_G \circ t^{-1/d} (\mathrm{Exp}^{1 \bmod d})^{[-1]} \circ t^{1/d} p_1).
\end{aligned}$$

The non-equivariant version of the $d = 2$ special case is:

$$\begin{aligned}
1 + \sum_{n\geq 1} \sum_{i\in\mathbb{Z}} \dim WH_i(Q_n^{1 \bmod 2}(G))\,(-t)^i \frac{x^n}{|G|^n n!} &= -t^{1/2} \tanh(\frac{1}{|G|} \mathrm{arcsinh}(t^{1/2}x)) \\
&\quad + \mathrm{sech}(\frac{1}{|G|} \mathrm{arcsinh}(t^{1/2}x)) \exp(\frac{t^{-1/2}}{|G|} \mathrm{arcsinh}(t^{1/2}x)).
\end{aligned} \tag{5.10}$$

In a forthcoming joint paper with Rains, (5.10) will be shown to imply the following formula for the Betti numbers of the real De Concini-Procesi models of type B:

$$\begin{aligned} 1+\frac{x}{2}+\sum_{n\geq 2}\sum_{i\in\mathbb{Z}}\dim H^i(\overline{M}_{B_n}(\mathbb{R});\mathbb{Q})\,(-t)^i\,\frac{x^n}{2^n n!} \\ = \operatorname{sech}(\frac{1}{2}\operatorname{arcsinh}(t^{1/2}x))\exp(\frac{t^{-1/2}}{2}\operatorname{arcsinh}(t^{1/2}x)). \end{aligned} \tag{5.11}$$

We will also give an equivariant version of this formula along the lines of Theorem 5.5.

Finally, the following result can be deduced from (4.17).

Theorem 5.6. *In $\mathbb{A}_G\otimes_{\mathbb{Q}}\mathbb{Q}[t,t^{-1}]$ we have the equation*

$$\begin{aligned} &1+\sum_{n\geq 1}\sum_{i\in\mathbb{Z}}\mathrm{ch}_{G\wr S_n}(WH_i(Q_n^{0\bmod d}(G)))\,(-t)^i \\ &= \mathrm{Exp}_G + t - (\sum_{j=0}^{d-1} t^{\frac{d-j}{d}}\,\mathrm{Exp}_G^{j\bmod d}\circ t^{1/d}p_1)(\mathrm{Exp}_G\circ(t^{-1}-1)L\circ(\mathrm{Exp}^{0\bmod d}-1)\circ t^{1/d}p_1). \end{aligned}$$

The non-equivariant version of the $d=2$ case is:

$$\begin{aligned} 1+\sum_{n\geq 1}\sum_{i\in\mathbb{Z}}\dim WH_i(Q_n^{0\bmod 2}(G))\,(-t)^i\,\frac{x^n}{|G|^n n!} \\ = \exp(\frac{x}{|G|})+t-(t\cosh(\frac{t^{1/2}x}{|G|})+t^{1/2}\sinh(\frac{t^{1/2}x}{|G|}))\cosh(t^{1/2}x)^{\frac{1}{|G|}(t^{-1}-1)}. \end{aligned} \tag{5.12}$$

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