New regular partial difference sets and strongly regular graphs with parameters (96,20,4,4) and (96,19,2,4)

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Abstract

New (96,20,4,4) and (96,19,2,4) regular partial difference sets are constructed, together with the corresponding strongly regular graphs. Our source are (96,20,4) regular symmetric designs.

Keywords: Difference set, partial difference set, Cayley graph, symmetric design.

1 Introduction and preliminaries

We start with defining objects to be constructed.

Definition 1 *Let H be a group of order v. A k-subset $S \subset H$ is called a (v, k, λ, μ) partial difference set if the multiset $\{xy^{-1} \mid x, y \in S, x \neq y\}$ contains each nonidentity element of S exactly λ times and it contains each nonidentity element of $H \setminus S$ exactly μ times.*

Using the notation of a group ring $\mathbb{Z}H$ (where $\underline{S} := \sum_{s\in S} s$), a (v, k, λ, μ) partial difference set $S \subset H$ in the group H can be defined as a subset for which the equation

$$\underline{S} \cdot \underline{S^{(-1)}} = k\underline{\{e\}} + \lambda\underline{S \setminus \{e\}} + \mu\underline{(H \setminus S) \setminus \{e\}} \tag{1.1}$$

holds; e denotes the group identity element.

Partial differential sets S_1 and S_2 in groups H_1 and H_2, respectively, we will call *equivalent* if there exists a group isomorphism $\varphi : H_1 \to H_2$ which maps S_1 onto S_2.

The notion of a partial difference set generalizes that of a difference set, well-known in group and design theory.

Definition 2 *A (v,k,λ) difference set is a k-element subset $\Delta \subseteq H$ in a group H of order v provided that the multiset $\{xy^{-1} \mid x,y \in \Delta, x \neq y\}$ contains each nonidentity element of H exactly λ times.*

In terms of a group ring, $\Delta \subseteq H$ is a difference set in a group H if and only if the relation $\underline{\Delta} \cdot \underline{\Delta^{(-1)}} = k\underline{\{e\}} + \lambda \underline{H \setminus \{e\}}$ holds in $\mathbb{Z}H$. In case a set $\Delta \subseteq H$ is a difference set in a group H, its so called "shift" Δx by each element $x \in H$ is a difference set in H as well, [1]. It is obvious that any (v,k,λ) difference set is a (v,k,λ,λ) partial difference set.

A partial difference set (PDS for short) S is *reversible* if $S = S^{(-1)}$. A reversible partial difference set S is called *regular* if $e \notin S$. It is easy to see (cf. [7]) that the following assertions hold.

Proposition 1 *Suppose that S is a reversible (v,k,λ,μ) PDS in a group H, such that $e \in S$. Then $(S - e)$ is a regular $(v,k-1,\lambda-2,\mu)$ PDS in H. Conversely, if S is a regular PDS in H, then $(S + e)$ is a reversible PDS with corresponding parameters.*

Proposition 2 *Suppose that Δ is a (v,k,λ) difference set in H, $x \in H$. Then*

(i) Δx is a regular (v,k,λ,λ) PDS if and only if $x^{-1} \notin \Delta$ and Δx is a reversible set;

(ii) $\Delta x - e$ is a regular $(v,k-1,\lambda-2,\lambda)$ PDS if and only if $x^{-1} \in \Delta$ and Δx is a reversible set.

The *development* of a difference set $\Delta \subseteq H$ is the incidence structure $dev\Delta = (H, \{\Delta g \mid g \in H\}, \in)$. By this structure difference sets and symmetric designs are interrelated, as shows the following important result, [1].

Theorem 1 *Let H be a finite group of order v and Δ a proper, non-empty k-element subset of H. Then Δ is a (v,k,λ) difference set in H if and only if $dev\Delta$ is a symmetric (v,k,λ) design on which H acts regularly.*

Let's repeat, a *symmetric block design* with parameters (v,k,λ) is a finite incidence structure $\mathcal{D} = (\mathcal{V},\mathcal{B},\mathcal{I})$ consisting of $|\mathcal{V}| = v$ points and $|\mathcal{B}| = v$ blocks, where each block is incident with k points and any two distinct points are incident with exactly λ common blocks. An *automorphism* of a symmetric block design $\mathcal{D}$ is a permutation on $\mathcal{V}$ which sends blocks to blocks. The set of all automorphisms of $\mathcal{D}$ forms its full automorphism group denoted by $Aut\mathcal{D}$. If a subgroup $H \leq Aut\mathcal{D}$ acts regularly on $\mathcal{V}$ and $\mathcal{B}$, then $\mathcal{D}$ is called *regular* and H is called a *Singer group* of $\mathcal{D}$.

Two difference sets Δ^1 (in H^1) and Δ^2 (in H^2) are *isomorphic* if the designs $dev\Delta^1$ and $dev\Delta^2$ are isomorphic; Δ^1 and Δ^2 are *equivalent* if there exists a group isomorphism $\varphi : H^1 \to H^2$ such that $\varphi(\Delta^1) = \Delta^2 g$ for a suitable $g \in H^2$. It is easy to see that equivalent difference sets Δ^1 and Δ^2 give rise to isomorphic symmetric designs $dev\Delta^1$ and $dev\Delta^2$. Depending on the respective property of H, a difference set (and PDS as well) is called *abelian, cyclic,* or *nonabelian.*

The so far introduced notions and observations are connected to graph theory. More precisely, regular partial difference sets and strongly regular graphs are closely related through the concept of Cayley graphs.

Definition 3 *A strongly regular graph (SRG) with parameters (v, k, λ, μ) is a graph with v vertices which is regular of valency k, i.e. every vertex is incident with k edges, such that any pair of adjacent vertices have exactly λ common neighbours and any pair of non-adjacent vertices have exactly μ common neighbours.*

Definition 4 *For a group H and a set $S \subset H$ with the property that $e \notin S$ and $S = S^{(-1)}$, the Cayley graph $\Gamma = Cay(H, S)$ over H with connection set S is the graph with vertex set H so that the vertices x and y are adjacent if and only if $x^{-1}y \in S$. Then Γ is undirected graph without loops.*

Accordingly, the edge set of a Cayley graph $\Gamma = Cay(H, S)$ over H with connection set S is $E := \{\{x, sx\} \mid x \in H, s \in S\}$. Our construction of strongly regular graphs (cf. [5]) will be based on the following important assertion about Cayley graphs, [1] p. 230 or [6].

Theorem 2 *A Cayley graph $Cay(H, S)$ is a (v, k, λ, μ) strongly regular graph if and only if S is a (v, k, λ, μ) regular partial difference set in H.*

In this sense, equivalent regular PDS's obviously correspond to isomorphic strongly regular Cayley graphs. Note that for two inequivalent partial difference sets S_1 and S_2 in a group H, the graphs $Cay(H, S_1)$ and $Cay(H, S_2)$ can be isomorphic. Similarly, for two inequivalent partial difference sets S_1 and S_2 in groups H_1 and H_2, $|H_1| = |H_2|$, the graphs $Cay(H_1, S_1)$ and $Cay(H_2, S_2)$ can be isomorphic. Several examples of both such cases are shown in Section 2.

In our computation we use GAP, the well-known system for computational group theory, [9]. Moreover, because we deal with a rather large number of groups, for identifying groups we use GAP-catalogue number whenever it is available. Namely, the order of some groups that appear in our considerations exceeds the scope of the GAP Library Small Groups. A GAP-catalogue number is of the form $[m, n]$ and it stands for n-th group of order m in the catalogue. For graph exploring we use GRAPE [8], a package which is a part of GAP.

2 Construction of regular partial difference sets and graphs

Following the theoretical background highlighted in Section 1, it can easily be verified that the procedure for the search of regular partial difference sets, starting from a known difference set $\Delta \subseteq H$, can be performed in the next two steps:

(i) construction of all shifts Δx of $\Delta, x \in H$,

(ii) selection of those shifts which are reversible sets in H.

Then, each reversible shift which does not contain e is a regular (v, k, λ, λ) PDS, while each reversible shift that contains e yields a regular $(v, k-1, \lambda-2, \lambda)$ PDS $\Delta x - e$.

To this procedure of "surveyed shifting" we have submitted about seventy (96,20,4) difference sets in approximately 30 groups presented in [3] and [4]. The cited papers contain detailed description of the difference sets construction from 9 regular (96,20,4) symmetric designs. The procedure ended in construction of 59 regular PDS's in 9 groups. After GAP-testing on group automorphisms, final result boiled down to 31 inequivalent regular PDS in 9 groups. Regarding isomorphism of the corresponding strongly regular Cayley graphs, these 31 PDS's split into eight nonisomorphic SRG-classes. Their representatives we denote by Γ_j, $j = 1, 2, \ldots, 8$. It turned out that difference set shifts being or yielding regular PDS's are connected with three designs only, all three given in [3] and there denoted by D_1, D_6, and D_8. Sticking to that labelling, in presentation of the obtained regular PDS's we indicate the originating design in the superscript of a concerned difference set $\Delta^k_{[96,n]}$, $k \in \{1, 6, 8\}$. The subscript refers to GAP-cn of the host group. The results we give group by group. All nine groups are nonabelian. We use the notation $p^q = qpq^{-1}$ for p, q arbitrary elements of a group.

1. In the group:

$$H_{[96,64]} = \langle x, y, z, a \mid x^4 = y^4 = z^2 = [x, y] = 1,\ x^z = y, y^z = x, a^3 = 1, \\ x^a = x^{-1}y^{-1}, y^a = x, a^z = a^{-1} \rangle$$

two nonisomorphic difference sets enable construction of two inequivalent regular PDS's.

The shift of

$$\Delta^1_{[96,64]} = 1 + a^2x^2z + a^2x^2axaz + axa^2x^2z + axax + a^2xz + x^2axaz + \\ x^2axz + ax^2a^2x + xa^2z + xaxz + x^2axa + a^2xaxz + axa^2z + axax^2az + \\ axa^2x^2 + ax^2ax + a^2xax^2 + x^2a^2 + ax^2z$$

by axa^2x^2z, e extracted, is a regular PDS. The corresponding SRG with parameters (96,19,2,4) we denote by Γ_1. Using GRAPE one finds $|Aut\Gamma_1| = 9216$.

The shift of

$$\Delta^8_{[96,64]} = 1 + x^2axaz + ax^2axa + xa^2z + a^2x^2axz + x^3z + xz+ \\ ax^2a^2xz + x^2a^2z + x^2a^2xz + x^3 + axa^2x^2 + xaxz + x^2ax + ax^2z+ \\ a^2x^2axa + a^2xax^2a + a^2x^2a^2xz + xaz + ax^2$$

by a^2xax^2a is a regular PDS. The corresponding SRG with parameters (96,20,4,4) we denote by Γ_8, $|Aut\Gamma_8| = 138240$.

2. In the group:

$$H_{[96,70]} = \langle x, y, z, t, a \mid x^2 = y^2 = z^2 = t^2 = [x, y] = [x, z] = [x, t] = [y, z] = 1, \\ [y, t] = [z, t] = a^6 = 1, x^a = yt, y^a = xz, z^a = xyz, t^a = yzt \rangle$$

two nonisomorphic difference sets enable construction of three inequivalent regular PDS's.

The shifts of

$$\Delta^1_{[96,70]} = 1 + a^2x + xaya^2 + a^3xa + ay + a^4xaya^2 + a^5xa^2 + a^5xya + a^5ya+ \\ axa + a^5xa + a^5xaya^2 + a^2xa^2 + a^2xa^2y + a^3xay + xa^2ya+ \\ a^4xa^2ya^2 + a^2xay + ya + a^3$$

by a^3xay and a^2xa^2, e extracted, are regular PDS's. Using GAP they are checked to be equivalent. The corresponding SRG's with parameters (96,19,2,4) are isomorphic to Γ_1.

The shift of

$$\Delta^8_{[96,70]} = 1 + a^3xa^2ya + a^5xa + a^4xaya + ay + axy + xa + a^5xa^2 + a^2xa + a^5xa^2ya^2 + a^2ya + a^2xay + a^4xa^2ya^2 + a^2 + xa^2 + a^5xaya^2 + a^3xya^2 + a^5xy + a^4ya + a^5$$

by $a^4xa^2ya^2$ is a regular PDS. The corresponding SRG with parameters (96,20,4,4) is isomorphic to Γ_8.

$\Delta^8_{[96,70]}$ multiplied by xa^2 gives another regular PDS, not isomorphic to the previous one. The corresponding SRG with parameters (96,20,4,4) we denote by Γ_7. GRAPE reveals $|Aut\Gamma_7| = 3072$.

Now we already have examples of pairs of inequivalent regular PDS's that correspond to isomorphic strongly regular Cayley graphs: $\Delta^1_{[96,64]}axa^2x^2z \setminus \{e\}$ and $\Delta^1_{[96,70]}a^3xay \setminus \{e\}$ (or $\Delta^1_{[96,70]}a^2xa^2 \setminus \{e\}$); $\Delta^8_{[96,64]}a^2xax^2a$ and $\Delta^8_{[96,70]}a^4xa^2ya^2$. More examples follow.

3. In the group:

$$H_{[96,71]} = \langle x, y, z, a \mid x^4 = y^4 = z^2 = [x, y] = 1,\ x^z = xy^2, y^z = x^2y^{-1}, a^3 = 1,\ x^a = x^{-1}y^{-1},\ y^a = x, [a, z] = 1\rangle$$

two nonisomorphic difference sets enable construction of two inequivalent regular PDS's.

The shift of

$$\Delta^1_{[96,71]} = 1 + ax^2az + a^2x^2ax + xax + xax^2a + xa^2z + a^2xz + ax^2a^2z + a^2xax + axax^2z + axa^2z + axax^2 + a^2xa^2 + a^2xax^2a + a^2xa^2x^2 + ax^2a^2xz + ax^2a + axa^2xz + ax^2ax + ax^2axa,$$

by $axax^2$, e extracted, is a regular PDS. The corresponding SRG with parameters (96,19,2,4) is isomorphic to Γ_1.

The shift of

$$\Delta^8_{[96,71]} = 1 + ax^2a^2xz + a^2x^2ax + x^2z + axax^2z + xa^2xz + x^2axaz + axaz + a^2xaxz + ax^2axaz + ax^2axa + a^2xax + ax^2 + xax^2 + x^2az + a^2xa^2xz + axz + xax + a^2xa^2x^2z + x^2ax$$

by ax^2 is a regular PDS. The corresponding SRG with parameters (96,20,4,4) is isomorphic to Γ_8.

4. In the group:

$$H_{[96,186]} = S_4 \times C_4 = \langle x, y, z, a \mid x^4 = y^2 = z^2 = [x, y] = [x, z] = (zy)^3 = 1, a^3 = 1, [a, x] = 1, (za)^2 = (zya)^2 = 1\rangle$$

a single difference set enables construction of only one, up to equivalency, regular PDS.

The shifts of

$$\Delta^1_{[96,186]} = 1 + a^2x^2 + ax^3yay + axya^2y + ay + ax^2y + a^2x^2ya^2yay + ax^2yay + x^3ya^2yay + ax^3ya^2yay + xya + a^2xya^2 + a^2x + y + aya + aya^2 + aya^2y + x^3 + a^2ya^2yay + a^2y$$

by aya^2y and ax^2ya^2y are equivalent regular PDS's. The corresponding isomorphic SRG's with parameters (96,20,4,4) are denoted by Γ_2. Using GRAPE we checked $|Aut\Gamma_2| = 11520$.

5. In the group:

$$H_{[96,190]} = \quad \langle x,y,a \mid x^8=y^2=1, x^y=x^5, a^3=1, (xa)^2=1, [y,a]=1\rangle$$

two nonisomorphic difference sets enable construction of three inequivalent regular PDS's.

The shifts of

$\Delta^1_{[96,190]} = 1+ax^2a^2xy+ax^2a^2y+ax^4+a^2xa^2y+x^3y+a^2y+axa^2+$
$a^2xa^2x+a^2+axa^2x^2y+xa^2xay+a+x^2ay+axa^2xa^2y+a^2x^3+$
$xy+ax^3a^2y+a^2x^2+y$

by a^2y and a^2x^4y are equivalent regular PDS's. The corresponding SRG's with parameters (96,20,4,4) are isomorphic to Γ_2.

The shift of

$\Delta^8_{[96,190]} = 1+a^2x^3ay+xa^2xa^2y+ax^3y+a^2xa^2+ax^3ay+a^2xa^2y+$
$ax^3a+a^2x^2a^2xy+axy+axa^2xay+ax^2a^2+ax^3a^2+$
$a^2x^4y+xa^2xay+xay+ax^2a+axa^2x+axa^2x^2+x^3ay$

by x^2a^2y is a regular PDS. The corresponding SRG with parameters (96,20,4,4) is isomorphic to Γ_8.

$\Delta^8_{[96,190]}axa^2xy$ is another regular PDS, not isomorphic to the previous one. The corresponding SRG with parameters (96,20,4,4) is isomorphic to Γ_7.

6. In the group:

$$\begin{aligned} H_{[96,195]} = \quad & \langle x,y,z,t,w,a \mid x^2=y^2=z^2=t^2=[x,y]=[x,z]=[x,t]=1,\\ & [y,z]=[y,t]=[z,t]=1, w^2=a^3=(wa)^2=1,\ x^w=yt,\\ & y^w=xz, z^w=t, t^w=z, x^a=xz, y^a=yt, z^a=t, t^a=zt\rangle \end{aligned}$$

seven inequivalent difference sets originating from three nonisomorphic symmetric designs enable construction of 12 inequivalent regular PDS's. Here $AutD_1$ and $AutD_6$ have more than one conjugacy class of subgroups isomorphic to $H_{[96,195]}$.

The shifts of

$\Delta^1_{[96,195],1} = 1+axayw+a^2xaxaw+xw+xa^2y+axay+xa^2w+xaw+a^2xax+$
$axa+axw+a^2y+axa^2xyw+a^2xaxyw+axaxaxw+axa^2xy+axaxw+$
$a^2xa+a^2xw+axyw$

by w and axa^2yw are equivalent regular PDS's. The corresponding SRG's with parameters (96,20,4,4) are isomorphic to Γ_2.

The shift of

$\Delta^1_{[96,195],2} = 1+xaxaw+a^2xayw+a^2xa^2y+axw+a^2xaxaxyw+aw+axa^2y+$
$a^2xaxy+xa^2xw+axaxax+a^2w+a^2xaxa+a^2xaxay+a^2xa^2+$
$w+axaxw+a^2xaw+a^2xaxayw+yw$

by a^2w, e extracted, is a regular PDS. The corresponding SRG with parameters (96,19,2,4) is isomorphic to Γ_1.

$\Delta^1_{[96,195],2}$ multiplied by xa^2yw is another regular PDS. The corresponding SRG with parameters (96,20,4,4) we denote by Γ_3. Using GRAPE one finds $|Aut\Gamma_3| = 1536$.

The shifts of

$\Delta^1_{[96,195],3} = 1+axax+xaxyw+a^2xaxa+a^2yw+axayw+xa^2xw+y+xy+$

$xaxaxy + x + xa^2 + ay + xa + a^2xa^2y + a^2xa^2xyw + a^2y+$
$ayw + xaxaxw + a^2xaxayw$

by a^2xa^2xy and ax, e extracted, are equivalent regular PDS's. The corresponding SRG's with parameters (96,19,2,4) are isomorphic to Γ_1.

Six regular PDS's are obtained starting from difference set

$$\Delta^1_{[96,195],4} = 1 + xa + axaxaw + a^2xaxaw + xa^2xy + axa + a^2 + y+ xy + a^2xaxa + x + a^2xa^2y + xaxw + aw + a^2xa^2xw+ a^2xayw + ay + xw + xaxaxy + axa^2xyw.$$

$\Delta^1_{[96,195],4}$-shifts by a^2, axa^2x, and a^2xaxa are equivalent regular PDS's that correspond to SRG's with parameters (96,20,4,4) isomorphic to Γ_3. On the other side, $\Delta^1_{[96,195],4}xy \setminus \{e\}$, $\Delta^1_{[96,195],4}xa^2y \setminus \{e\}$, and $\Delta^1_{[96,195],4}xay \setminus \{e\}$ are equivalent regular PDS's that correspond to SRG's with parameters (96,19,2,4) isomorphic to Γ_1.

Six regular PDS's are obtained starting from difference set

$$\Delta^6_{[96,195],1} = 1 + a^2xay + axaxay + axa^2 + a^2 + a^2xy + a^2y + a^2x + a^2w+ xaxaw + axaxw + xa^2xw + a + xa + a^2xa^2xy + a^2xaxaxy+ aw + a^2xa^2yw + xaxyw + a^2xaxaxw.$$

$\Delta^6_{[96,195],1}$-shifts by: e, a, a^2, e extracted, are three equivalent regular PDS's that correspond to isomorphic SRG's with parameters (96,19,2,4). We denote them by Γ_4 and explore ([8]) $Aut\Gamma_4 = [288, 1026]$. Further three $\Delta^6_{[96,195],1}$-shifts by a^2xa^2y, axa^2y, and xa^2y are equivalent regular PDS's corresponding to isomorphic SRG's with parameters (96,20,4,4). We denote them by Γ_5 and using [8] find: $Aut\Gamma_5 = [96, 195]$.

Six regular PDS's are obtained starting from difference set

$$\Delta^6_{[96,195],2} = 1 + axa^2 + a^2xay + axaxay + a^2xa^2x + xay + ay + a^2xaxax+ w + axa^2yw + axaxaxw + a^2xaxyw + xaxa + xaxaxy+ xaxax + xaxay + a^2xa^2xw + a^2xaxaw + xaxw + aw.$$

$\Delta^6_{[96,195],2}$-shifts by: e, a^2xa^2x, and $xaxa$, e extracted, are equivalent regular PDS's that correspond to SRG's with parameters (96,19,2,4) isomorphic to Γ_4. Further three $\Delta^6_{[96,195],2}$-shifts by $axay$, axa^2y, and axy are equivalent regular PDS's corresponding to three isomorphic SRG's with parameters (96,20,4,4). We denote them by Γ_6; $Aut\Gamma_6 = [96, 195]$. Nevertheless, $\Gamma_5 \not\cong \Gamma_6$.

Finally, two shifts of

$$\Delta^8_{[96,195]} = 1 + axa^2x + axaxa + a^2xax + axax + a^2xy + a^2x + axaxy+ a^2xa^2xw + a^2xaxaxw + a^2xxaxyw + a^2xa^2xyw + a^2xaxa+ a^2xa^2xy + ax + xay + a^2w + xaxaxyw + a^2xw + xaxayw$$

are inequivalent regular PDS's. $\Delta^8_{[96,195]}$-shift by $axaxaxyw$ corresponds to SRG with parameters (96,20,4,4) isomorphic to Γ_7, while $\Delta^8_{[96,195]}$- shift by a^2xaxw corresponds to SRG with parameters (96,20,4,4) isomorphic to Γ_8.

Note that the group $H_{[96,195]}$ provides examples of mutually inequivalent regular PDS's that correspond to isomorphic strongly regular Cayley graphs. For instance, $\Delta^1_{[96,195],2}a^2w \setminus \{e\}$, $\Delta^1_{[96,195],3}ax \setminus \{e\}$, and $\Delta^1_{[96,195],4}xy \setminus \{e\}$ are isomorphic to Γ_1; $\Delta^6_{[96,195],1} \setminus \{e\}$ and $\Delta^6_{[96,195],2} \setminus \{e\}$ are isomorphic to Γ_4.

7. In the group:

$$\begin{aligned} H_{[96,197]} = \quad & A_4 \times D_8 = \langle x, y, a \mid x^2 = y^2 = a^3 = 1, x^a = y, y^a = xy \rangle \times \\ & \times \langle z, t \mid z^4 = t^2 = (tz)^2 = 1 \rangle \end{aligned}$$

one difference set enables construction of, up to equivalency, one regular PDS.

The shifts of

$$\begin{aligned} \Delta^1_{[96,197]} = & 1 + a^2 + a^2xaz^3 + a^2xz^3 + a^2xa^2z^3 + a^2xa^2z + az + xz^2 + a^2xazt + \\ & a^2xz^3t + axa^2z^3t + xa^2zt + xa^2z^3 + a^2xz^2t + xa^2z^2t + axa^2t + axaz^2 + \\ & axa^2z^3 + az^3 + a^2xat \end{aligned}$$

by a^2z^3t and a^2zt are equivalent regular PDS's. Two corresponding SRG's with parameters (96,20,4,4) are isomorphic to Γ_2.

8. In the group:

$$\begin{aligned} H_{[96,226]} = \quad & S_4 \times Z_2^2 = \langle x, y, a \mid x^2 = y^2 = [x, y] = a^3 = (xy)^3 = (ya)^2 = 1, \\ & (xya)^2 = 1 \rangle \times \langle z, t \mid z^2 = t^2 = [z, t] = 1 \rangle \end{aligned}$$

the difference set

$$\begin{aligned} \Delta^1_{[96,226]} = & 1 + xt + xa^2 + xaxzt + xaz + xa^2xaxz + xa + axaxzt + a^2xa^2xz + \\ & axat + axa^2xat + axa + a^2z + xa^2xa + axa^2xt + a^2xa + x + xa^2zt + \\ & xa^2xax + a^2xazt \end{aligned}$$

enables construction of, up to equivalency, one regular PDS. The $\Delta^1_{[96,226]}$-shifts by xa^2xa, xa^2xaz, xa^2xat, and xa^2xazt are mutually equivalent regular PDS's. The corresponding SRG's with parameters (96,20,4,4) are isomorphic to Γ_2.

9. In the group:

$$\begin{aligned} H_{[96,227]} = \quad & \langle x, y, z, t, w, a \mid x^2 = y^2 = z^2 = t^2 = [x, y] = [x, z] = [x, t] = [y, z] = 1, \\ & [y, t] = [z, t] = 1, w^2 = a^3 = (wa)^2 = 1, \; x^w = y, \; y^w = x, z^w = t, \\ & t^w = z, x^a = xy, y^a = x, z^a = t, t^a = zt \rangle \end{aligned}$$

four inequivalent difference sets originating from three nonisomorphic symmetric designs enable construction of 6 inequivalent regular PDS's. $AutD_1$ has more than one conjugacy class of subgroups isomorphic to $H_{[96,227]}$.

The shift of

$$\begin{aligned} \Delta^1_{[96,227],1} = & 1 + za^2w + xaza^2w + xa + xa^2za^2w + axw + axa^2zaw + axa^2 + \\ & a^2xa + xa^2zw + x + axzaw + a^2xa^2z + xa^2za^2 + a^2xaza + \\ & a^2zaw + a^2xw + zw + xazw + xw \end{aligned}$$

by axa^2za^2w is a regular PDS. It corresponds to SRG with parameters (96,20,4,4) isomorphic to Γ_3. Another regular PDS can be obtained as $\Delta^1_{[96,227],1}$-shift by a^2xw and e extracted. The corresponding SRG with parameters (96,19,2,4) is isomorphic to Γ_1.

The shifts of

$$\begin{aligned} \Delta^1_{[96,227],2} = & 1 + xaza + axa^2zaw + xw + axaza^2 + a^2za^2 + a^2xa^2 + xz + \\ & axa^2z + axa + a^2xa + aza + aza^2w + a^2xzaw + axa^2zw + \\ & axaza^2w + a^2x + axw + xa + axzw \end{aligned}$$

by a^2xa, a^2xa^2, a^2x, a^2za^2, axa^2z, and $xaza$, e extracted, are equivalent regular PDS's. The corresponding six SRG's with parameters (96,19,2,4) are isomorphic to Γ_1.

The shifts of

$$\Delta^6_{[96,227]} = 1 + axa^2z + xz + a^2xa + axa + a^2x + xaza + aza + axa^2w + axazaw + axza^2w + axa^2zw + a^2xa^2 + axaza^2 + xa + a^2za^2 + a^2xw + axzaw + za^2w + xa^2zw$$

by e, a^2xa^2, and axa, then e extracted, are three equivalent regular PDS's that correspond to SRG's with parameters (96,19,2,4) isomorphic to Γ_4.

The shift of

$$\Delta^8_{[96,227]} = 1 + z + aza^2 + a^2za + a^2xa^2z + xza + a^2za^2 + ax + azw + axzw + xzaw + a^2xazaw + axa^2w + a^2xazw + zw + xw + axa^2za^2 + xa^2z + a^2z + a^2xaza^2$$

by a^2xaza^2w is a regular PDS corresponding to SRG with parameters (96,20,4,4) isomorphic to Γ_8. $\Delta^8_{[96,227]}$-shift by $azaw$ is a regular PDS corresponding to SRG with parameters (96,20,4,4) isomorphic to Γ_7.

In $H_{[96,227]}$ we have again the case of two mutually inequivalent regular partial difference sets that correspond to isomorphic strongly regular Cayley graphs.

Regarding the obtained SRG's with 96 vertices, our results can be summarized in (2.1).

Graph	Valency	Corresp. design	$\|Aut\Gamma_i\|$	Vertex group id. number
Γ_1	19	D_1	9216	[64; 70; 71; 195; 227]
Γ_2	20	D_1	11520	[186; 190; 195; 197; 226]
Γ_3	20	D_1	1536	[195; 227]
Γ_4	19	D_6	288	[195; 227]
Γ_5	20	D_6	96	[195]
Γ_6	20	D_6	96	[195]
Γ_7	20	D_8	3072	[70; 190; 195; 227]
Γ_8	20	D_8	138240	[64; 70; 71; 190; 195; 227]

(2.1)

Six graphs are with parameters (96,20,4,4) and two with parameters (96,19,2,4). Two of the obtained graphs proved to be isomorphic to the already known graphs. Up to isomorphism (GRAPE-tested), Γ_8 is the collinearity graph of $GQ(5,3)$ and Γ_2 is the graph denoted by K' in [2]. Note that each Γ_j, $j = 1, 2, \ldots, 8$ can be represented as a PDS in the group $H_{[96,195]}$. By single horizontal lines in (2.1) a kind of "design-equivalence" is framed, but this, of course, regards only parameters (96,20,4,4). The GRAPE-files determining $\Gamma_i, i = 1, 2, \ldots, 8$ are available at http://www.pmfst.hr/~vucicic/8srg96v.txt.

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