A short proof of a theorem of Kano and Yu on factors in regular graphs

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Abstract

In this note we present a short proof of the following result, which is a slight extension of a nice 2005 theorem by Kano and Yu. Let e be an edge of an r-regular graph G. If G has a 1-factor containing e and a 1-factor avoiding e, then G has a k-factor containing e and a k-factor avoiding e for every $k \in \{1, 2, \ldots, r-1\}$.

Keywords: Regular graph; Regular factor; 1-factor; k-factor.

We consider finite and undirected graphs with vertex set $V(G)$ and edge set $E(G)$, where multiple edges and loops are admissible. A graph is called r-regular if every vertex has degree r. A k-factor F of a graph G is a spanning subgraph of G such that every vertex has degree k in F. A classical theorem of Petersen [3] says:

Theorem 1 (Petersen [3] 1891) Every $2p$-regular graph can be decomposed into p disjoint 2-factors.

Theorem 2 (Katerinis [2] 1985) Let p, q, r be three odd integers such that $p < q < r$. If a graph has a p-factor and an r-factor, then it has a q-factor.

Using Theorems 1 and 2, Katerinis [2] could prove the next attractive result easily.

Corollary 1 (Katerinis [2] 1985) Let G be an r-regular graph. If G has a 1-factor, then G has a k-factor for every $k \in \{1, 2, \ldots, r\}$.

Proofs of Theorems 1 and 2 as well as of Corollary 1 can also be found in [4]. The next result is also a simple consequence of Theorems 1 and 2.
Theorem 3 Let e be an edge of an r-regular graph G with $r \geq 2$. If G has a 1-factor

containing e and a 1-factor avoiding e, then G has a k-factor containing e and a k-factor avoiding e for every $k \in \{1, 2, \ldots, r-1\}$.

Proof. Let F and F_e be two 1-factors of G containing e and avoiding e, respectively.

Case 1: Assume that $r = 2m+1$ is odd. According to Theorem 1, the $2m$-regular graphs $G - E(F)$ and $G - E(F_e)$ can be decomposed into 2-factors. Thus there exist all even regular factors of G containing e or avoiding e, respectively. If F_{2k} is a $2k$-factor of G containing e or avoiding e, then $G - E(F_{2k})$ is a $(2m+1-2k)$-factor avoiding e or containing e, respectively. Hence the statement is valid in this case.

Case 2: Assume that $r = 2m$ is even. In view of Theorem 1, G has all regular even factors containing e or avoiding e, respectively.

Since G has a 1-factor avoiding e, the graph $G-e$ has a 1-factor. In addition, $G-E(F)$ is an $(r-1)$-regular factor of G avoiding e, and so $G-e$ has an $(r-1)$-factor. Applying Theorem 2, we deduce that $G-e$ has all regular odd factors between 1 and $r-1$, and these are regular odd factors of G avoiding e.

If F_{2k+1} is a $(2k+1)$-factor of G avoiding e, then $G - E(F_{2k+1})$ is a $(2m-(2k+1))$-factor containing e, and the proof is complete.

Corollary 2 (Kano and Yu [1] 2005) Let G be a connected r-regular graph of even order. If for every edge e of G, G has a 1-factor containing e, then G has a k-factor containing e and another k-factor avoiding e for all integers k with $1 \le k \le r-1$.

The following example will show that Theorem 3 is more general than Corollary 2.

Example Let G consists of 6 vertices u, v, w, x, y, z, the edges ux, vx, wy, zy, three parallel edges between u and v, three parallel edges between w and z and two parallel edges e and e' connecting x and y. Then G is a 4-regular graph, and G has a 1-factor containing e and a 1-factor avoiding e. According to Theorem 3, G has a k-factor containing e and a k-factor avoiding e for every $k \in \{1, 2, 3\}$. However, Corollary 2 by Kano and Yu does not work, since the edges ux, vx, wy and zy are not contained in any 1-factor.

References

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