Eigenvectors and Reconstruction

Hongyu He *

Department of Mathematics
Louisiana State University, Baton Rouge, USA
hongyu@math.lsu.edu

Submitted: Jul 6, 2006; Accepted: Jun 14, 2007; Published: Jul 5, 2007
Mathematics Subject Classification: 05C88

Abstract

In this paper, we study the simple eigenvectors of two hypomorphic matrices using linear algebra. We also give new proofs of results of Godsil and McKay.

1 Introduction

We start by fixing some notations ([HE1]). Let A be a $n \times n$ real symmetric matrix. Let A_i be the matrix obtaining by deleting the i-th row and i-th column of A. We say that two symmetric matrices A and B are hypomorphic if, for each i, B_i can be obtained by simultaneously permuting the rows and columns of A_i. Let Σ be the set of permutations. We write $B = \Sigma(A)$.

If M is a symmetric real matrix, then the eigenvalues of M are real. We write

$$eigen(M) = (\lambda_1(M) \geq \lambda_2(M) \geq \ldots \geq \lambda_n(M)).$$

If α is an eigenvalue of M, we denote the corresponding eigenspace by $eigen_\alpha(M)$. Let $\mathbf{1}$ be the n-dimensional vector $(1, 1, \ldots, 1)$. Put $\mathbf{J} = \mathbf{1}^t\mathbf{1}$. In [HE1], we proved the following theorem.

Theorem 1 ([HE1]) *Let B and A be two real $n \times n$ symmetric matrices. Let Σ be a hypomorphism such that $B = \Sigma(A)$. Let t be a real number. Then there exists an open interval T such that for $t \in T$ we have*

1. $\lambda_n(A + t\mathbf{J}) = \lambda_n(B + t\mathbf{J})$;

2. $eigen_{\lambda_n}(A + t\mathbf{J})$ *and* $eigen_{\lambda_n}(B + t\mathbf{J})$ *are both one dimensional;*

*I would like to thank the referee for his valuable comments.

3. $eigen_{\lambda_n}(A + t\mathbf{J}) = eigen_{\lambda_n}(B + t\mathbf{J})$.

As proved in [HE1], our result implies Tutte's theorem which says that $eigen(A + tJ) = eigen(B + tJ)$. So $\det(A + tJ - \lambda I) = \det(B + tJ - \lambda I)$.

In this paper, we shall study the eigenvectors of A and B. Most of the results in this paper are not new. Our approach is new. We apply Theorem 1 to derive several well-known results. We first prove that the squares of the entries of simple unit eigenvectors of A can be reconstructed as functions of $eigen(A)$ and $eigen(A_i)$. This yields a proof of a Theorem of Godsil-McKay. We then study how the eigenvectors of A change after a perturbation of rank 1 symmetric matrices. Combined with Theorem 1, we prove another result of Godsil-McKay which states that the simple eigenvectors that are perpendicular to $\mathbf{1}$ are reconstructible. We further show that the orthogonal projection of $\mathbf{1}$ onto higher dimensional eigenspaces is reconstructible.

Our investigation indicates that the following conjecture could be true.

Conjecture 1 *Let A be a real $n \times n$ symmetric matrix. Then there exists a subgroup $G(A) \subseteq O(n)$ such that a real symmetric matrix B satisfies the properties that $eigen(B) = eigen(A)$ and $eigen(B_i) = eigen(A_i)$ for each i if and only if $B = UAU^t$ for some $U \in G(A)$.*

This conjecture is clearly true if $rank(A) = 1$. For $rank(A) = 1$, the group $G(A)$ can be chosen as $\mathbb{Z}_2^n$, all in the form of diagonal matrices. In some other cases, $G(A)$ can be a subgroup of the permutation group S_n.

2 Reconstruction of Square Functions

Theorem 2 *Let A be a $n \times n$ real symmetric matrix. Let $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ be the eigenvalues of A. Suppose λ_i is a simple eigenvalue of A. Let $\mathbf{p}_i = (p_{1,i}, p_{2,i}, \ldots, p_{n,i})^t$ be a unit vector in $eigen_{\lambda_i}(A)$. Then for every m, $p_{m,i}^2$ can be expressed as a function of $eigen(A)$ and $eigen(A_m)$.*

Proof: Let λ_i be a simple eigenvalue of A. Let $\mathbf{p}_i = (p_{1,i}, p_{2,i}, \ldots, p_{n,i})^t$ be a unit vector in $eigen_{\lambda_i}(A)$. There exists an orthogonal matrix P such that $P = (\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n)$ and $A = PDP^t$ where

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Then

$$A - \lambda_i I = PDP^t - \lambda_i I = P(D - \lambda_i I)P^t = \sum_{j \neq i} (\lambda_j - \lambda_i)\mathbf{p}_j \mathbf{p}_j^t.$$

which equals

$$
\begin{pmatrix}
p_{1,1} & \cdots & \widehat{p_{1,i}} & \cdots & p_{1,n} \\
p_{2,1} & \cdots & \widehat{p_{2,i}} & \cdots & p_{2,n} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
p_{n,1} & \cdots & \widehat{p_{n,i}} & \cdots & p_{n,n}
\end{pmatrix}
\begin{pmatrix}
\lambda_1 - \lambda_i & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & \widehat{\lambda_i - \lambda_i} & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & \lambda_n - \lambda_i
\end{pmatrix}
$$

$$
\begin{pmatrix}
p_{1,1} & p_{2,1} & \cdots & p_{n,1} \\
\vdots & \vdots & \ddots & \vdots \\
\widehat{p_{1,i}} & \widehat{p_{2,i}} & \cdots & \widehat{p_{n,i}} \\
\vdots & \vdots & \ddots & \vdots \\
p_{1,n} & p_{2,n} & \cdots & p_{n,n}
\end{pmatrix}.
$$

Deleting the m-th row and m-th column, we obtain

$$
\begin{pmatrix}
p_{1,1} & \cdots & \widehat{p_{1,i}} & \cdots & p_{1,n} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\widehat{p_{m,1}} & \cdots & \widehat{p_{m,i}} & \cdots & \widehat{p_{m,n}} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
p_{n,1} & \cdots & \widehat{p_{n,i}} & \cdots & p_{n,n}
\end{pmatrix}
\begin{pmatrix}
\lambda_1 - \lambda_i & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & \widehat{\lambda_i - \lambda_i} & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & \lambda_n - \lambda_i
\end{pmatrix}
$$

$$
\begin{pmatrix}
p_{1,1} & \cdots & \widehat{p_{m,1}} & \cdots & p_{n,1} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\widehat{p_{1,i}} & \cdots & \widehat{p_{m,i}} & \cdots & \widehat{p_{n,i}} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
p_{1,n} & \cdots & \widehat{p_{m,n}} & \cdots & p_{n,n}
\end{pmatrix}.
$$

This is $A_m - \lambda_i I_{n-1}$. Notice that P is orthogonal. Let $P_{m,i}$ be the matrix obtained by deleting the m-th row and i-th column. Then $\det P_{m,i}^2 = p_{m,i}^2$ where $p_{m,i}$ is the (m,i)-th entry of P. Taking the determinant, we have

$$\det(A_m - \lambda_i I_{n-1}) = p_{m,i}^2 \prod_{j \neq i} (\lambda_j - \lambda_i).$$

It follows that

$$p_{m,i}^2 = \frac{\prod_{j=1}^{n-1}(\lambda_j(A_m) - \lambda_i)}{\prod_{j \neq i}(\lambda_j - \lambda_i)}.$$

Q.E.D.

Corollary 1 *Let A and B be two $n \times n$ real symmetric matrices. Suppose that $eigen(A) = eigen(B)$ and $eigen(A_i) = eigen(B_i)$. Let λ_i be a simple eigenvalue of A and B. Let*

$\mathbf{p}_i = (p_{1,i}, p_{2,i}, \ldots, p_{n,i})^t$ *be a unit vector in* $eigen_{\lambda_i}(A)$ *and* $\mathbf{q}_i = (q_{1,i}, q_{2,i}, \ldots, q_{n,i})^t$ *be a unit vector in* $eigen_{\lambda_i}(B)$*. Then*

$$p_{j,i}^2 = q_{j,i}^2 \ \forall j \in [1, n].$$

Corollary 2 (Godsil-McKay, see Theorem 3.2, [GM]) *Let A and B be two $n \times n$ real symmetric matrices. Suppose that A and B are hypomorphic. Let λ_i be a simple eigenvalue of A and B. Let* $\mathbf{p}_i = (p_{1,i}, p_{2,i}, \ldots, p_{n,i})^t$ *be a unit vector in* $eigen_{\lambda_i}(A)$ *and* $\mathbf{q}_i = (q_{1,i}, q_{2,i}, \ldots, q_{n,i})^t$ *be a unit vector in* $eigen_{\lambda_i}(B)$*. Then*

$$p_{j,i}^2 = q_{j,i}^2 \ \forall j \in [1, n].$$

3 Eigenvalues and Eigenvectors under the perturbation of a rank one symmetric matrix

Let A be a $n \times n$ real symmetric matrix. Let x be a n-dimensional row column vector. Let $M = xx^t$. Now consider $A + tM$. We have

$$A + tM = PDP^t + tM = P(D + tP^tMP)P^t = P(D + tP^txx^tP)P^t.$$

Let $P^tx = q$. So $q_i = (\mathbf{p}_i, x)$ for each $i \in [1, n]$. Then

$$A + tM = P(D + tqq^t)P^t.$$

Put $D(t) = D + tqq^t$.

Lemma 1 $\det(D + tqq^t - \lambda I) = \det(A - \lambda I)(1 + \sum_i \frac{tq_i^2}{\lambda_i - \lambda})$.

Proof: $\det(D - \lambda I + tqq^t)$ can be written as a sum of products of $\lambda_i - \lambda$ and q_iq_j. For each S a subset of $[1, n]$, combine the terms containing only $\prod_{i \in S}(\lambda_i - \lambda)$. Since the rank of qq^t is one, only for $|S| = n, n-1$, the coefficients may be nonzero. We obtain

$$det(D + tqq^t - \lambda I) = \prod_{i=1}^{n}(\lambda_i - \lambda) + \sum_{i=1}^{n} tq_i^2 \prod_{j \neq i}(\lambda_i - \lambda).$$

The Lemma follows. □

Put $P_t(\lambda) = 1 + \sum_i \frac{tq_i^2}{\lambda_i - \lambda}$.

Lemma 2 *Fix $t < 0$. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct and $q_i \neq 0$ for every i. Then $P_t(\lambda)$ has exactly n roots $(\mu_1, \mu_2, \cdots, \mu_n)$ satisfying an interlacing relation:*

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \mu_{n-1} > \lambda_n > \mu_n.$$

Proof: Clearly, $\frac{dP_t(\lambda)}{d\lambda} = \sum_i \frac{tq_i^2}{(\lambda_i - \lambda)^2} < 0$. So $P_t(\lambda)$ is always decreasing. On the interval $(-\infty, \lambda_n)$, $\lim_{\lambda \to -\infty} P_t(\lambda) = 1$ and $\lim_{\lambda \to \lambda_n^-} P_t(\lambda) = -\infty$. So $P_t(\lambda)$ has a unique root $\mu_n \in (-\infty, \lambda_n)$. Similar statement holds for each $(\lambda_{i-1}, \lambda_i)$. On (λ_1, ∞), $\lim_{\lambda \to \infty} P_t(\lambda) = 1$ and $\lim_{\lambda \to \lambda_1^+} P_t(\lambda) = \infty$. So $P_t(\lambda)$ does not have any roots in (λ_1, ∞). Q.E.D.

Theorem 3 *Fix $t < 0$ and $x \in \mathbb{R}^n$. Let $M = xx^t$. Let l be the number of distinct eigenvalues satisfying $(x, eigen_\lambda(A)) \neq 0$. Choose an orthonormal basis of each eigenspace of A so that one of the eigenvectors is a multiple of the orthogonal projection of x onto the eigenspace if this projection is nonzero. Denote this basis by $\{\mathbf{p}_i\}$ and let $P = (\mathbf{p}_1, \mathbf{p}_2, \dots, p_n)$. Let*

$$S = \{i_1 > i_2 > \cdots > i_l\}$$

such that $(x, \mathbf{p}_i) \neq 0$ for every $i \in S$ and $(x, \mathbf{p}_i) = 0$ for every $i \notin S$. Then there exists $(\mu_1, \dots, \mu_l)$ such that

$$\lambda_{i_1} > \mu_1 > \lambda_{i_2} > \mu_2 > \cdots > \lambda_{i_l} > \mu_l$$

and

$$eigen(A + tM) = \{\lambda_i(A) \mid i \notin S\} \cup \{\mu_1, \mu_2 \dots, \mu_l\}.$$

Furthermore, $eigen_{\mu_j}(A + tM)$ contains

$$\sum_{i \in S} \mathbf{p}_i \frac{q_i}{\lambda_i - \mu_j}.$$

Here the index set $\{i_1, i_2, \cdots, i_l\}$ may not be unique. I shall also point out a similar statement holds for $t > 0$ with

$$\mu_1 > \lambda_{i_1} > \mu_2 > \lambda_{i_2} > \cdots > \mu_l > \lambda_{i_l}.$$

Proof: Recall that $q_i = (\mathbf{p}_i, x)$. Since $(x, eigen_{\lambda_{i_j}}(A)) \neq 0$, $q_{i_j} \neq 0$. For $i \notin S$, $q_i = 0$. Notice

$$P_t(\lambda) = 1 + \sum_{j=1}^{l} \frac{tq_{i_j}^2}{\lambda_{i_j} - \lambda}.$$

Applying Lemma 2 to S, we obtain the roots of $P_t(\lambda)$, $\{\mu_1, \mu_2, \dots, \mu_l\}$, satisfying

$$\lambda_{i_1} > \mu_1 > \lambda_{i_2} > \mu_2 > \cdots > \lambda_{i_l} > \mu_l.$$

It follows that the roots of $\det(A + tM - \lambda I) = P_t(\lambda) \prod_{i=1}^n (\lambda_i - \lambda)$ can be obtained from $eigen(A)$ be changing $\{\lambda_{i_1} > \lambda_{i_2} > \cdots > \lambda_{i_l}\}$ to $\{\mu_1, \mu_2 \dots, \mu_l\}$. Therefore,

$$eigen(A + tM) = \{\lambda_i(A) \mid i \notin S\} \cup \{\mu_1, \mu_2 \dots, \mu_l\}.$$

Fix a μ_j. Let $\{\mathbf{e}_i\}$ be the standard basis for $\mathbb{R}^n$. Notice that

$$\begin{aligned}
&(A+tM)\sum_{i\in S}\frac{q_i}{\lambda_i-\mu_j}\mathbf{p}_i\\
=&P(D+tqq^t)P^t\sum_{i\in S}\frac{q_i}{\lambda_i-\mu_j}\mathbf{p}_i\\
=&P(D+tqq^t)\sum_{i\in S}\frac{q_i}{\lambda_i-\mu_j}\mathbf{e}_i\\
=&P\left(\sum_{i\in S}\frac{\lambda_i q_i}{\lambda_i-\mu_j}\mathbf{e}_i+t\begin{pmatrix}q_1\\ \vdots\\ q_n\end{pmatrix}\sum_{i\in S}\frac{q_i^2}{\lambda_i-\mu_j}\right) \qquad (1)\\
=&P\left(\sum_{i\in S}\frac{\lambda_i q_i}{\lambda_i-\mu_j}\mathbf{e}_i-\sum_{i\in S}q_i\mathbf{e}_i\right)\\
=&P\sum_{i\in S}\frac{\mu_j q_i}{\lambda_i-\mu_j}\mathbf{e}_i\\
=&\mu_j\sum_{i\in S}\frac{q_i}{\lambda_i-\mu_j}\mathbf{p}_i
\end{aligned}$$

Notice that here we use the fact that $P_t(\mu_j)=\sum_{i\in S}\frac{tq_i^2}{\lambda_i-\mu_j}+1=0$. We have obtained that $(A+tM)\sum_{\lambda_i\in S}\frac{q_i}{\lambda_i-\mu_j}\mathbf{p}_i=\mu_j\sum_{i\in S}\frac{q_i}{\lambda_i-\mu_j}\mathbf{p}_i$. Therefore,

$$\sum_{i\in S}\frac{q_i}{\lambda_i-\mu_j}\mathbf{p}_i\in eigen_{\mu_j}(A+tM).$$

Q.E.D.

4 Reconstruction of Simple Eigenvectors not perpendicular to 1

Now let $M=\mathbf{J}=\mathbf{1}\mathbf{1}^t$. Theorem 3 applies to $A+t\mathbf{J}$ and $B+t\mathbf{J}$.

Theorem 4 (Godsil-McKay, [GM]) *Let B and A be two real $n\times n$ symmetric matrices. Let Σ be a hypomorphism such that $B=\Sigma(A)$. Let $S\subseteq[1,n]$, $A=PDP^t$ and $B=UDU^t$ be as in Theorem 3. For $i\in S$, we have $\mathbf{p}_i=\mathbf{u}_i$ or $\mathbf{p}_i=-\mathbf{u}_i$. In particular, if λ_i is a simple eigenvalue of A and $(eigen_{\lambda_i}(A),\mathbf{1})\neq 0$, then $eigen_{\lambda_i}(A)=eigen_{\lambda_i}(B)$.*

Proof: • By Tutte's theorem, $eigen(A)=eigen(B)$. Let $A=PDP^t$ and $B=UDU^t$. Since $\det(A+t\mathbf{J}-\lambda I)=\det(B+t\mathbf{J}-\lambda I)$, by Lemma 1,

$$\det(A-\lambda I)(1+\sum_i\frac{t(\mathbf{1},\mathbf{p}_i)^2}{\lambda_i-\lambda})=\det(B-\lambda I)(1+\sum_i\frac{t(\mathbf{1},\mathbf{u}_i)^2}{\lambda_i-\lambda}).$$

It follows that for every λ_i, $\sum_{\lambda_j=\lambda_i}(\mathbf{1},\mathbf{p}_j)^2=\sum_{\lambda_j=\lambda_i}(\mathbf{1},\mathbf{u}_j)^2$. Consequently, the l for A is the same as the l for B. Let S be as in Theorem 3 for both A and B. Without loss of generality, suppose that $A=PDP^t$ and $B=UDU^t$ as in Theorem 3. In particular, for every $i\in[1,n]$, we have

$$(\mathbf{p}_i,\mathbf{1})^2=(\mathbf{u}_i,\mathbf{1})^2. \tag{2}$$

• Let T be as in the proof of Theorem 1 in [HE1] for A and B. Without loss of generality, suppose $T=(t_1,t_2)\subseteq\mathbb{R}^-$. Let $t\in T$ and let $\mu_l(t)$ be the μ_l in Theorem 3 for A and B. Notice that the lowest eigenvectors of $A+t\mathbf{J}$ and $B+t\mathbf{J}$ are in $\mathbb{R}^{+n}$ (see Lemma 1, Theorem 7 and Proof of Theorem 2 in [HE1]). So they are not perpendicular to $\mathbf{1}$. By Theorem 3, $\mu_l(t)=\lambda_n(A+t\mathbf{J})=\lambda_n(B+t\mathbf{J})$. By Theorem 1,

$$eigen_{\mu_1(t)}(A+t\mathbf{J})=eigen_{\mu_l(t)}(B+t\mathbf{J})\cong\mathbb{R}.$$

So $\sum_{i\in S}\mathbf{p}_i\frac{(\mathbf{p}_i,\mathbf{1})}{\lambda_i-\mu_l(t)}$ is parallel to $\sum_{i\in S}\mathbf{u}_i\frac{(\mathbf{u}_i,\mathbf{1})}{\lambda_i-\mu_l(t)}$. Since $\{\mathbf{p}_i\}$ and $\{\mathbf{u}_i\}$ are orthonormal, by Equation 2,

$$\|\sum_{i\in S}\mathbf{p}_i\frac{(\mathbf{p}_i,\mathbf{1})}{\lambda_i-\mu_l(t)}\|^2=\|\sum_{i\in S}\mathbf{u}_i\frac{(\mathbf{u}_i,\mathbf{1})}{\lambda_i-\mu_l(t)}\|^2.$$

It follows that for every $t\in T$,

$$\sum_{i\in S}\mathbf{p}_i\frac{(\mathbf{p}_i,\mathbf{1})}{\lambda_i-\mu_l(t)}=\pm\sum_{i\in S}\mathbf{u}_i\frac{(\mathbf{u}_i,\mathbf{1})}{\lambda_i-\mu_l(t)}.$$

• Recall that $-\frac{1}{t}=\sum_i\frac{q_i^2}{\lambda_i-\mu_l(t)}$. Notice that the function $\rho\to\sum_i\frac{q_i^2}{\lambda_i-\rho}$ is a continuous and one-to-one mapping from $(-\infty,\lambda_n)$ onto $(0,\infty)$. There exists a nonempty interval $T_0\subseteq(-\infty,\lambda_n)$ such that if $\rho\in T_0$, then $\sum_i\frac{q_i^2}{\lambda_i-\rho}\in(-\frac{1}{t_1},-\frac{1}{t_2})$. So every $\rho\in T_0$ is a $\mu_l(t)$ for some $t\in(t_1,t_2)$. It follow that for every $\rho\in T_0$,

$$\sum_{i\in S}\mathbf{p}_i\frac{(\mathbf{p}_i,\mathbf{1})}{\lambda_i-\rho}=\pm\sum_{i\in S}\mathbf{u}_i\frac{(\mathbf{u}_i,\mathbf{1})}{\lambda_i-\rho}.$$

Notice that both vectors are nonzero and depend continuously on ρ. Either,

$$\sum_{i\in S}\mathbf{p}_i\frac{(\mathbf{p}_i,\mathbf{1})}{\lambda_i-\rho}=\sum_{i\in S}\mathbf{u}_i\frac{(\mathbf{u}_i,\mathbf{1})}{\lambda_i-\rho}\qquad\forall\ (\rho\in T_0);$$

or,

$$\sum_{i\in S}\mathbf{p}_i\frac{(\mathbf{p}_i,\mathbf{1})}{\lambda_i-\rho}=-\sum_{i\in S}\mathbf{u}_i\frac{(\mathbf{u}_i,\mathbf{1})}{\lambda_i-\rho}\qquad\forall\ (\rho\in T_0);$$

•. Notice that the functions $\{\rho\to\frac{1}{\lambda_{i_j}-\rho}\}|_{i_j\in S}$ are linearly independent. For every $i\in S$, we have

$$\mathbf{p}_i(\mathbf{p}_i,\mathbf{1})=\pm\mathbf{u}_i(\mathbf{u}_i,\mathbf{1}).$$

Because $\mathbf{p}_i$ and $\mathbf{u}_i$ are both unit vectors, $\mathbf{p}_i=\pm\mathbf{u}_i$. In particular, for every simple λ_i with $(\mathbf{p}_i,\mathbf{1})\neq 0$ we have $eigen_{\lambda_i}(A)=eigen_{\lambda_i}(B)$. Q.E.D.

Corollary 3 *Let B and A be two real $n \times n$ symmetric matrices. Suppose that $B = \Sigma(A)$ for a hypomorphism Σ. Let λ_i be an eigenvalue of A such that $(eigen_{\lambda_i}(A), \mathbf{1}) \neq 0$. Then the orthogonal projection of $\mathbf{1}$ onto $eigen_{\lambda_i}(A)$ equals the orthogonal projection of $\mathbf{1}$ onto $eigen_{\lambda_i}(B)$.*

Proof: Notice that the projections are $\mathbf{p}_i(\mathbf{p}_i, \mathbf{1})$ and $\mathbf{u}_i(\mathbf{u}_i, \mathbf{1})$. Whether $\mathbf{p}_i = \mathbf{u}_i$ or $\mathbf{p}_i = -\mathbf{u}_i$, we always have

$$\mathbf{p}_i(\mathbf{p}_i, \mathbf{1}) = \mathbf{u}_i(\mathbf{u}_i, \mathbf{1}).$$

Q.E.D.

Conjecture 2 *Let A and B be two hypomorphic matrices. Let λ_i be a simple eigenvalue of A. Then there exists a permutation matrix τ such that $\tau eigen_{\lambda_i}(A) = eigen_{\lambda_i}(B)$.*

This conjecture is apparently true if $eigen_{\lambda_i}(A)$ is not perpendicular to $\mathbf{1}$.

References

[Tutte] W. T. Tutte, "All the King's Horses (A Guide to Reconstruction)", *Graph Theory and Related Topics*, Academic Press, 1979, (15-33).

[GM] C. D. Godsil and B. D. McKay, "Spectral Conditions for the Reconstructiblity of a graph", *J. Combin. Theory Ser. B* 30. 1981, No. 3, (285-289).

[HE1] H. He, "Reconstruction and Higher Dimensional Geometry", *Journal of Combinatorial Theory, Series B* 97, No 3 (421-429).

[Ko] W. Kocay, "Some New Methods in Reconstruction Theory", *Combinatorial mathematics, IX (Brisbane, 1981)*, LNM 952, (89-114).