Compositions of Graphs Revisited

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Abstract

The idea of graph compositions, which was introduced by A. Knopfmacher and M. E. Mays, generalizes both ordinary compositions of positive integers and partitions of finite sets. In their original paper they developed formulas, generating functions, and recurrence relations for composition counting functions for several families of graphs. Here we show that some of the results involving compositions of bipartite graphs can be derived more easily using exponential generating functions.

Keywords: compositions, bipartite graph, stirling number.

1 Introduction

A composition of a graph G is a partition of the vertex set of G into vertex sets of connected induced subgraphs of G. Knopfmacher and Mays [2] found an explicit formula for $C(K_{m,n})$, the number of compositions of the complete bipartite graph $K_{m,n}$ in the form

$$C(K_{m,n}) = \sum_{i=1}^{m+1} a_{m,i} i^n, \tag{1}$$

where $(a_{i,j})$ is an array defined via the recurrences $a_{m,0} = 0$ for any nonnegative integer m, $a_{0,1} = 1$, $a_{0,n} = 0$ for any $n > 1$, and otherwise

$$a_{m,n} = \sum_{i=0}^{m-1} \binom{m-1}{i} a_{m-1-i,n-1} - \sum_{i=1}^{m-1} \binom{m-1}{i} a_{m-1-i,n-1}.$$

We will derive this result using exponential generating functions and also show that we can express the coefficients $a_{m,i}$ explicitly in terms of the Stirling numbers of the second kind. We first need to describe some basic properties of exponential generating functions in two variables. We will use Stanley's notation [4] throughout this paper.

2 Exponential generating function in two variables

Proposition 1. *Given functions $f, g : \mathbb{N} \times \mathbb{N} \to K$, where K is a field of characteristic 0, we define a new function $h : \mathbb{N} \times \mathbb{N} \to K$ by*

$$h(\#X, \#Y) = \sum f(\#S, \#T)g(\#U, \#V)$$

where X and Y are finite sets and the sum is over all S, T, U and V such that $X = S \uplus U$ and $Y = T \uplus V$; i.e., X and Y are disjoint unions of S, U and T, V respectively.

Then

$$E_h(x, y) = E_f(x, y)E_g(x, y), \tag{2}$$

where the exponential generating function of f is defined by

$$E_f(x, y) = \sum_{m,n=0}^{\infty} f(m, n)\frac{x^m}{m!}\frac{y^n}{n!}.$$

Proof. If $\#X = m$ and $\#Y = n$, then there are $\binom{m}{i}$ pairs (S, U) and $\binom{n}{j}$ pairs (T, V), where $X = S \uplus U$ and $Y = T \uplus V$ with $\#S = i, \#T = j, \#U = m - i$ and $\#V = n - j$. Then $h(m, n)$ is given by

$$h(m, n) = \sum_{i,j=0}^{m,n} \binom{m}{i}\binom{n}{j} f(i, j)g(m - i, n - j)$$

and this is equivalent to (2). $\square$

Corollary. Given functions $f_1, \ldots, f_k : \mathbb{N} \times \mathbb{N} \to K$, we can define a new function $h : \mathbb{N} \times \mathbb{N} \to K$ by

$$h(\#X, \#Y) = \sum \prod_{i=1}^{k} f_i(\#S_i, \#T_i)$$

where the sum is over all $(S_1, \ldots, S_k)$ and $(T_1, \ldots, T_k)$ such that $X = \uplus_{i=1}^{k} S_i$ and $Y = \uplus_{i=1}^{k} T_i$. Then

$$E_h(x, y) = \prod_{i=1}^{k} E_{f_i}(x, y). \tag{3}$$

Proposition 2. *Given the function $f : \mathbb{N} \times \mathbb{N} \to K$, where K is a field of characteristic 0 and $f(0, 0) = 0$, define a new function $h : \mathbb{N} \times \mathbb{N} \to K$ such that for disjoint finite sets X and Y,*

$$h(\#X, \#Y) = \sum_{\{S_1,\ldots,S_k\}} \prod_{i=1}^{k} f(\#(S_i \cap X), \#(S_i \cap Y)). \tag{4}$$

where the sum is taken over all partitions $\{S_1, \ldots, S_k\}$ of the set $X \cup Y$. Then

$$E_h(x, y) = \exp(E_f(x, y)).$$

Proof. Let k be fixed. Then the blocks of the partition $\{S_1, \dots, S_k\}$ are all distinct and so are the pairs $(S_i \cap X, S_i \cap Y)$ for $i = 1, \dots, k$. So there are $k!$ ways of linearly ordering them. If we define $h_k(\#X, \#Y)$ by

$$h_k(\#X, \#Y) = \sum_{\{S_1,\dots,S_k\}} \prod_{i=1}^{k} f(\#(S_i \cap X), \#(S_i \cap Y))$$

for a fixed value of k then by the Corollary to Proposition 1 we get

$$E_{h_k}(x, y) = \frac{(E_f(x, y))^k}{k!}.$$

Therefore summing over all $k \geq 0$ gives the desired result. □

Example. Let X and Y be disjoint sets with $\#X = m$ and $\#Y = n$ and let $C(m, n)$ be the number of connected bipartite graphs between the sets X and Y. Then

$$\exp\left(\sum_{m,n=0}^{\infty} C(m, n) \frac{x^m}{m!} \frac{y^n}{n!} \right) = \sum_{m,n=0}^{\infty} 2^{mn} \frac{x^m}{m!} \frac{y^n}{n!}. \tag{5}$$

This can be seen easily because the coefficient of $\frac{x^m}{m!}\frac{y^n}{n!}$ in the right-hand side of (5) is the number of bipartite graphs with bipartition (X, Y). On the other hand the number of such graphs in which the vertex sets of the connected components are $\{S_1, S_2, \dots, S_k\}$ is $\prod_{i=1}^{k} C(\#(S_i \cap X), \#(S_i \cap Y))$. So summing over all partition $\{S_1, S_2, \dots, S_k\}$ of $X \cup Y$ and applying Proposition 2 shows that the number of bipartite graphs with bipartition (X, Y) is the coefficient of $\frac{x^m}{m!}\frac{y^n}{n!}$ in the left-hand side.

3 Compositions of bipartite graphs

Let G be a labelled graph with vertex set $V(G)$. A composition of G is a partition of $V(G)$ into vertex sets of connected induced subgraphs of G. Thus a composition provides a set of connected induced subgraphs of G, $\{G_1, G_2, \cdots, G_m\}$, with the properties that $\bigcup_{i=1}^{m} V(G_i) = V(G)$ and for $i \neq j$, $V(G_i) \bigcap V(G_j) = \emptyset$.

Let $C(G)$ denote the number of distinct compositions of the graph G. For example, the complete bipartite graph $K_{2,3}$ has exactly 34 compositions. In this section we will consider complete bipartite graph only.

Consider a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ as follows: Given $m, n \in \mathbb{N}$ we define

$$f(m, n) = \begin{cases} 1, & \text{if } m > 0 \text{ and } n > 0 \\ & \text{or } m = 1 \text{ and } n = 0 \\ & \text{or } m = 0 \text{ and } n = 1 \\ 0, & \text{otherwise.} \end{cases}$$

In other words $f(m,n)=1$ if $K_{m,n}$ is connected and 0 if $K_{m,n}$ is not connected. We also define $h:\mathbb{N}\times\mathbb{N}\to\mathbb{Q}$ by

$$h(m,n)=\sum_{\{S_1,\ldots,S_k\}}\prod_{i=1}^{k} f(\#(S_i\cap X),\#(S_i\cap Y)),$$

where $\#X=m$ and $\#Y=n$. Then $h(m,n)=C(K_{m,n})$ and thus by Proposition 2 we get

$$\sum_{m,n=0}^{\infty} C(K_{m,n})\frac{x^m}{m!}\frac{y^n}{n!}=\exp(E_f(x,y)). \tag{6}$$

But from the definition of $E_f(x,y)$ we have

$$\begin{aligned} E_f(x,y)=\sum_{m,n=0}^{\infty} f(m,n)\frac{x^m}{m!}\frac{y^n}{n!} &= \sum_{m,n=1}^{\infty}\frac{x^m}{m!}\frac{y^n}{n!}+x+y\\ &=\sum_{m=1}^{\infty}\frac{x^m}{m!}\sum_{n=1}^{\infty}\frac{y^n}{n!}+x+y\\ &=(e^x-1)(e^y-1)+x+y. \end{aligned}$$

So

$$\sum_{m,n=0}^{\infty} C(K_{m,n})\frac{x^m}{m!}\frac{y^n}{n!}=e^{(e^x-1)(e^y-1)+x+y}. \tag{7}$$

Knopfmacher and Mays [2] showed that

$$C(K_{m,n})=\sum_{i=1}^{m+1} a_{m,i}i^n. \tag{8}$$

for some integers $a_{m,i}$. We will derive the same result here from (7).

We start by defining integers $a_{m,i}$ by

$$\lambda e^{(e^x-1)(\lambda-1)+x}=\sum_{m=0}^{\infty}\frac{x^m}{m!}\sum_{i=0}^{\infty} a_{m,i}\lambda^i. \tag{9}$$

Now equating the constant term in x on both sides we get

$$\lambda=\sum_{i=0}^{\infty} a_{0,i}\lambda^i,$$

which shows that $a_{0,1}=1$ and $a_{0,i}=0$ for $i\neq 1$. We observe that $a_{m,0}=0$. Now we equate the coefficients of λ^n on both sides of (9).

On the left we have

$$[\lambda^n]\lambda e^{(e^x-1)\lambda}e^{x-(e^x-1)} = [\lambda^{n-1}]e^{(e^x-1)\lambda}e^{x-(e^x-1)}$$
$$= \frac{(e^x-1)^{n-1}}{(n-1)!}e^{x-(e^x-1)}.$$

On the right we have

$$[\lambda^n]\sum_{m=0}^{\infty}\frac{x^m}{m!}\sum_{i=0}^{\infty}a_{m,i}\lambda^i = \sum_{m=0}^{\infty}\frac{x^m}{m!}a_{m,n}.$$

So

$$\frac{(e^x-1)^{n-1}}{(n-1)!}e^{x-(e^x-1)} = \sum_{m=0}^{\infty}\frac{x^m}{m!}a_{m,n}.$$

Thus $a_{m,n}=0$ for $m<n-1$, i.e., $n>m+1$. So we may write (9) as

$$\lambda e^{(e^x-1)(\lambda-1)+x} = \sum_{m=0}^{\infty}\frac{x^m}{m!}\sum_{i=1}^{m+1}a_{m,i}\lambda^i. \tag{10}$$

Letting $\lambda=e^y$ in equation (10) and using (7) we get

$$\sum_{m,n}C(K_{m,n})\frac{x^m}{m!}\frac{y^n}{n!} = \sum_{m=0}^{\infty}\frac{x^m}{m!}\sum_{i=1}^{m+1}a_{m,i}e^{iy}.$$

Equating coefficients of $\dfrac{x^m}{m!}\dfrac{y^n}{n!}$ we get

$$C(K_{m,n}) = \sum_{i=1}^{m+1}a_{m,i}i^n,$$

which is the desired result.

4 Relation with Stirling numbers of the second kind

The Stirling number of the second kind $S(n,m)$ counts the number of ways of partitioning a set of n elements into m nonempty sets. We can also get an expression for $a_{m,i}$ involving the Stirling numbers of the second kind. To do this let

$$\rho_m(z) = \sum_{i=1}^{m+1}a_{m,i}z^i.$$

Then setting $\lambda=z+1$ in (10) we get

$$(1+z)e^xe^{(e^x-1)z} = \sum_{m=0}^{\infty}\frac{x^m}{m!}\rho_m(1+z)$$

or

$$e^x e^{(e^x-1)z} = \sum_{m=0}^{\infty} \frac{x^m}{m!} \frac{\rho_m(z+1)}{z+1}.$$

Using the previous equation gives

$$\frac{d}{dx} e^{(e^x-1)z} = z e^x e^{(e^x-1)z} = \sum_{m=0}^{\infty} \frac{x^m}{m!} \frac{z}{z+1} \rho_m(z+1). \tag{11}$$

The generating function for the Stirling numbers of the second kind is

$$\sum_{m,k} S(m,k) \frac{x^m}{m!} z^k = e^{(e^x-1)z}. \tag{12}$$

So from (11) and (12) we get

$$\sum_{m,k} S(m+1,k) \frac{x^m}{m!} z^k = \frac{d}{dx} \sum_{m,k} S(m,k) \frac{x^m}{m!} z^k = \sum_{m=0}^{\infty} \frac{x^m}{m!} \frac{z}{z+1} \rho_m(z+1).$$

Equating coefficients of $\frac{x^m}{m!}$ we get

$$\sum_{k} S(m+1,k) z^k = \frac{z}{z+1} \rho_m(z+1).$$

So

$$\rho_m(z) = z \sum_{k} S(m+1,k)(z-1)^{k-1}.$$

From this we can easily extract the coefficient of z^i to get

$$a_{m,i} = \sum_{k} \binom{k-1}{i} (-1)^{k-i} S(m+1,k).$$

5 Generalization

The generalization of (7) to complete multipartite graphs is easy if we use the generating function method. A complete multipartite graph is a multipartite graph such that any two vertices that are not in the same part have an edge connecting them. The number of edges for such graphs are given by the formula $a_1(a_2+\cdots+a_n)+a_2(a_3+\cdots+a_n)+\cdots+a_{n-1}a_n$, where each a_i is the number of vertices in that part. If $K_{a_1,a_2,\ldots,a_n}$ is a complete multipartite graph with $a_1+a_2+\cdots+a_n$ vertices then the number of compositions for this graph is given by the generating function

$$\sum_{a_1,a_2,\ldots,a_n=0}^{\infty} C(K_{a_1,a_2,\ldots,a_n}) \frac{{x_1}^{a_1}}{a_1!} \frac{{x_2}^{a_2}}{a_2!} \cdots \frac{{x_n}^{a_n}}{a_n!} = y_1 y_2 \cdots y_n e^{y_1 y_2 \cdots y_n - y_1 - y_2 \cdots - y_n + n - 1}, \tag{13}$$

where $y_i = e^{x_i}$.

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