A Note on The Rogers-Fine Identity

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Abstract

In this paper, we derive an interesting identity from the Rogers-Fine identity by applying the q-exponential operator method.

1 Introduction and main result

Following Gasper and Rahman [7], we write

$$(a;q)_0 = 1, \quad (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), n = 1, \cdots, \infty,$$

$${}_r\Phi_s\left(\begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array}; \ q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_r; q)_n}{(q, b_1, \cdots, b_s; q)_n} \left[(-1)^n q^{n(n-1)/2}\right]^{1+s-r} x^n.$$

For convenience, we take $|q| < 1$ in this paper.

Recall that the Rogers-Fine identity [1, 2, 6, 10] is expressed as follows:

$$\sum_{n=0}^{\infty} \frac{(\alpha;q)_n}{(\beta;q)_n}\tau^n = \sum_{n=0}^{\infty} \frac{(\alpha;q)_n (q\alpha\tau/\beta;q)_n (1-\alpha\tau q^{2n})}{(\beta;q)_n (\tau;q)_{n+1}} (\beta\tau)^n q^{n^2-n}. \tag{1}$$

This identity (1) is one of the fundamental formulas in the theory of the basic hypergeometric series. In this paper, we derive an interesting identity from (1) by applying the q-exponential operator method. As application, we give an extension of the terminating very-well-poised ${}_6\Phi_5$ summation formula. The main result of this paper is:

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Theorem 1.1. Let $a_{-1}, a_0, a_1, a_2, \cdots, a_{2t+2}$ be complex numbers, $|a_{2i}| < 1$ with $i = 0, 1, 2, \cdots, t+1$, then for any non-negative integer M, we have

$$\begin{aligned}
&\sum_{n=0}^{M} \frac{(q^{-M}, c, a_2, a_4, \cdots, a_{2t+2}; q)_n}{(\beta, b, a_1, a_3, \cdots, a_{2t+1}; q)_n} \tau^n \\
&\quad = \sum_{m=0}^{M} \frac{(q^{-M}; q)_m (\tau q^{1-M}/\beta; q)_m (1 - \tau q^{2m-M})}{(\beta; q)_m (\tau; q)_{m+1}} (\beta\tau)^m \, q^{m^2-m} \\
&\quad \times \prod_{j=0}^{t+1} \frac{(a_{2j}; q)_m}{(a_{2j-1}; q)_m} \sum_{m_1=0}^{m} \frac{(q^{-m}, q^{1-m}/\beta, b/c; q)_{m_1}}{(q, q^{1-M}\tau/\beta, q^{1-m}/c; q)_{m_1}} \sum_{0 \le m_{t+2} \le m_{t+1} \le \cdots \le m_2 \le m_1} \\
&\quad \prod_{i=1}^{t+1} \frac{(q^{-m_i}, q^{1-m}/a_{2i-3}, a_{2i-1}/a_{2i}; q)_{m_{i+1}}}{(q, q^{1-m}/a_{2i}, q^{1-m_i} a_{2i-2}/a_{2i-3}; q)_{m_{i+1}}} q^{m_1+m_2+\cdots+m_{t+2}},
\end{aligned} \tag{2}$$

where $t = -1, 0, 1, 2, \cdots, \infty$, $c = a_0$ and $b = a_{-1}$.

2 The proof of the theorem and its application

Before our proof, let's first make some preparations. The q-differential operator D_q and q-shifted operator η (see [3, 4, 8, 9]), acting on the variable a, are defined by:

$$D_q \{f(a)\} = \frac{f(a) - f(aq)}{a} \quad \text{and} \quad \eta \{f(a)\} = f(aq).$$

Rogers [9] first used them to construct the following q-operator

$$E(d\theta) = (-d\theta; q)_\infty = \sum_{n=0}^{\infty} \frac{q^{(n-1)n/2} (d\theta)^n}{(q; q)_n}, \tag{3}$$

where $\theta = \eta^{-1} D_q$. Note that, Rogers used the symbol $q\delta$ to denote θ [9]. Then he applied it to derive relationships between special functions involving certain fundamental q-symmetric polynomials. This operator theory was developed by Chen and Liu [4] and Liu [8]. They employed (3) to obtain many classical q-series formulas. Later Bowman [3] studied further results of this operator and gave convergence criteria. He used it to obtain results involving q-symmetric expansions and q-orthogonal polynomials. Inspired by their work, we constructed the following q-exponential operator [5]

Definition 2.1. Let $\theta = \eta^{-1} D_q$, b, c are complex numbers. We define

$${}_1\Phi_0 \left(\begin{matrix} b \\ - \end{matrix} ; \; q, -c\theta \right) = \sum_{n=0}^{\infty} \frac{(b; q)_n (-c\theta)^n}{(q; q)_n}. \tag{4}$$

In [5], we have applied it to obtain some formal extensions of q-series formulas. Notice that the operator $E(d\theta)$ follows (4) by setting $c = dh$, $b = 1/h$, and taking $h = 0$. The following operator identities were given in [5]:

Lemma 2.1. If $s/\omega = q^{-N}$, $|cst/\omega| < 1$, where N is a non-negative integer, then

$$
\begin{aligned}
&{}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix} ;\ q, -c\theta\right)\left\{\frac{(as, at; q)_\infty}{(a\omega; q)_\infty}\right\} \\
&= \frac{(as, at, bct; q)_\infty}{(a\omega, ct; q)_\infty}\ {}_3\Phi_2\left(\begin{matrix} b, & s/\omega, & q/at \\ & q/ct, & q/a\omega \end{matrix} ;\ q, q\right). \qquad (5)
\end{aligned}
$$

Lemma 2.2. For $|cs| < 1$,

$$
{}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix} ;\ q, -c\theta\right)\{(as; q)_\infty\} = \frac{(as, bcs; q)_\infty}{(cs; q)_\infty}. \qquad (6)
$$

Now let's return to the proof of Theorem 1.1. Employing

$$
(q/a; q)_n = (-a)^{-n} q^{n(n+1)/2} \frac{(q^{-n}a; q)_\infty}{(a; q)_\infty}, \qquad (7)
$$

and setting $\alpha = q^{-M}$ in (1), we rewrite the new expression as follows:

$$
\begin{aligned}
\sum_{n=0}^{M} \left(q^{-M}; q\right)_n (\beta q^n; q)_\infty \tau^n &= \sum_{n=0}^{M} \frac{(q^{-M}; q)_n (1 - q^{2n-M}\tau)}{(\tau; q)_{n+1}} \left(-q^{-M}\tau^2\right)^n \\
&\quad \times q^{n(3n-1)/2} \frac{(\beta q^{M-n}/\tau, \beta q^n; q)_\infty}{(\beta q^M/\tau; q)_\infty}. \qquad (8)
\end{aligned}
$$

Applying the operator ${}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix} ;\ q, -c\theta\right)$ to both sides of (8) with respect to the variable β then we have

$$
\begin{aligned}
&\sum_{n=0}^{M} \left(q^{-M}; q\right)_n \tau^n\ {}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix} ;\ q, -c\theta\right)\{(\beta q^n; q)_\infty\} \\
&= \sum_{n=0}^{M} \frac{(q^{-M}; q)_n (1 - \tau q^{2n-M})}{(\tau; q)_{n+1}} \left(-q^{-M}\tau^2\right)^n \\
&\quad \times q^{n(3n-1)/2}\ {}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix} ;\ q, -c\theta\right)\left\{\frac{(\beta q^{M-n}/\tau, \beta q^n; q)_\infty}{(q^M\beta/\tau; q)_\infty}\right\}.
\end{aligned}
$$

By (5) and (6), we have the relation

$$
\begin{aligned}
\sum_{n=0}^{M} \frac{(q^{-M}, c; q)_n}{(\beta, bc; q)_n}\tau^n &= \sum_{n=0}^{M} \frac{(q^{-M}; q)_n (q^{1-M}\tau/\beta, c; q)_n (1 - \tau q^{2n-M})}{(\beta, bc; q)_n (\tau; q)_{n+1}} (\beta\tau)^n q^{n^2-n} \\
&\quad \times\ {}_3\Phi_2\left(\begin{matrix} q^{-n}, b, q^{1-n}/\beta \\ q^{1-M}\tau/\beta, q^{1-n}/c \end{matrix} ;\ q, q\right). \qquad (9)
\end{aligned}
$$

Using (7) again , we rewrite (9) as follows:

$$\sum_{n=0}^{M}\frac{(q^{-M},c;q)_n}{(\beta;q)_n}\tau^n(bcq^n;q)_\infty = \sum_{n=0}^{M}\frac{(q^{-M};q)_n(q^{1-M}\tau/\beta,c;q)_n(1-\tau q^{2n-M})}{(\beta;q)_n(\tau;q)_{n+1}}(\beta\tau)^n q^{n^2-n} \times\sum_{n_1=0}^{n}\frac{(q^{-n},q^{1-n}/\beta;q)_{n_1}q^{n_1}}{(q,q^{1-M}\tau/\beta,q^{1-n}/c;q)_{n_1}}\frac{(b,bcq^n;q)_\infty}{(bq^{n_1};q)_\infty}. \tag{10}$$

Applying the operator ${}_1\Phi_0\left(\begin{array}{c} a_1 \\ - \end{array};\ q,-a_2\theta\right)$ to both sides of (10) with respect to the variable b, from (5) and (6) and simplifying then we have

$$\sum_{n=0}^{M}\frac{(q^{-M},c,a_2c;q)_n}{(\beta,bc,a_1a_2c;q)_n}\tau^n = \sum_{n=0}^{M}\frac{(q^{-M};q)_n(q^{1-M}\tau/\beta,c,a_2c;q)_n(1-\tau q^{2n-M})}{(\beta,bc,a_1a_2c;q)_n(\tau;q)_{n+1}}(\beta\tau)^n q^{n^2-n} \times\sum_{n_1=0}^{n}\frac{(q^{-n},q^{1-n}/\beta,b;q)_{n_1}}{(q,q^{1-M}\tau/\beta,q^{1-n}/c;q)_{n_1}}q^{n_1}\sum_{n_2=0}^{n_1}\frac{(q^{-n_1},q^{1-n}/bc,a_1;q)_{n_2}}{(q,q^{1-n}/a_2c,q^{1-n_1}/b;q)_{n_2}}q^{n_2}. \tag{11}$$

Replacing bc by b in (9), we have the case of $t=-1$. If we replace (bc,a_2c,a_1a_2c) by (b,a_2,a_1) in (11) respectively, we obtain the case of $t=0$.

By induction, similar proof can be performed to get the equation (2).

Letting $t=-1$ in (2), and then setting $b=q^{1-M}\tau/\beta$, we have the following identity:

Corollary 2.1. If $|c|<1$, then

$$\sum_{n=0}^{M}\frac{(q^{-M},c;q)_n}{(\beta,q^{1-M}c\tau/\beta;q)_n}\tau^n = \sum_{n=0}^{M}\frac{(q^{-M};q)_n(q^{1-M}\tau/\beta,\beta/c;q)_n(1-\tau q^{2n-M})}{(\beta,q^{1-M}c\tau/\beta;q)_n(\tau;q)_{n+1}}(-c\tau)^n q^{n(n-1)/2}. \tag{12}$$

Combined with (12), we can get the following extension of the terminating very-well-poised ${}_6\Phi_5$ summation formula:

Theorem 2.1. For $|c|<1$, $|e|<1$ and $|\tau|<1$

$$\sum_{n=0}^{M}\frac{(1-\tau q^{2n})(\tau,q^{-M};q)_n}{(1-\tau)(q,\tau q^{M+1};q)_n}(-c\tau q^M)^n q^{n(n-1)^2}\frac{(q/c,e\tau;q)_n}{(c\tau,de\tau;q)_n} \times {}_3\Phi_2\left(\begin{array}{ccc} q^{-n}, & q^{1-n}/c\tau, & d \\ & q^{1-n}/e\tau, & q/c \end{array};\ q,q\right) = \frac{(\tau q,e\tau;q)_M}{(c\tau,de\tau;q)_M}. \tag{13}$$

Proof. Setting $\beta=q$ and replacing τ by τq^M in (12), we have

$$\frac{(\tau q;q)_M}{(c\tau;q)_M} = \sum_{n=0}^{M}\frac{(1-\tau q^{2n})(\tau,q/c,q^{-M};q)_n}{(1-\tau)(q,c\tau,\tau q^{M+1};q)_n}(-c\tau q^M)^n q^{n(n-1)/2}. \tag{14}$$

Employing (7), we rewrite (14) as follows:

$$(\tau q;q)_M(c\tau q^M;q)_\infty=\sum_{n=0}^{M}\frac{(1-\tau q^{2n})(\tau,q^{-M};q)_n}{(1-\tau)(q,\tau q^{M+1};q)_n}(\tau q^M)^n q^{n^2}\frac{(cq^{-n},c\tau q^n;q)_\infty}{(c;q)_\infty}. \tag{15}$$

Applying the operator ${}_1\Phi_0\left(\begin{matrix} d \\ - \end{matrix}\;;\;q,-e\theta\right)$ to both sides of (15) with respect to the variable c, using (5) and (6) and simplifying then we complete the proof.

Taking $d=q/c$ then setting $e=cf/q$ in (13), we have

Corollary 2.2 (The terminating very-well-poised ${}_6\Phi_5$ summation formula).

$${}_6\Phi_5\left(\begin{matrix} q^{-M},\tau,q\sqrt{\tau},-q\sqrt{\tau},q/c,q/f \\ \tau q^{M+1},\sqrt{\tau},-\sqrt{\tau},c\tau,f\tau \end{matrix}\;;\;q,cf\tau q^{M-1}\right)=\frac{(\tau q,cf\tau/q;q)_M}{(c\tau,f\tau;q)_M}.$$

Remark: In the context of this paper, convergence of the basic hypergeometric series is no issue at all because they are terminating q-series.

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