On Directed Triangles in Digraphs*

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Abstract

Using a recent result of Chudnovsky, Seymour, and Sullivan, we slightly improve two bounds related to the Caccetta-Haggkvist Conjecture. Namely, we show that if $\alpha \geq 0.35312$, then each n-vertex digraph D with minimum outdegree at least αn has a directed 3-cycle. If $\beta \geq 0.34564$, then every n-vertex digraph D in which the outdegree and the indegree of each vertex is at least βn has a directed 3-cycle.

1 Introduction

In this note we follow the notation of [5]. For a vertex u in a digraph $D = (V, E)$, let $N^+(u) = \{v \in V : (u, v) \in E\}$ and $N^-(u) = \{v \in V : (v, u) \in E\}$. Every digraph in this note has no parallel or antiparallel edges.

Caccetta and Häggkvist [2] conjectured that each n-vertex digraph with minimum outdegree at least d contains a directed cycle of length at most $\lceil n/d \rceil$. The following important case of the conjecture is still open: *Each n-vertex digraph with minimum outdegree at least $n/3$ contains a directed triangle.* Caccetta and Häggkvist [2] proved the following weakening of the conjecture.

Theorem 1. [2] *If $\alpha \geq (3-\sqrt{5})/2 \sim 0.38196\ldots$, then each n-vertex digraph D with minimum outdegree at least αn has a directed 3-cycle.*

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Then Bondy [1] relaxed the restriction on α in Theorem 1 to $\alpha \geq (2\sqrt{6}-3)/5 \sim 0.37979$ and Shen [5] relaxed it to $\alpha \geq 3 - \sqrt{7} \sim 0.354248$.

De Graaf, Schrijver, and Seymour [4] considered the corresponding problem for digraphs in which both the outdegrees and indegrees are bounded from below. They proved that every n-vertex digraph in which the outdegree and the indegree of each vertex is at least $0.34878n$ has a directed 3-cycle. Shen's bound [5] on α implies an improvement of the de Graaf–Schrijver–Seymour bound to $0.347785n$. Here we use a recent result of Chudnovsky, Seymour, and Sullivan [3] to somewhat improve these results as follows.

Theorem 2. *If $\alpha \geq 0.35312$, then each n-vertex digraph D with minimum outdegree at least αn has a directed 3-cycle.*

Theorem 3. *If $\beta \geq 0.34564$, then each n-vertex digraph D in which both minimum outdegree and minimum indegree is at least βn has a directed 3-cycle.*

In the next section, we cite the Chudnovsky–Seymour–Sullivan result and a conjecture of theirs, and derive a useful consequence. In Section 3, we outline Shen's proof of his bound on α in [5]. In Sections 4 and 5 we prove Theorem 2. In Section 6 we outline a part of the proof in [4] and prove Theorem 3.

2 A result on dense digraphs

Chudnovsky, Seymour, and Sullivan [3] proved the following fact.

Lemma 4. *If a digraph D is obtained from a tournament by deleting k edges and has no directed triangles, then one can delete from D an additional k edges so that the resulting digraph D' is acyclic.*

We use this fact for the following lemma.

Lemma 5. *If a digraph D is obtained from a tournament by deleting k edges and has no directed triangles, then it has a vertex with outdegree less than $\sqrt{2k}$ (and a vertex with indegree less than $\sqrt{2k}$).*

Proof. Let $m = \lceil \sqrt{2k} \rceil$. By Lemma 4, D contains an acyclic digraph D' with at least $|E(D)| - k$ edges. Arrange the vertices of D' in an order $u_1, u_2, \ldots, u_q$ so that there are no backward edges. If D has no vertices with outdegree less than m, then for each $i = 0, 1, \ldots, m$, the set $E(D) - E(D')$ contains at least $m - i$ edges starting at vertex u_{q-i}. Hence

$$k \geq 1 + 2 + \ldots + m = \binom{m+1}{2} > \frac{m^2}{2} \geq k,$$

a contradiction. □

In fact, Chudnovsky, Seymour, and Sullivan [6, Conjecture 6.27] conjectured the following improvement of Lemma 4.

Conjecture 6. *If a digraph D is obtained from a tournament by deleting k edges and has no directed triangles, then one can delete from D at most $k/2$ additional edges so that the resulting digraph D' is acyclic.*

If true, this conjecture would imply the following strengthening of Lemma 5: *Each digraph D obtained from a tournament by deleting k edges, that has no directed triangles, has a vertex with outdegree less than $\sqrt{k}$.* This in turn would imply some improvements in the bounds of Theorems 2 and 3.

3 A sketch of Shen's proof

In this section, we outline the proof in [5]. Assume that there exists an n-vertex digraph $D = (V, E)$ without directed triangles with $\deg^+(u) = r = \lceil n\alpha \rceil$ for all $u \in V(D)$. We may assume that D has the fewest vertices among digraphs with this property.

For each arc $(u, v) \in E$, set
$P(u, v) := N^+(v) \setminus N^+(u)$,
$p(u, v) := |P(u, v)|$, the number of induced directed 2-paths whose first edge is (u, v);
$Q(u, v) := N^-(u) \setminus N^-(v)$,
$q(u, v) := |Q(u, v)|$, the number of induced directed 2-paths whose last edge is (u, v);
$T(u, v) := N^+(u) \cap N^+(v)$,
$t(u, v) := |T(u, v)|$, the number of transitive triangles having edge (u, v) as "base."

Let t be the number of *transitive* triangles in D. Note that

$$t = \sum_{(u,v)\in E(D)} t(u, v). \tag{1}$$

It was proved in [5] that

$$n > 2r + \deg^-(v) + q(u, v) - \alpha t(u, v) - p(u, v) \tag{2}$$

for every $(u, v) \in E(D)$. The idea is the following: the sets $N^+(v)$, $N^-(v)$, and $Q(u, v)$ are disjoint. Moreover, every vertex in $T(u, v)$ cannot have outneighbors in $N^-(v) \cup Q(u, v)$. By the minimality of D, some vertex $w \in T(u, v)$ (if $T(u, v)$ is non-empty) has fewer than $\alpha t(u, v)$ outneighbors in $T(u, v)$. Hence w has at least $r - p(u, v) - \alpha t(u, v)$ outneighbors outside of $N^-(v) \cup Q(u, v)$. This yields (2).

Summing inequalities (2) over all edges in D and observing that

$$\sum_{(u,v)\in E(D)} (2r - n) = rn(2r - n),$$

$$\sum_{(u,v)\in E(D)} \deg^-(v) = \sum_{v\in V(D)} (\deg^-(v))^2 \geq r^2 n, \tag{3}$$

$$\sum_{(u,v)\in E(D)} q(u, v) = \sum_{(u,v)\in E(D)} p(u, v), \tag{4}$$

by (1), Shen concludes that

$$\alpha t > rn(3r - n). \tag{5}$$

Noting that $t \leq n\binom{r}{2}$, Shen derives the inequality $\alpha^2 - 6\alpha + 2 > 0$ and concludes that $\alpha < 3 - \sqrt{7}$.

4 Preliminaries

In this and the next sections, we will follow Shen's scheme and use Lemma 5 to prove Theorem 2.

So, let $\alpha \geq 0.35312$ and let D be the smallest counterexample to Theorem 2. Below we use notation from the previous section.

Lemma 7. *If $|V(D)| = n$, then $t > 0.476r^2n$.*

Proof. If $t \leq 0.476r^2n$, then by (5)

$$0.476r^2n\alpha > rn(3r - n).$$

Dividing by r^2n and rearranging we get

$$0.476\alpha + \frac{n}{r} > 3.$$

Since $\frac{n}{r} \leq \frac{1}{\alpha}$ and $\alpha > 0$ we have

$$0.476\alpha^2 - 3\alpha + 1 > 0.$$

This means that $\alpha < 0.35312$, a contradiction. □

Lemma 8. *For every $v \in V(D)$, $|N^-(v)| < 1.186r$.*

Proof. Suppose that $|N^-(v)| \geq 1.186r$. By the minimality of D, some vertex $w \in N^+(v)$ has fewer than αr outneighbors in $N^+(v)$. Since $N^+(w)$ and $N^-(v)$ are disjoint,

$$n > |N^-(v)| + 2r - \alpha r \geq r(3.186 - \alpha).$$

Hence $\alpha^2 - 3.186\alpha + 1 > 0$ and therefore, $\alpha < 1.593 - \sqrt{1.593^2 - 1} < 0.353$, a contradiction. □

For each $(u, v) \in E(D)$, let $f(u, v)$ be the number of missing edges in $N^+(u) \cap N^+(v)$. Similarly, for each $u \in V(D)$, let

$$f(u) = \binom{r}{2} - |E(D(N^+(u)))| \qquad \text{and} \qquad t(u) = |E(D(N^+(u)))|.$$

Clearly, $f(u)$ is the number of missing edges in $N^+(u)$ and $t(u)$ is the number of transitive triangles in D with source vertex u. By definition, $t(u) + f(u) = \binom{r}{2}$ for each $u \in V(D)$, and $t = \sum_{u \in V(D)} t(u)$. Let $f = \sum_{u \in V(D)} f(u)$ and $\gamma = \frac{f}{r^2n}$. Then

$$t = \binom{r}{2}n - f = \binom{r}{2}n - \gamma r^2 n \leq (0.5 - \gamma)r^2 n,$$

and by Lemma 7,

$$\gamma \leq 0.5 - \frac{t}{r^2 n} < 0.5 - 0.476 = 0.024. \tag{6}$$

Lemma 9.

$$\sum_{(u,v)\in E(D)} f(u,v) < \frac{1.172}{2} r\ f = 0.586r \sum_{u\in V(D)} f(u).$$

Proof. Let $\overline{E}(D)$ denote the set of *non-edges* of D, that is, the pairs $xy \in \binom{V(D)}{2}$ such that neither (x,y) nor (y,x) is an edge in D. Note that $\sum_{u\in V(D)} f(u) = \sum_{xy\in\overline{E}(D)} |N^-(x) \cap N^-(y)|$ and that $\sum_{(u,v)\in E(D)} f(u,v) = \sum_{xy\in\overline{E}(D)} |E(D(N^-(x)\cap N^-(y))|$. Therefore, the statement of the lemma holds if for every $xy \in \overline{E}(D)$,

$$|E(D(N^-(x) \cap N^-(y)))| < 0.586r|N^-(x) \cap N^-(y)|. \tag{7}$$

Let $|N^-(x) \cap N^-(y)| = q$. Since $|E(D(N^-(x) \cap N^-(y)))| \leq \binom{q}{2} = \frac{q-1}{2}q$, we see that (7) is clearly true when $q < r$. Therefore we assume that $q \geq r$. Let k denote the number of edges missing from $D(N^-(x) \cap N^-(y))$. Note that any acyclic digraph on q vertices, with maximum outdegree at most r, has at most $\binom{r}{2} + r(q-r) = \binom{q}{2} - \binom{q-r}{2}$ edges. Since $D(N^-(x) \cap N^-(y))$ itself contains no directed triangle and has maximum outdegree at most r, by Lemma 4 it contains an acyclic subgraph with at least $\binom{q}{2} - 2k$ edges. Therefore

$$\binom{q}{2} - 2k \leq \binom{q}{2} - \binom{q-r}{2},$$

implying that $k \geq \frac{1}{2}\binom{q-r}{2}$. Therefore we find $|E(D(N^-(x) \cap N^-(y)))| \leq \binom{q}{2} - \frac{1}{2}\binom{q-r}{2}$. To verify (7) then, we simply need to check that for $q \geq r$ we have

$$\binom{q}{2} - \frac{1}{2}\binom{q-r}{2} < 0.586rq.$$

Suppose the contrary. Then

$$\begin{aligned} \binom{q}{2} - \frac{1}{2}\binom{q-r}{2} &\geq 0.586rq \\ 2q(q-1) - (q-r)(q-r-1) &\geq 2.344rq \\ q^2 + (2r - 1 - 2.344r)q - r(r+1) &\geq 0 \\ q^2 - 0.344rq - r^2 &> 0. \end{aligned}$$

But this implies $q > (0.344r + r\sqrt{4.118336})/2 > 1.1866r$, contradicting Lemma 8. □

5 Proof of Theorem 2

Let $(u,v) \in E(D)$. By Lemma 5, some vertex $w \in N^+(u) \cap N^+(v)$ has at most $\sqrt{2f(u,v)}$ outneighbors in $N^+(u) \cap N^+(v)$. Other outneighbors of w are in $V(D) \setminus (T(u,v) \cup Q(u,v) \cup N^-(v) \cup \{u\})$. Thus, we have

$$n > 2r + \deg^-(v) + q(u,v) - p(u,v) - \sqrt{2f(u,v)}. \tag{8}$$

Summing over all $(u,v) \in E(D)$, we get

$$r \cdot n^2 > 2r^2 n + \sum_{(u,v)\in E(D)} \deg^-(v) + \sum_{(u,v)\in E(D)} (q(u,v) - p(u,v)) - \sum_{(u,v)\in E(D)} \sqrt{2f(u,v)}.$$

Applying (3) and (4), we get

$$r \cdot n^2 > 3r^2 n - \sum_{(u,v)\in E(D)} \sqrt{2f(u,v)} \geq 3r^2 n - rn\sqrt{\frac{2\sum_{(u,v)\in E(D)} f(u,v)}{rn}}. \tag{9}$$

By Lemma 9,

$$rn\sqrt{\frac{2\sum_{(u,v)\in E(D)} f(u,v)}{rn}} \leq rn\sqrt{\frac{1.172r \cdot f}{rn}} = rn\sqrt{\frac{1.172\gamma r^2 n}{n}} = r^2 n\sqrt{1.172\gamma}.$$

Plugging this in (9) and dividing both sides by $r^2 n$, we get

$$\frac{n}{r} > 3 - \sqrt{1.172\gamma}. \tag{10}$$

From this and (6), we have

$$\frac{r}{n} < \frac{1}{3 - \sqrt{1.172 \cdot 0.024}} \leq 0.35307,$$

a contradiction.

6 Digraphs with bounded indegrees and outdegrees

Let $k = \lceil n\beta \rceil$ and assume that there exists an n-vertex digraph $D = (V, E)$ without directed triangles with $\deg^+(u) \geq k$ and $\deg^-(u) \geq k$ for all $u \in V(D)$. We may assume that after deleting any edge, some vertex will have either indegree or outdegree less than k.

For each edge $(u,v) \in E$, set $T^+(u,v) := N^+(u) \cap N^+(v)$, $T^-(u,v) := N^-(u) \cap N^-(v)$, $t^+(u,v) := |T^+(u,v)|$, $t^-(u,v) := |T^-(u,v)|$.

Let $s = 1/\alpha$, where α is the smallest positive real such that for each n every n-vertex digraph with minimum outdegree greater than αn has a directed triangle. By Theorem 2, $\alpha \leq 0.35312$.

The following properties of D are proved in [4].

(i) There exists a vertex v' with both indegree and outdegree equal to k (see Equation (4) on p. 280).

(ii) For all $u, v, w \in V$, if $(u,v), (v,w), (u,w) \in E(D)$, then

$$t^-(u,v) + t^+(v,w) \geq 4k - n \quad \textit{(see Equation (5) on p. 281)}. \tag{11}$$

(iii) For each edge $(u, v) \in E$,

$$t^-(u,v) \geq (3k-n)s = \frac{3k-n}{\alpha} \text{ and } t^+(u,v) \geq (3k-n)s = \frac{3k-n}{\alpha} \text{ (see (6) on p. 281).} \tag{12}$$

(iv) $k^2 > 2(3k-n)(5k-n-2(3k-n)s)s$ *(see the equation between (14) and (16) on p. 282).*

In fact, the k^2 on the left-hand side of the last inequality is simply the upper bound for the total number of edges, $|E(D(N^-(v')))| + |E(D(N^+(v')))|$, in the in-neighborhood and the out-neighborhood of v'. Thus, if the total number of edges in the in-neighborhood and the out-neighborhood of v' is $(1-\gamma)k^2$, then instead of (iv) we can write

$$(1-\gamma)k^2 > 2(3k-n)(5k-n-2(3k-n)s)s. \tag{13}$$

Dividing both sides of (13) by k^2 and rearranging, we get the following slight variation of Inequality (16) in [4]:

$$(4s^2 - 2s)(n/k)^2 - (24s^2 - 16s)(n/k) + (36s^2 - 30s + 1 - \gamma) > 0.$$

Note that there is a misprint in [4]: the last summand in (16) is $(36s^2 - 20s + 1)$ instead of $(36s^2 - 30s + 1)$. Letting $x = n/k$ and $\lambda = 2s = 2/\alpha$, we have

$$(\lambda^2 - \lambda)x^2 - 2(3\lambda^2 - 4\lambda)x + (9\lambda^2 - 15\lambda + 1 - \gamma) > 0. \tag{14}$$

The roots of (14) are

$$\begin{aligned} x_{1,2} &= \frac{3\lambda^2 - 4\lambda \pm \sqrt{(3\lambda^2 - 4\lambda)^2 - (\lambda^2 - \lambda)(9\lambda^2 - 15\lambda + 1 - \gamma)}}{\lambda^2 - \lambda} \\ &= \frac{3\lambda^2 - 4\lambda \pm \sqrt{\gamma\lambda^2 + (1-\gamma)\lambda}}{\lambda^2 - \lambda} = 3 - \frac{1 \pm \sqrt{\gamma + (1-\gamma)/\lambda}}{\lambda - 1}. \end{aligned}$$

Since $x = n/k$ and we know from [4] that $n/k > 2.85$, we conclude that

$$x > 3 - \frac{1 - \sqrt{\gamma + (1-\gamma)/\lambda}}{\lambda - 1}. \tag{15}$$

Let f_1 be the number of non-edges in $N^+(v')$ and f_2 be the number of non-edges in $N^-(v')$. Then, by the definition of γ, $f_1 + f_2 + (1-\gamma)k^2 = k^2 - k$, and hence

$$\gamma k^2 > f_1 + f_2.$$

Comparing Lemma 5 with (iii), we have

$$\sqrt{2f_1} \geq (3k-n)s \quad \text{and} \quad \sqrt{2f_2} \geq (3k-n)s.$$

Hence

$$\gamma k^2 > f_1 + f_2 \geq (3k-n)^2 s^2 = k^2\big((3-x)^2 s^2\big). \tag{16}$$

Assume now that $\beta \geq 0.34564$. Then $x = n/\lceil \beta n \rceil \leq 1/\beta \leq 2.893184$. By Theorem 2, $s \geq 1/0.35312$. Then by (16),

$$\gamma > \left(\frac{3 - 2.893184}{0.35312} \right)^2 \geq 0.302492^2 > 0.0915.$$

Since the right-hand side of (15) grows with γ, plugging $\gamma = 0.0915$ and $\lambda = 2s = 2/0.35312$ into (15) gives a lower bound on x, namely

$$x > 3 - \frac{1 - \sqrt{0.0915 + (1 - 0.0915)0.35312/2}}{(2/0.35312) - 1} = 3 - \frac{1 - \sqrt{0.0915 + 0.9085 \cdot 0.17656}}{(2 - 0.35312)/0.35312}$$

$$= 3 - 0.35312 \frac{1 - \sqrt{0.25190476}}{1.64688} \geq 3 - 0.35312 \frac{1 - 0.5019}{1.64688} > 2.89319,$$

a contradiction to our assumption. This proves Theorem 3. □

We conclude with a remark on the explicit relation between α and β that we use here. Combining (16) with (14) and simplifying, we obtain

$$(3 - 2\alpha)x^2 - (18 - 16\alpha)x + 27 - 30\alpha + \alpha^2 > 0.$$

This implies

$$x > \frac{9 - 8\alpha + \alpha\sqrt{1 + 2\alpha}}{3 - 2\alpha}$$

so since $\beta \leq 1/x$ we find

$$\beta < \frac{3 - 2\alpha}{9 - 8\alpha + \alpha\sqrt{1 + 2\alpha}}. \tag{17}$$

Observe that even if we knew the best possible value $\alpha = 1/3$ for α, the bound on β given by this formula is only .34498.

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