The spectral radius of subgraphs of regular graphs

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Abstract

Let $\mu(G)$ and $\mu_{\min}(G)$ be the largest and smallest eigenvalues of the adjacency matrix of a graph G. Our main results are:

(i) Let G be a regular graph of order n and finite diameter D. If H is a proper subgraph of G, then

$$\mu(G) - \mu(H) > \frac{1}{nD}.$$

(ii) If G is a regular nonbipartite graph of order n and finite diameter D, then

$$\mu(G) + \mu_{\min}(G) > \frac{1}{nD}.$$

Keywords: *smallest eigenvalue, largest eigenvalue, diameter, connected graph, nonbipartite graph*

Main results

Our notation follows [1]. Specifically, $\mu(G)$ and $\mu_{\min}(G)$ stand for the largest and smallest eigenvalues of the adjacency matrix of a graph G.

The aim of this note is to improve some recent results on eigenvalues of subgraphs of regular graphs. Cioabă ([2], Corollary 2.2) showed that if G is a regular graph of order n and e is an edge of G such that $G - e$ is a connected graph of diameter D, then

$$\mu(G) - \mu(G - e) > \frac{1}{nD}.$$

The approach of [3] helps improve this assertion in a natural way:

Theorem 1 *Let G be a regular graph of order n and finite diameter D. If H is a proper subgraph of G, then*

$$\mu(G) - \mu(H) > \frac{1}{nD}. \tag{1}$$

Since $\mu(H) \leq \mu(H')$ whenever $H \subset H'$, we may assume that H is a maximal proper subgraph of G, that is to say, $V(H) = V(G)$ and H differs from G in a single edge. Thus, we can deduce Theorem 1 from the following assertion.

Theorem 2 *Let G be a regular graph of order n and finite diameter D. If uv is an edge of G, then*

$$\mu(G) - \mu(G - uv) > \begin{cases} 1/(nD), & \text{if } G - uv \text{ is connected;} \\ 1/(n-3)(D-1), & \text{otherwise.} \end{cases}.$$

Furthermore, Theorem 1 implies a result about nonbipartite graphs.

Theorem 3 *If G is a regular nonbipartite graph of order n and finite diameter D, then*

$$\mu(G) + \mu_{\min}(G) > \frac{1}{nD}.$$

Finally, we note the following more general version of the lower bound in Corollary 2.2 in [2].

Lemma 4 *Let G be a connected regular graph and e be an edge of G. If H is a component of $G - e$ with $\mu(H) = \mu(G - e)$, then*

$$\mu(G) - \mu(H) > \frac{1}{Diam(H)\,|H|}.$$

This lemma follows easily from Theorem 2.1 of [2] and its proof is omitted.

Proofs

Proof of Theorem 2 Write $dist_F(s,t)$ for the length of a shortest path joining two vertices s and t in a graph F. Write d for the degree of G, let $H = G - uv$, and set $\mu = \mu(H)$.

Case (a): H is connected.

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a unit eigenvector to μ and let x_w be a maximal entry of $\mathbf{x}$; we thus have $x_w^2 \geq 1/n$. We can assume that $w \neq v$ and $w \neq u$. Indeed, if $w = v$, we see that

$$\mu x_v = \sum_{vi \in E(G)} x_i \leq (d-1)\, x_v,$$

and so $d - \mu \geq 1$, implying (1). We have

$$d - \mu = d \sum_{i \in V(G)} x_i^2 - 2 \sum_{ij \in E(G)} x_i x_j = \sum_{ij \in E(G)} (x_i - x_j)^2 + x_u^2 + x_v^2.$$

Assume first that $dist_H(w,u) \leq D-1$. Select a shortest path $u = u_1, \ldots, u_k = w$ joining u to w in H. We see that

$$\begin{aligned} d-\mu &= \sum_{ij \in E(G)} (x_i - x_j)^2 + x_u^2 + x_v^2 > \sum_{i=1}^{k-1} \left(x_{u_i} - x_{u_{i+1}}\right)^2 + x_u^2 \\ &\geq \frac{1}{k-1}\left(x_{u_i} - x_{u_{i+1}}\right)^2 + x_u^2 = \frac{1}{k-1}(x_w - x_u)^2 + x_u^2 \geq \frac{1}{k}x_w^2 \geq \frac{1}{nD}, \end{aligned}$$

completing the proof.

Hereafter, we assume that $dist_H(w,u) \geq D$ and, by symmetry, $dist_H(w,v) \geq D$.

Let $P(u,w)$ and $P(v,w)$ be shortest paths joining u and v to w in G. If $u \in P(v,w)$, then there exists a path of length at most $D-1$, joining w to u in G, and thus in H, a contradiction. Hence, $u \notin P(v,w)$ and, by symmetry, $v \notin P(u,w)$. Therefore, the paths $P(u,w)$ and $P(v,w)$ belong to H, and we have

$$dist_H(w,u) = dist_H(w,v) = D.$$

Let $Q(u,z)$ and $Q(v,z)$ be the longest subpaths of $P(u,w)$ and $P(v,w)$ having no internal vertices in common. Clearly $Q(u,z)$ and $Q(v,z)$ have the same length. Write $Q(z,w)$ for the subpath of $P(u,w)$ joining z to w and let

$$Q(u,z) = u_1, \ldots, u_k, \quad Q(v,z) = v_1, \ldots, v_k, \quad Q(z,w) = w_1, \ldots, w_l,$$

where

$$u_1 = u, \quad u_k = v_k = w_1 = z, \quad w_l = w, \quad k + l - 2 = D.$$

The following argument is borrowed from [2]. Using the AM-QM inequality, we see that

$$\begin{aligned} d-\mu &\geq \sum_{i=1}^{k-1}\left(x_{v_i} - x_{v_{i+1}}\right)^2 + x_v^2 + \sum_{i=1}^{k-1}\left(x_{u_i} - x_{u_{i+1}}\right)^2 + x_u^2 + \sum_{i=1}^{l-1}\left(x_{w_i} - x_{w_{i+1}}\right)^2 \\ &\geq \frac{2}{D-l+2}x_z^2 + \frac{1}{l-1}(x_w - x_z)^2 \geq \frac{2}{D+l-1}x_w^2 \geq \frac{1}{Dn}, \end{aligned}$$

completing the proof.

Case (b): H is disconnected.

Since G is connected, H is union of two connected graphs H_1 and H_2 such that $u \in H_1$, $v \in H_2$. Assume $\mu = \mu(H_1)$, set $|H_1| = k$ and let $\mathbf{x} = (x_1, \ldots, x_k)$ be a unit eigenvector to μ. Since $d \geq 2$, we see that $|H_2| \geq 3$, and so, $k \leq n-3$.

Let x_w be a maximal entry of $\mathbf{x}$; we thus have $x_w^2 \geq 1/k \geq 1/(n-3)$. Like in the previous case, we see that $w \neq u$. Since $d \geq 2$, there is a vertex $z \in H_2$ such that $z \neq v$. Select a shortest path $u = u_1, u_2, \ldots, u_l = w$ joining u to w in H_1. Since $dist_G(z,w) \leq diam\ G = D$, we see that $l \leq D-1$. As above, we have

$$\begin{aligned} d-\mu &= \sum_{ij \in E(G)} (x_i - x_j)^2 + x_u^2 + x_v^2 > \sum_{i=1}^{l-1}\left(x_{u_i} - x_{u_{i+1}}\right)^2 + x_u^2 \\ &\geq \frac{1}{l-1}(x_{u_1} - x_{u_k})^2 + x_u^2 = \frac{1}{l-1}(x_w - x_u)^2 + x_u^2 \geq \frac{1}{l}x_w^2 \geq \frac{1}{(n-3)(D-1)}, \end{aligned}$$

completing the proof. $\square$

Proof of Theorem 3 Let $\mathbf{x} = (x_1, \ldots, x_n)$ be an eigenvector to $\mu_{\min}(G)$ and let $U = \{u : x_u < 0\}$. Write H for the bipartite subgraph of G containing all edges with exactly one vertex in U; note that H is a proper subgraph of G and $\mu_{\min}(H) < \mu_{\min}(G)$. Hence,

$$\mu(G) + \mu_{\min}(G) > \mu(G) + \mu_{\min}(H) = \mu(G) - \mu(H),$$

and the assertion follows from Theorem 1. $\square$

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