Regular spanning subgraphs of bipartite graphs of high minimum degree

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Abstract

Let G be a simple balanced bipartite graph on $2n$ vertices, $\delta = \delta(G)/n$, and $\rho_0 = \frac{\delta+\sqrt{2\delta-1}}{2}$. If $\delta \geq 1/2$ then G has a $\lfloor \rho_0 n \rfloor$-regular spanning subgraph. The statement is nearly tight.

1 Introduction

In this paper we will consider regular spanning subgraphs of simple graphs. We mostly use standard graph theory notation: $V(G)$ and $E(G)$ will denote the vertex and the edge set of a graph G, respectively. The degree of $x \in V(G)$ is denoted by $deg_G(x)$ (we may omit the subscript), $\delta(G)$ is the minimum degree of G. We call a bipartite graph $G(A,B)$ with color classes A and B *balanced* if $|A| = |B|$. For $X, Y \subset V(G)$ we denote the number of edges of G having one endpoint in X and the other endpoint in Y by $e(X,Y)$. If $T \subset V(G)$ then $G|_T$ denotes the subgraph we get after deleting every vertex of $V - T$ and the edges incident to them. Finally, $K_{r,s}$ is the complete bipartite graph on color classes of size r and s for two positive integers r and s.

If $f : V(H) \to \mathrm{Z}^+$ is a function, then an f-factor is a subgraph H' of the graph H such that $deg_{H'}(x) = f(x)$ for every $x \in V(H)$. Notice, that when $f \equiv r$ for some $r \in \mathrm{Z}^+$, then H' is an r-regular subgraph of H.

There are several results concerning f-factors of graphs. Perhaps the most notable among them is the theorem of Tutte [7]. Finding f-factors is in general not an easy task even for the case f is a constant and the graph is regular (see eg., [1]). In this paper we look for f-factors in (not necessarily regular) bipartite graphs with large minimum degree, for $f \equiv r$.

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Theorem 1 *Let $G(A,B)$ be a balanced bipartite graph on $2n$ vertices, and assume that $\delta = \delta(G)/n \geq 1/2$. Set $\rho_0 = \frac{\delta+\sqrt{2\delta-1}}{2}$. Then*
(I) G has a $\lfloor \rho_0 n \rfloor$–regular spanning subgraph;
(II) moreover, for every $\delta > 1/2$ if n is sufficiently large and δn is an integer then there exists a balanced bipartite graph G_δ having minimum degree δ such that it does not admit a spanning regular subgraph of degree larger than $\lceil \rho_0 n \rceil$.

The above theorem plays a crucial role in the proof of some results in extremal graph theory ([2, 3]).

2 The main tool

Let F be a bipartite graph with color classes A and B. By the well-known König–Hall theorem there is a perfect matching in F if and only if $|N(S)| \geq |S|$ for every $S \subset A$. We are going to need a far reaching generalization of this result, due to Gale and Ryser [6, 4] (one can find the proof in [5] as well). It gives a necessary and sufficient condition for the existence of an f-factor in a bipartite graph:

Proposition 2 *Let F be a bipartite graph with bipartition $\{A, B\}$, and $f(x) \geq 0$ an integer valued function on $A \cup B$. F has an f–factor if and only if*

$$(i) \sum_{x \in A} f(x) = \sum_{y \in B} f(y)$$

and

$$(ii) \sum_{x \in X} f(x) \leq e(X,Y) + \sum_{y \in B-Y} f(y)$$

for all $X \subset A$ and $Y \subset B$.

3 Proof of Theorem 1

We will show the two parts of the theorem in separate subsections.

3.1 Proof of part I

Observe, that since we are looking for a spanning regular subgraph, the f function of Proposition 2 will be identically ρn for some constant ρ. We start with some notation: for $X \subset A$ let $\xi = |X|/n$, and for $Y \subset B$ let $\sigma = |Y|/n$. We will normalize $e(X,Y)$: $\eta(X,Y) = e(X,Y)/n^2$. Let

$$\eta_m(\xi, \sigma) = \min\{\eta(X,Y) : X \subset A,\ Y \subset B,\ |X|/n = \xi, |Y|/n = \sigma\}.$$

Since f is identically ρn, condition (i) of Proposition 2 is satisfied. Moreover, if ρn is an integer and

$$\rho(\xi + \sigma - 1) \leq \eta_m(\xi, \sigma)$$

for some ρ and for every $0 \leq \xi, \sigma \leq 1$, then (ii) is satisfied, hence, G has a ρn–regular spanning subgraph. In the rest of this section we will show that the above inequality is valid for $\rho = \lfloor \rho_0 n \rfloor / n$.

Clearly, $e(X,Y) \geq |X|(\delta n - |B - Y|)$ and $e(X,Y) \geq |Y|(\delta n - |A - X|)$ for arbitrary sets $X \subset A$ and $Y \subset B$. Hence, we have that $\eta_m(\xi, \sigma) \geq \max(\xi(\delta + \sigma - 1), \sigma(\delta + \xi - 1))$. (In fact we always have that $\eta_m \geq 0$, since it is the edge density between the two color classes in G.)

First consider the case $\xi = \sigma$. We are looking for a ρ for which $\rho(2\xi - 1) \leq \xi(\delta + \xi - 1)$. In another form, we need that

$$p_\rho(\xi) = \xi^2 + (\delta - 2\rho - 1)\xi + \rho \geq 0.$$

The discriminant of the above polynomial is the polynomial $dcr(\rho) = 4\rho^2 - 4\delta\rho + \delta^2 - 2\delta + 1$. Clearly, if $dcr(\rho) \leq 0$ for some ρ, then $p_\rho(\xi) \geq 0$.

One can directly find the roots of $dcr(\rho)$: $\frac{\delta \pm \sqrt{2\delta - 1}}{2}$. At this point we have to be careful, since the degrees in a graph are non-negative integers, so ρn has to be a natural number. We will show that $dcr(\rho) \leq 0$ for $\rho = \lfloor (\delta + \sqrt{2\delta - 1}) n/2 \rfloor / n$.

Clearly, $dcr(x) \leq 0$ in $I = [(\delta - \sqrt{2\delta - 1})/2, (\delta + \sqrt{2\delta - 1})/2]$, the length of this interval is $\sqrt{2\delta - 1}$. Divide the $[0,1]$ interval into n disjoint subintervals each of length $1/n$, denote the set of the endpoints of these subintervals by S. Observe that is $I \cap S \geq 1$, then we can pick the largest point of this intersection, this is $\rho \in I$, and we are done with proving that $p_\rho(\xi) \geq 0$.

We will investigate two cases: first, if $\delta > 1/2$, and second, if $\delta = 1/2$.

First case: $\delta > 1/2$. We know that δn is an integer, it is larger than $n/2$, hence, $\delta n \geq \frac{n+1}{2}$. If the length of I is at least $1/n$, it will intersect with S. Assuming that $\sqrt{2\delta - 1} < \frac{1}{n}$ we would get $\delta n < \frac{n^2+1}{2n}$, but the latter expression is less than $\frac{n+1}{2}$. Hence, in this case $|I \cap S| \geq 1$.

Let $g(\xi, \sigma) = \sigma(\delta + \xi - 1) - \rho(\xi + \sigma - 1)$. We will show, that $g(\xi, \sigma) \geq 0$ for $0 \leq \sigma \leq \xi \leq 1$, in the lower right triangle T of the unit square. This will prove that (ii) of Proposition 2 is satisfied. Notice, that g is bounded in the triangle above, $-2 \leq g(\xi, \sigma) \leq \eta_m(\xi, \sigma) - \rho(\xi + \sigma - 1)$, and continously differentiable.

Let us check the sign of g on the border of the triangle. Since $\rho = \lfloor (\delta + \sqrt{2\delta - 1}) n/2 \rfloor / n$, we have that $g(\xi, \xi) \geq 0$. $g(\xi, 0) = -\rho(\xi - 1) \geq 0$, and $g(1, \sigma) = \sigma(\delta - \rho) \geq 0$, because $\delta \geq (\delta + \sqrt{2\delta - 1})/2$. Let us check the partial derivatives of g:

$$\frac{\partial g}{\partial \xi} = \sigma - \rho,$$

and

$$\frac{\partial g}{\partial \sigma} = \delta + \xi - 1 - \rho.$$

Assuming that g achieves its minimum inside the triangle at the point (ξ', σ') the partial derivatives of g have to diminish at (ξ', σ'). It would then follow that $\sigma' = \rho$ and $\xi' = 1+\rho-\delta$, therefore, $g(\xi', \sigma') = \rho^2 - \rho(2\rho-\delta) = \delta\rho - \rho^2$. Hence g is non-negative in T. The same reasoning works for the triangle $0 \le \xi \le \sigma \le 1$, this follows easily by symmetry. With this we finished the proof for the case $\delta > 1/2$.

Second case: $\delta = 1/2$. If δn is even (n is divisible by 4), we are done, since in this case I contains the point $\delta/2 = \lfloor n/4 \rfloor / n$, and $\delta n/2$ is an integer. Therefore we have that $p_{1/4}(\xi) \ge 0$, and as above, one can check that g is non-negative in every point of T.

There is only one case left: if δn is odd, that is, n is of the form $4k+2$ for some natural number k. In this case we want to prove, that that the spanning subgraph is $\frac{k}{4k+2}$-regular.

First observe, that for our purposes it is sufficient if $g(\xi, \sigma) \ge 0$ in a discrete point set: in the points (ξ, σ) belonging to $(S \times S) \cap T$, since $|X|$ and $|Y|$ are natural numbers. Set $\rho = \frac{k}{4k+2}$ and analyze the polynomial $p_\rho(\xi)$. It is an easy exercise to check that it has two distinct roots: $1/2 = \frac{2k+1}{4k+2}$ and $1/2 - 1/n = \frac{2k}{4k+2}$. Hence, $p_\rho(\xi) \ge 0$ for $\xi \notin (1/2-1/n, 1/2)$.

We will cut out a small open triangle T_s from T. T_s has vertices $(1/2, 1/2)$, $(1/2, 1/2 - 1/n)$ and $(1/2 - 1/n, 1/2 - 1/n)$. Clearly, $T - T_s$ is closed and $T_s \cap (S \times S) = \emptyset$.

Recall, that $g(\xi, \sigma) = \sigma(\delta + \xi - 1) - \rho(\xi + \sigma - 1)$ for $0 \le \sigma \le \xi \le 1$. We will check the sign of g on the border of $T - T_s$. There are two line segments for which we cannot apply our earlier results concerning g. The first is

$$L_1 = \left\{ (\xi, \sigma) : \frac{n-2}{2n} \le \xi \le \frac{1}{2}, \sigma = \frac{n-2}{2n} \right\},$$

the second is

$$L_2 = \left\{ (\xi, \sigma) : \xi = \frac{1}{2}, \frac{n-2}{2n} \le \sigma \le \frac{1}{2} \right\}.$$

On L_1 we get that

$$g\left(\xi, \frac{n-2}{2n}\right) = \frac{n-2}{2n}\left(\xi - \frac{1}{2}\right) - \frac{k}{4k+2}\left(\xi - \frac{n-2}{2n}\right) = \frac{k}{n}\xi - \frac{k}{2n} + \frac{k}{n^2}.$$

It is easy to see that the above expression is non-negative for every $(n-2)/(2n) \le \xi \le 1/2$.

For L_2 we have

$$g\left(\frac{1}{2}, \sigma\right) = \sigma\left(\frac{1}{2} + \frac{1}{2} - 1\right) - \frac{k}{4k+2}\left(\frac{1}{2} + \sigma - 1\right) = \frac{k}{4k+2}\left(\frac{1}{2} - \sigma\right) \ge 0$$

for $(n-2)/(2n) \le \sigma \le 1/2$.

In order to finish proving that g is non-negative in every point of $(S \times S) \cap (T - T_s)$ it is sufficient to show that the minimum of g inside $T - T_s$ is at least as large as the minimum of g on the border of $T - T_s$. This can be shown along the same lines as previously. By symmetry we will get that condition (ii) of Proposition 2 is satisfied in every point of $S \times S$.

3.2 Proof of part II

For proving part II of the theorem we want to construct a class of balanced bipartite graphs the elements of which cannot have a large regular spanning subgraph. We will achieve this goal in two steps. First, we will consider a simple linear function, which, as we will see later, is closely related to our task. In the second step we will construct those bipartite graph which satisfy part II of Theorem 1.

Set $\gamma' = \frac{1-\sqrt{2\delta-1}}{2}$ and let $0 < p < 1$. Consider the following equation:

$$(1-p)(1-\gamma') = \gamma'(1-p) + \delta - \gamma'. \qquad (1)$$

It is easy to see that $p' = \frac{\delta+\gamma'-1}{2\gamma'-1}$ is its solution. We have that

$$(1-p')(1-\gamma') = \left(1 - \frac{\delta+\gamma'-1}{2\gamma'-1}\right)(1-\gamma') = \frac{\gamma'-\delta}{2\gamma'-1}(1-\gamma').$$

Substituting $\gamma' = \frac{1-\sqrt{2\delta-1}}{2}$ we get

$$\frac{\delta - \frac{1-\sqrt{2\delta-1}}{2}}{\sqrt{2\delta-1}}\left(1 - \frac{1-\sqrt{2\delta-1}}{2}\right) = \frac{2\delta-1+\sqrt{2\delta-1}}{\sqrt{2\delta-1}}\frac{1+\sqrt{2\delta-1}}{2} =$$

$$\frac{1+\sqrt{2\delta-1}}{2}\frac{1+\sqrt{2\delta-1}}{2} = \frac{\delta+\sqrt{2\delta-1}}{2}.$$

We promised to define a class of bipartite graphs for $\delta > 1/2$ which exist for every sufficiently large value of n if δn is a natural number, such that these graphs do not admit spanning regular graphs with large degree.

For that let $\gamma = \lceil \gamma' n \rceil / n$. Then γn is an integer, and $\gamma' \leq \gamma \leq \gamma' + 1/n$. Let $G = (A, B, E)$ be a balanced bipartite graph on $2n$ vertices. A is divided into two disjoint subsets, A_l and A_e, we also divide B into B_l and B_e. We will have that $|A_l| = |B_l| = \gamma n$ and $|A_e| = |B_e| = (1-\gamma)n$. There are no edges in between the vertices of A_l and B_l. The subgraphs $G|_{A_l \cup B_e}$ and $G|_{B_l \cup A_e}$ are isomorphic to $K_{\gamma n, (1-\gamma)n}$, therefore, every vertex in $A_l \cup B_l$ has degree $(1-\gamma)n$. We require that every vertex in $A_e \cup B_e$ has degree δn, hence, $G|_{A_e \cup B_e}$ will be a $(\delta-\gamma)n$-regular graph. Observe, that $\gamma < \delta < 1-\gamma$, thus, $\delta(G) = \delta n$.

Let us consider a simple method for edge removal from G: given $0 < p < 1$ discard $p(1-\gamma)n$ incident edges for every vertex in $A_l \cup B_l$, and no edge from $G|_{A_e \cup B_e}$. Of course, we need that $p(1-\gamma)n$ is an integer.

Then a vertex in $A_l \cup B_l$ will have degree $(1-p)(1-\gamma)n$, and the average degree of the vertices of $A_e \cup B_e$ will be $\gamma(1-p)n + (\delta-\gamma)n$. Choose $\tilde{p}$ to be the solution of the following equation:

$$(1-p)(1-\gamma)n = \gamma(1-p)n + (\delta-\gamma)n. \qquad (2)$$

Notice, that the only difference between (1) and (2) is that we subsituted γ' by γ. One can see that if $p < \tilde{p}$ then there is a vertex $x \in A_e \cup B_e$ such that every vertex of $A_l \cup B_l$

will have degree larger than $deg(x)$. That is, for finding a regular subgraph more edges have to be discarded among those which are incident to the vertices of $A_l \cup B_l$.

The solution of (2) is $\widetilde{p} = \frac{\delta+\gamma-1}{2\gamma-1}$ (here $\widetilde{p}(1-\gamma)n$ is not necessarily an integer). Computing the derivative shows that $\gamma \geq \gamma'$ implies $p' \geq \widetilde{p}$. Let us show that $p' - \widetilde{p}$ is small:

$$p' - \widetilde{p} = \frac{\delta + \gamma' - 1}{2\gamma' - 1} - \frac{\delta + \gamma - 1}{2\gamma - 1} =$$
$$\frac{(\delta + \gamma' - 1)(2\gamma - 1) - (\delta + \gamma - 1)(2\gamma' - 1)}{(2\gamma - 1)(2\gamma' - 1)} = \frac{2\gamma\delta - 2\gamma'\delta + \gamma' - \gamma}{(2\gamma - 1)(2\gamma' - 1)}.$$

Observe, that $1 - 2\gamma' = \sqrt{2\delta - 1}$, and that $1 - 2\gamma \geq 1 - 2\gamma' - 2/n > 0$ whenever n is sufficiently large. Therefore,

$$\frac{2\gamma\delta - 2\gamma'\delta + \gamma' - \gamma}{(2\gamma - 1)(2\gamma' - 1)} = \frac{(\gamma' - \gamma)\sqrt{2\delta - 1}}{2\gamma - 1} = (\gamma - \gamma')\frac{\sqrt{2\delta - 1}}{1 - 2\gamma} \leq$$
$$\frac{(\gamma - \gamma')\sqrt{2\delta - 1}}{1 - 2\gamma' - 2/n} \leq \frac{1}{n}\left(1 + \frac{2}{n\sqrt{2\delta - 1} - 2}\right) = \frac{1}{n}\left(1 + O(1/n)\right).$$

Above we used the fact that $\gamma \leq \gamma' + \frac{1}{n}$. Since $\widetilde{p}(1-\gamma)n$ is not necessarily an integer, we introduce p_0: $p_0 = \lceil \widetilde{p}(1-\gamma)n \rceil / ((1-\gamma)n)$. Clearly, the least number of edges one has to remove from the vertices of $A_l \cup B_l$ in order to find a spanning regular subgraph of G is at least $p_0(1-\gamma)n$. With this choice of p_0 every degree in $A_l \cup B_l$ will be $(1-p_0)(1-\gamma)n$ after the edge removal process.

Finally, we show that $(1-p_0)(1-\gamma)$ is very close to $\frac{\delta+\sqrt{2\delta-1}}{2}$:

$$(1 - p_0)(1 - \gamma) - \frac{\delta + \sqrt{2\delta - 1}}{2} = (1 - p_0)(1 - \gamma) - (1 - p')(1 - \gamma') \leq$$
$$(1 - \widetilde{p})(1 - \gamma') - (1 - p')(1 - \gamma') = (1 - \gamma')(1 - \widetilde{p} - 1 + p') =$$
$$(1 - \gamma')(p' - \widetilde{p}) = (1 - \gamma')\frac{1}{n}\left(1 + O(1/n)\right).$$

If n is sufficiently large, then $(1-\gamma')(1+O(1/n)) < 1$, since $0 < \gamma' < 1/2$. Hence, if $H \subset G$ is an r-regular spanning subgraph, then

$$\rho_0 n = \left\lfloor \frac{\delta + \sqrt{2\delta - 1}}{2} n \right\rfloor \leq r < (1 - p')(1 - \gamma')n + 1 = \frac{\delta + \sqrt{2\delta - 1}}{2} n + 1.$$

Since r is an integer which is less than $\frac{\delta+\sqrt{2\delta-1}}{2}n + 1$, we get that

$$\left\lfloor \frac{\delta + \sqrt{2\delta - 1}}{2} n \right\rfloor \leq r \leq \left\lceil \frac{\delta + \sqrt{2\delta - 1}}{2} n \right\rceil,$$

and this is what we wanted to prove.

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