Semicanonical basis generators of the cluster algebra of type $A_1^{(1)}$

Andrei Zelevinsky*

Department of Mathematics
Northeastern University, Boston, USA
andrei@neu.edu

Submitted: Jul 27, 2006; Accepted: Dec 23, 2006; Published: Jan 19, 2007
Mathematics Subject Classification: 16S99

Abstract

We study the cluster variables and "imaginary" elements of the semicanonical basis for the coefficient-free cluster algebra of affine type $A_1^{(1)}$. A closed formula for the Laurent expansions of these elements was given by P.Caldero and the author. As a by-product, there was given a combinatorial interpretation of the Laurent polynomials in question, equivalent to the one obtained by G.Musiker and J.Propp. The original argument by P.Caldero and the author used a geometric interpretation of the Laurent polynomials due to P.Caldero and F.Chapoton. This note provides a quick, self-contained and completely elementary alternative proof of the same results.

1 Introduction

The (coefficient-free) cluster algebra $\mathcal{A}$ of type $A_1^{(1)}$ is a subring of the field $\mathbb{Q}(x_1, x_2)$ generated by the elements x_m for $m \in \mathbb{Z}$ satisfying the recurrence relations

$$x_{m-1}x_{m+1} = x_m^2 + 1 \quad (m \in \mathbb{Z}) \ . \tag{1}$$

This is the simplest cluster algebra of infinite type; it was studied in detail in [2, 6]. Besides the generators x_m (called *cluster variables*), $\mathcal{A}$ contains another important family of elements $s_0, s_1, \ldots$ defined recursively by

$$s_0 = 1, \ s_1 = x_0x_3 - x_1x_2, \ \ s_n = s_1s_{n-1} - s_{n-2} \quad (n \geq 2). \tag{2}$$

*Research supported by NSF (DMS) grant # 0500534 and by a Humboldt Research Award

As shown in [2, 6], the elements $s_1, s_2, \ldots$ together with the *cluster monomials* $x_m^p x_{m+1}^q$ for all $m \in \mathbb{Z}$ and $p, q \geq 0$, form a $\mathbb{Z}$-basis of $\mathcal{A}$ referred to as the *semicanonical basis*.

As a special case of the *Laurent phenomenon* established in [3], $\mathcal{A}$ is contained in the Laurent polynomial ring $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$. In particular, all x_m and s_n can be expressed as integer Laurent polynomials in x_1 and x_2. These Laurent polynomials were explicitly computed in [2] using their geometric interpretation due to P. Caldero and F. Chapoton [1]. As a by-product, there was given a combinatorial interpretation of these Laurent polynomials, which can be easily seen to be equivalent to the one previously obtained by G. Musiker and J. Propp [5].

The purpose of this note is to give short, self-contained and completely elementary proofs of the combinatorial interpretation and closed formulas for the Laurent polynomial expressions of the elements x_m and s_n.

2 Results

We start by giving an explicit combinatorial expression for each x_m and s_n, in particular proving that they are Laurent polynomials in x_1 and x_2 with positive integer coefficients. By an obvious symmetry of relations (1), each element x_m is obtained from x_{3-m} by the automorphism of the ambient field $\mathbb{Q}(x_1, x_2)$ interchanging x_1 and x_2. Thus, we restrict our attention to the elements x_{n+3} for $n \geq 0$.

Following [2, Remark 5.7] and [4, Example 2.15], we introduce a family of *Fibonacci polynomials* $F(w_1, \ldots, w_N)$ given by

$$F(w_1, \ldots, w_N) = \sum_D \prod_{k \in D} w_k, \tag{3}$$

where D runs over all *totally disconnected* subsets of $\{1, \ldots, N\}$, i.e., those containing no two consecutive integers. In particular, we have

$$F(\emptyset) = 1, \ F(w_1) = w_1 + 1, \ F(w_1, w_2) = w_1 + w_2 + 1.$$

We also set

$$f_N = x_1^{-\lfloor \frac{N+1}{2} \rfloor} x_2^{-\lfloor \frac{N}{2} \rfloor} F(w_1, \ldots, w_N)|_{w_k = x_{\langle k+1 \rangle}^2}, \tag{4}$$

where $\langle k \rangle$ stands for the element of $\{1, 2\}$ congruent to k modulo 2. In view of (3), each f_N is a Laurent polynomial in x_1 and x_2 with positive integer coefficients. In particular, an easy check shows that

$$f_0 = 1, \quad f_1 = \frac{x_2^2 + 1}{x_1} = x_3, \quad f_2 = \frac{x_1^2 + x_2^2 + 1}{x_1 x_2} = s_1. \tag{5}$$

Theorem 2.1 *[2, Formula (5.16)] For every $n \geq 0$, we have*

$$s_n = f_{2n}, \quad x_{n+3} = f_{2n+1}. \tag{6}$$

In particular, all x_m and s_n are Laurent polynomials in x_1 and x_2 with positive integer coefficients.

Using the proof of Theorem 2.1, we derive the explicit formulas for the elements x_m and s_n.

Theorem 2.2 *[2, Theorems 4.1, 5.2] For every $n \geq 0$, we have*

$$\begin{aligned} x_{n+3} &= x_1^{-n-1} x_2^{-n} (x_2^{2(n+1)} + \sum_{q+r \leq n} \binom{n-r}{q} \binom{n+1-q}{r} x_1^{2q} x_2^{2r}); \quad (7) \\ s_n &= x_1^{-n} x_2^{-n} \sum_{q+r \leq n} \binom{n-r}{q} \binom{n-q}{r} x_1^{2q} x_2^{2r}. \quad (8) \end{aligned}$$

3 Proof of Theorem 2.1

In view of (3), the Fibonacci polynomials satisfy the recursion

$$F(w_1, \ldots, w_N) = F(w_1, \ldots, w_{N-1}) + w_N F(w_1, \ldots, w_{N-2}) \quad (N \geq 2). \quad (9)$$

Substituting this into (4) and clearing the denominators, we obtain

$$x_{\langle N \rangle} f_N = f_{N-1} + x_{\langle N-1 \rangle} f_{N-2} \quad (N \geq 2). \quad (10)$$

Thus, to prove (6) by induction on n, it suffices to prove the following identities for all $n \geq 0$ (with the convention $s_{-1} = 0$):

$$\begin{aligned} x_1 x_{n+3} &= s_n + x_2 x_{n+2}; \quad (11) \\ x_2 s_n &= x_{n+2} + x_1 s_{n-1}. \quad (12) \end{aligned}$$

We deduce (11) and (12) from (2) and its analogue established in [6, formula (5.13)]:

$$x_{m+1} = s_1 x_m - x_{m-1} \quad (m \in \mathbb{Z}). \quad (13)$$

(For the convenience of the reader, here is the proof of (13). By (1), we have

$$\frac{x_{m-2} + x_m}{x_{m-1}} = \frac{x_{m-1}^2 + x_m^2 + 1}{x_m x_{m-1}} = \frac{x_{m-1} + x_{m+1}}{x_m}.$$

So $(x_{m-1} + x_{m+1})/x_m$ is a constant independent of m; setting $m = 2$ and using (2), we see that this constant is s_1.)

We prove (11) and (12) by induction on n. Since both equalities hold for $n = 0$ and $n = 1$, we can assume that they hold for all $n < p$ for some $p \geq 2$, and it suffices to prove them for $n = p$. Combining the inductive assumption with (2) and (13), we obtain

$$\begin{aligned} x_1 x_{p+3} &= x_1 (s_1 x_{p+2} - x_{p+1}) \\ &= s_1 (s_{p-1} + x_2 x_{p+1}) - (s_{p-2} + x_2 x_p) \\ &= (s_1 s_{p-1} - s_{p-2}) + x_2 (s_1 x_{p+1} - x_p) \\ &= s_p + x_2 x_{p+2}, \end{aligned}$$

and

$$
\begin{aligned}
x_2 s_p &= x_2(s_1 s_{p-1} - s_{p-2}) \\
&= s_1(x_{p+1} + x_1 s_{p-2}) - (x_p + x_1 s_{p-3}) \\
&= (s_1 x_{p+1} - x_p) + x_1(s_1 s_{p-2} - s_{p-3}) \\
&= x_{p+2} + x_1 s_{p-1},
\end{aligned}
$$

finishing the proof of Theorem 2.1.

4 Proof of Theorem 2.2

Formulas (7) and (8) follow from (11) and (12) by induction on n. Indeed, assuming that, for some $n \geq 1$, formulas (7) and (8) hold for all the terms on the right hand side of (11) and (12), we obtain

$$
\begin{aligned}
x_{n+3} &= x_1^{-1}(s_n + x_2 x_{n+2}) \\
&= x_1^{-n-1} x_2^{-n} \Big(\sum_{q+r \leq n} \binom{n-r}{q} \binom{n-q}{r} x_1^{2q} x_2^{2r} \\
&\quad + (x_2^{2(n+1)} + \sum_{q+r \leq n-1} \binom{n-1-r}{q} \binom{n-q}{r} x_1^{2q} x_2^{2(r+1)}) \Big) \\
&= x_1^{-n-1} x_2^{-n} \Big(x_2^{2(n+1)} + \sum_{q+r \leq n} \binom{n-r}{q} \Big(\binom{n-q}{r} + \binom{n-q}{r-1} \Big) x_1^{2q} x_2^{2r} \Big) \\
&= x_1^{-n-1} x_2^{-n} \Big(x_2^{2(n+1)} + \sum_{q+r \leq n} \binom{n-r}{q} \binom{n+1-q}{r} x_1^{2q} x_2^{2r} \Big),
\end{aligned}
$$

and

$$
\begin{aligned}
s_n &= x_2^{-1}(x_{n+2} + x_1 s_{n-1}) \\
&= x_1^{-n} x_2^{-n} \Big(x_2^{2n} + \sum_{q+r \leq n-1} \binom{n-1-r}{q} \binom{n-q}{r} x_1^{2q} x_2^{2r} \\
&\quad + \sum_{q+r \leq n-1} \binom{n-1-r}{q} \binom{n-1-q}{r} x_1^{2(q+1)} x_2^{2r} \Big) \\
&= x_1^{-n} x_2^{-n} \sum_{q+r \leq n} \Big(\binom{n-1-r}{q} + \binom{n-1-r}{q-1} \Big) \binom{n-q}{r} x_1^{2q} x_2^{2r} \\
&= x_1^{-n} x_2^{-n} \sum_{q+r \leq n} \binom{n-r}{q} \binom{n-q}{r} x_1^{2q} x_2^{2r},
\end{aligned}
$$

as desired.

References

[1] P. Caldero, F. Chapoton, Cluster algebras as Hall algebras of quiver representations, *Comment. Math. Helv.* **81** (2006), 595-616.

[2] P. Caldero, A. Zelevinsky, Laurent expansions in cluster algebras via quiver representations, *Moscow Math. J.* **6** (2006), 411-429.

[3] S. Fomin and A. Zelevinsky, Cluster algebras I: Foundations, *J. Amer. Math. Soc.* **15** (2002), 497–529.

[4] S. Fomin, A. Zelevinsky, Y-systems and generalized associahedra, *Ann. in Math.* **158** (2003), 977–1018.

[5] G. Musiker, J. Propp, Combinatorial interpretations for rank-two cluster algebras of affine type, *Electron. J. Combin.* **14** (2007), R15.

[6] P. Sherman, A. Zelevinsky, Positivity and canonical bases in rank 2 cluster algebras of finite and affine types, *Moscow Math. J.* **4** (2004), 947–974.