Transversal and cotransversal matroids via their representations.

Federico Ardila*

Submitted: May 23, 2006; Accepted: Feb 27, 2007; Published: Mar 5, 2007
Mathematics Subject Classification: 05B35; 05C38; 05A99

Abstract.

It is known that the duals of transversal matroids are precisely the strict gammoids. We show that, by representing these two families of matroids geometrically, one obtains a simple proof of their duality.

0

This note gives a new proof of the theorem, due to Ingleton and Piff [3], that the duals of transversal matroids are precisely the strict gammoids. Section 1 defines the relevant objects. Section 2 presents explicit representations of the families of transversal matroids and strict gammoids. Section 3 uses these representations to prove the duality of these two families.

1

Matroids and duality. A *matroid* $M = (E, \mathcal{B})$ is a finite set E, together with a non-empty collection $\mathcal{B}$ of subsets of E, called the *bases* of M, which satisfy the following axiom: If B_1, B_2 are bases and e is in $B_1 - B_2$, there exists f in $B_2 - B_1$ such that $(B_1 - e) \cup f$ is a basis.

If $M = (E, \mathcal{B})$ is a matroid, then $\mathcal{B}^* = \{E - B \mid B \in \mathcal{B}\}$ is also the collection of bases of a matroid $M^* = (E, \mathcal{B}^*)$, called the *dual* of M.

Representable matroids. Matroids can be thought of as providing a combinatorial abstraction of linear independence. If V is a set of vectors in a vector space and $\mathcal{B}$ is the

*federico@math.sfsu.edu. Dept. of Mathematics, San Francisco State University, San Francisco, CA, USA.
Supported by NSF grant DMS-9983797.

collection of maximal linearly independent subsets of V, then $M = (V, \mathcal{B})$ is a matroid. Such a matroid is called *representable*, and V is called a *representation* of M.

Transversal matroids. Let $A_1, \ldots, A_r$ be subsets of $[n] = \{1, \ldots, n\}$. A *transversal* (or *system of distinct representatives*) of $(A_1, \ldots, A_r)$ is an r-subset of $[n]$ whose elements can be labelled $\{e_1, \ldots, e_r\}$ in such a way that e_i is in A_i for each i. The transversals of $(A_1, \ldots, A_r)$ are the bases of a matroid on $[n]$.

Such a matroid is called a *transversal matroid*, and $(A_1, \ldots, A_r)$ is called a *presentation* of the matroid. This presentation can be encoded in the bipartite graph H with "top" vertex set $T = [n]$, "bottom" vertex set $B = \{\widehat{1}, \ldots, \widehat{r}\}$, and an edge joining j and $\widehat{i}$ whenever j is in A_i. The transversals are the r-sets in T that can be matched to B. We will denote this transversal matroid by $M[H]$.

Strict gammoids. Let G be a directed graph with vertex set $[n]$, and let $A = \{v_1, \ldots, v_r\}$ be a subset of $[n]$. We say that an r-subset B of $[n]$ *can be linked to* A if there exist r vertex-disjoint directed paths, each of which has its initial vertex in B and its final vertex in A. Each individual path is allowed to have repeated vertices and edges. Such a collection of r paths is called a *routing* from B to A. The r-subsets which can be linked to A are the bases of a matroid denoted $L(G, A)$. We can assume that the vertices in A are sinks of G; that is, that there are no edges coming out of them. This is because the removal of those edges does not affect the matroid $L(G, A)$.

The matroids that arise in this way are called *strict gammoids* or *cotransversal matroids*. The purpose of this note is to give a new proof of the following theorem, due to Ingleton and Piff.

Theorem 1. [3] Strict gammoids are precisely the duals of transversal matroids.

2

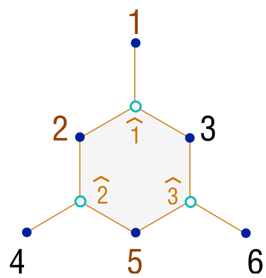
Let $\mathbb{K}$ be the field of fractions of the ring of formal power series in the indeterminates x_{ij} indexed by $1 \leq i \leq r$ and $1 \leq j \leq n$. We now show how transversal matroids and strict gammoids can be represented over $\mathbb{K}$.[1]

A representation of transversal matroids. Let M be a transversal matroid on the set $[n]$ with presentation $(A_1, \ldots, A_r)$. Let X be the $r \times n$ matrix whose (i, j) entry is $-x_{ij}$ if $j \in A_i$ and is 0 otherwise. The columns of X are a representation of M in the vector space $\mathbb{K}^r$. To see this, consider the columns $j_1, \ldots, j_r$. They are independent when their determinant is non-zero, and this happens as soon as one of the $r!$ summands of the determinant is non-zero. But $\pm X_{\sigma_1 j_1} \cdots X_{\sigma_r j_r}$ (where σ is a permutation of $[r]$) is non-zero if and only if $j_1 \in A_{\sigma_1}, \ldots, j_r \in A_{\sigma_r}$. So the determinant is non-zero if and only if $\{j_1, \ldots, j_r\}$ is a transversal.

We will find it convenient to choose a transversal $j_1 \in A_1, \ldots, j_r \in A_r$ at the outset, and normalize the rows to have the (i, j_i) entry be $-x_{ij_i} = 1$ for $1 \leq i \leq r$.

[1] It is possible to carry out the same constructions over $\mathbb{R}$, but special care is required to handle the issue of convergence of the infinite sums that will arise.

Example 1. Let $n = 6$ and $A_1 = \{1, 2, 3\}, A_2 = \{2, 4, 5\}, A_3 = \{3, 5, 6\}$. The corresponding bipartite graph H is shown below.



If we choose the transversal $1 \in A_1, 2 \in A_2, 3 \in A_3$, we obtain a representation for the transversal matroid $M[H]$, given by the columns of the following matrix:

$$X = \begin{pmatrix} 1 & -a & -b & 0 & 0 & 0 \\ 0 & 1 & 0 & -c & -d & 0 \\ 0 & 0 & 1 & 0 & -e & -f \end{pmatrix}$$

A representation of strict gammoids. Let $M = L(G, A)$ be a strict gammoid. Say G has vertex set $[n]$, and assume $A = \{r + 1, \ldots, n\}$. Any edge $i \to j$ of G has $i \leq r$, so we can assign to it weight x_{ij}. Let the weight of a path in G be the product of the weights on its edges. For each vertex i of G and each sink a in A, let p_{ia} be the sum of the weights of all paths in G which start at vertex i and end at sink a. We allow paths with repeated vertices and edges in this sum, so there may be infinitely many such paths; however, the number of paths of a given weight is finite, so p_{ia} is a well-defined element of $\mathbb{K}$.[2]

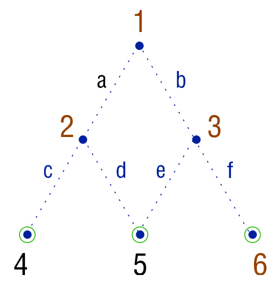
Let Y be the $(n - r) \times n$ matrix whose (a, i) entry is p_{ia}. The columns of Y are a representation of M. To see this, recall the *Lindström-Gessel-Viennot theorem*, which states that the determinant of the submatrix with columns $i_1, \ldots, i_{n-r}$ is equal to the signed sum of the weights of the routings from $\{i_1, \ldots, i_{n-r}\}$ to A.[3]

Notice that two routings R_1 and R_2 having the same weight must have the same multiset of edges. They can only differ in the order in which the kth path of R_1 and the kth path of R_2 traverse the same multiset of edges, as they go from i_k to j_k. But then R_1 and R_2 will also have the same sign. We conclude that there is no cancellation in the signed sum under consideration. Therefore the determinant of the submatrix with columns $i_1, \ldots, i_{n-r}$ is non-zero if and only if the signed sum is non-empty; that is, if and only if $\{i_1, \ldots, i_{n-r}\}$ can be linked to A.

Example 2. Consider the directed graph G shown below, where all edges point down, and the set $A = \{4, 5, 6\}$.

[2] In fact, p_{ia} is a rational function in the x_{ij}s. For a proof, see [10, Theorem 4.7.2].

[3] If the kth path in a routing starts at i_k and ends at j_k, then the sign of the routing is the sign of the permutation $j_1 \ldots j_{n-r}$ of A.



The representation we obtain for the strict gammoid $L(G, A)$ is given by the columns of the following matrix:

$$Y = \begin{pmatrix} ac & c & 0 & 1 & 0 & 0 \\ ad+be & d & e & 0 & 1 & 0 \\ bf & 0 & f & 0 & 0 & 1 \end{pmatrix}$$

Representations of dual matroids. If a rank-r matroid M is represented by the columns of an $r \times n$ matrix A, we can think of M as being represented by the r-dimensional subspace $V = \text{rowspace}(A)$ in $\mathbb{K}^n$. The reason is that, if we consider any other $r \times n$ matrix A' with $V = \text{rowspace}(A')$, the columns of A' also represent M.

This point of view is very amenable to matroid duality. If M is represented by the r-dimensional subspace V of $\mathbb{K}^n$, then the dual matroid M^* is represented by the $(n-r)$-dimensional orthogonal complement $V^\perp$ of $\mathbb{K}^n$.

Notice that the rowspaces of the matrices X and Y in the examples above are orthogonally complementary. That is, essentially, the punchline of this story.

3

Directed graphs with sinks and bipartite graphs with complete matchings. Given a directed graph G and a subset A of its set of sinks, we construct an undirected graph H as follows. We first split each vertex v not in A into a "top, incoming" vertex v and a "bottom, outgoing" vertex $\widehat{v}$, and draw an edge between them. Then we replace each edge $u \to v$ of G with an edge between the outgoing $\widehat{u}$ and the incoming v.

More concretely, given a directed graph G with vertex set V, and given a set A of sinks of G, we construct a bipartite graph H, together with a fixed bipartition and a fixed complete matching. The top vertex set in the bipartition is V, and the bottom vertex set is a copy $\widehat{V} - \widehat{A}$ of $V - A$. The complete matching is obtained by joining the top u and the bottom $\widehat{u}$ for each u in $V - A$. Then, for $u \neq v$, we join the bottom $\widehat{u}$ and the top v in H if and only if $u \to v$ is an edge of G.

Conversely, if we are given the bipartite graph H, a bipartition of H [4] and a complete matching of H, it is clear how to recover G and A. The resulting G and A will depend on which bipartition and matching are used. Observe that if we start with the directed graph G and sinks A of Example 2, we obtain the bipartite graph H of Example 1.

[4] which is unique if H is connected

Having laid the necessary groundwork, we can now present our proof of Theorem 1.

Proof of Theorem 1: Duality of transversal matroids and strict gammoids. We constructed a correspondence between a directed graph G with a specified subset A of its set of sinks, and a bipartite graph H with a specified bipartition and a specified complete matching. Now we show that, in this correspondence, the strict gammoid $L(G,A)$ is dual to the transversal matroid $M[H]$. We have constructed a subspace of $\mathbb{K}^n$ representing each one of them. By the remarks at the end of Section 2, it suffices to show that these two subspaces are orthogonally complementary, as observed in Examples 1 and 2.

Our representation of $M[H]$ is given by the columns of the $r \times n$ matrix X whose (i,i) entry is 1, and whose (i,j) entry, for $i \neq j$, is $-x_{ij}$ if $i \to j$ is an edge of G and is 0 otherwise. Think of the x_{ij}s as weights on the edges of G. For a vector $y \in \mathbb{K}^n$, the ith entry of the column vector Xy is $y_i - \sum_{j \in N(i)} x_{ij} y_j$, where the sum is over the set $N(i)$ of vertices j such that $i \to j$ is an edge of G. It follows that y is in the nullspace of X when, for each vertex i of G,

$$y_i = \sum_{j \in N(i)} x_{ij} y_j.$$

As before, let p_{ia} be the sum of the weights of the paths from i to a in G. Now we observe that

$$p_{ia} = \sum_{j \in N(i)} x_{ij} p_{ja}.$$

To see this, notice that the left hand side enumerates all paths from i to a, and the right hand side enumerates the same paths by grouping them according to the first vertex j that they visit after i. Therefore $(p_{1a}, \ldots, p_{na})$, the ath row of our representation Y of $L(G,A)$, is in the nullspace of X. Since each row of Y is in the nullspace of X, $\text{rowspace}(Y) \subseteq \text{nullspace}(X)$. But

$$\begin{aligned}
\dim(\text{rowspace}(Y)) &= \text{rank}(L(G,A)) = n - r, \text{ and} \\
\dim(\text{nullspace}(X)) &= n - \dim(\text{rowspace}(X)) = n - \text{rank}(M[H]) = n - r,
\end{aligned}$$

so in fact these two subspaces are equal. It follows that $\text{rowspace}(X)$ and $\text{rowspace}(Y)$ are orthogonal complements. This completes our proof of Theorem 1. $\square$

4

For more information on matroid theory, Oxley's book [8] is a wonderful place to start. The representation of transversal matroids shown here is due to Mirsky and Perfect [7]. The representation of strict gammoids that we use was constructed by Mason [6] and further explained by Lindström.[5][5]

This note is a small side project of [1]. While studying the combinatorics of generic flag arrangements and its implications on the Schubert calculus, we became interested in

[5]It is in this context that he discovered what is now commonly known as the Lindström-Gessel-Viennot theorem [2]. This theorem was also discovered and used earlier by Karlin and MacGregor [4].

the strict gammoid of Example 2 and its representations; we proved that it is the matroid of the arrangement of lines determined by intersecting three generic complete flags in $\mathbb{C}^3$. Similarly, the analogous strict gammoid in a triangular array of size n is the matroid of the line arrangement determined by three generic flags in $\mathbb{C}^n$.

I would like to thank Sara Billey for several helpful discussions, and Laci Lovász and Jim Oxley for help with the references. I also thank the referee for several suggestions which improved the presentation.

References

[1] F. Ardila and S. Billey. Flag arrangements and triangulations of products of simplices. Preprint, `math.CO/0605598`, 2006.

[2] I. Gessel and X. Viennot. Binomial determinants, paths and hook formulae. *Adv. Math* **58** (1985) 300-321.

[3] A. Ingleton and M. Piff. Gammoids and transversal matroids. *J. Combinatorial Theory Ser. B* **15** (1973) 51-68.

[4] S. Karlin and G. MacGregor. Coincidence probabilities. *Pacific J. Math.* **9** (1959) 1141-1164.

[5] B. Lindström. On the vector representations of induced matroids. *Bull. London Math. Soc.* **5** (1973) 85-90.

[6] J. Mason. On a class of matroids arising from paths in graphs. *Proc. London Math. Soc.* (3) **25** (1972) 55-74.

[7] L. Mirsky and H. Perfect. Applications of the notion of independence to problems of combinatorial analysis. *J. Combinatorial Theory* **2** (1967) 327-357.

[8] J. G. Oxley. *Matroid theory.* Oxford University Press. New York, 1992.

[9] M. J. Piff and D. J. A. Welsh. On the vector representation of matroids. *J. London Math. Soc. (2)* **2** 284-288.

[10] R. P. Stanley. *Enumerative combinatorics, vol. 1.* Cambridge University Press. Cambridge, 1997.