Clusters in a multigraph with elevated density

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Abstract

In this paper, we prove that in a multigraph whose density Γ exceeds the maximum vertex degree Δ, the collection of minimal clusters (maximally dense sets of vertices) is cycle-free. We also prove that for multigraphs with $\Gamma > \Delta+1$, the size of any cluster is bounded from the above by $(\Gamma-3)/(\Gamma-\Delta-1)$. Finally, we show that two well-known lower bounds for the chromatic index of a multigraph are equal.

1 Introduction

The **chromatic index** $\chi'(G)$ of a multigraph $G(V,E)$ is the minimal number of colors needed to color all edges of G so that no two edges incident to the same vertex have the same color. A trivial lower bound for $\chi'(G)$ is

$$\Delta(G) \leq \chi'(G),$$

where $\Delta(G)$ is the maximal vertex degree in G. A remarkable result discovered by Vizing (see [16]) gives the upper bound $\chi' = \chi'(G) \leq \Delta(G) + p(G)$, where $p(G)$ is the maximal number of parallel edges in G. Thus, for multigraphs without parallel edges (graphs), there are just two possible values for χ': either Δ, or $\Delta + 1$.

For general multigraphs, $p(G) \geq 1$, Shannon proved in [14] that $\chi'(G) \leq \lfloor (3\Delta)/2 \rfloor$, which, taking Vizing's bound into account, is strengthened to $\chi' \leq \Delta + \min\{p, \lfloor \Delta/2 \rfloor\}$. The basic question in multigraph edge-coloring is: "*what properties of a multigraph cause its chromatic index* χ' *to exceed* Δ?" Although to decide if $\chi'(G) = \Delta(G)$ is NP-complete, as proved by Hoyler ([8]), it is suspected that for multigraphs with $\chi'(G) > \Delta(G) + 1$, $\chi'(G)$ can be completely characterized in terms of their **density** $\Gamma(G)$, defined by

$$\Gamma(G) = \max_{H \subseteq G} \lceil \frac{e(H)}{\lfloor v(H)/2 \rfloor} \rceil,$$

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where H is a sub-multigraph of G of order at least two and $v(H)$ (resp. $e(H)$) denotes the number of vertices (resp. edges) in H. It is easy to see that, for every multigraph G,

$$\Gamma(G) \leq \chi'(G). \tag{1}$$

Currently, no multigraph is known with $\chi'(G) > \max(\Delta + 1, \Gamma)$. Conjectures connecting the chromatic index, maximal degree, and the density of a multigraph were independently proposed by Goldberg ([6]) and Seymour ([13]) more than 25 years ago (see also [9]).

Conjecture 1 (Goldberg ([6]) For every multigraph G, if $\chi'(G) > \Delta + 1$, then $\chi'(G) = \Gamma(G)$.

Conjecture 2 (Seymour [13]) For every multigraph G, $\chi(G) \leq \max\{\Delta(G), \Gamma(G)\} + 1$.

An extension of the conjectures above was proposed by Goldberg in [7]:

Conjecture 3 If $\Delta \neq \Gamma$, then $\chi' = \max\{\Delta, \Gamma\}$, else $\chi' \leq \Delta + 1$.

Since all three conjectures are closely related to each other, we globally refer to them as the GS-conjecture. See [2, 7, 10, 11, 15, 5] for some results towards the conjecture; in particular, Nishizeki and Kashiwagi ([11]) proved $\chi' = \Gamma$ for multigraphs with $\chi' > (11\Delta + 8)/10$, and Favrholdt, Steibitz, and Toft, ([5]) proved $\chi' = \Gamma$ for multigraphs with $\chi' > (13\Delta + 10)/12$; the latter is based on the Tashkinov's result from [15].

The GS-conjecture motivates the study of the multigraphs with $\Gamma > \Delta$; we call them multigraphs with elevated density. The properties of the multigraphs with elevated density presented here are formulated in terms of two new notions: set-cycles and multigraph clusters.

Definition 1 *A sequence $\mathcal{S} = \{S_i\}_{i=1}^k$ of sets is called a set-cycle if*

$$\forall i \in [1, k], S_i \cap S_{i+1} \neq \emptyset \ \& \ S_i \cap S_{i+1} \cap S_{i+2} = \emptyset.$$

Here $S_{k+1} = S_1$ and $S_{k+2} = S_2$.

A collection $\mathcal{T} = \{T_j\}_{j=1}^m$ is called a set-forest, if no sequence of sets from $\mathcal{T}$ is a set-cycle.

Definition 2 *Given a multigraph $G(V, E)$, a set $S \subseteq V$ is called maximally dense, or a cluster, if $e(S) > (\Gamma - 1)\lfloor |S|/2 \rfloor$. A cluster S is called minimal if no proper subset of S is a cluster.*

Thus, the clusters are subgraphs for which the lower bound for $\chi'(G)$ is achieved. Note that the notion of a cluster is close to that of an *overfull* graph introduced by A.J.W. Hilton. A simple graph is called *overfull*, if $|E(G)| > \Delta(H)\lfloor |V(H)|/2 \rfloor$. Clearly, if G contains an overfull subgraph H with $\Delta(G) = \Delta(H)$, then $\chi'(G) = \Delta(G) + 1$. A.J.W. Hilton asked if the reversed is true for graphs with $\Delta(G) > |V(G)|/3$ (see [3, 4] and [9] for the history of the question).

Our main result (Section 2) establishes that the collection of minimal clusters in a multigraph with $\Gamma > \Delta$ has a simple structure: it is a set-forest. We also prove that in a multigraph with $\Gamma > \Delta + 1$, the size of any cluster is bounded by a function which depends on Γ and Δ only (not on the number of the vertices of the multigraph). This bound matches the upper bound of the size of a critical multigraph which was proved in [6] under the assumption of the GS-conjecture.

A lower bound for $\chi'(G)$, which is sometimes stronger than $\Gamma(G)$, can be formulated in terms of maximum matchings of subgraphs of G.

Definition 3 *Let $F \subseteq E$ and let $m(F)$ denote the maximal size of a matching comprised of edges in F. Then,*

$$\Omega(G) = \max_{F \subseteq E(G)} \lceil \frac{|F|}{m(F)} \rceil.$$

It is easy to see that

$$\Gamma(G) \leq \Omega(G) \leq \chi'(G). \tag{2}$$

A star is an example of a multigraph with $\Gamma(G) < \Omega(G)$. If there were multigraphs with $\Delta \leq \Gamma < \Omega$, the GS-conjecture would be disproved. However, in Section 3, we use Tutte's matching theorem to prove that for every multigraph G,

$$\Omega(G) = \max\{\Delta(G), \Gamma(G)\}. \tag{3}$$

The notion of $\Omega(G)$ is close to that of the *fractional edge chromatic number* χ'_f introduced by Berge in [1] (see also Chapter 4 in [12]):

Definition 4 *A fractional edge coloring of G is an assignment of a non-negative weight w_M to each matching M in G so that for every edge $e \in E(G)$, $\sum_{M: e \in M} w_M \geq 1$. The fractional edge chromatic number, $\chi'_f(G)$, is then defined by*

$$\chi'_f(G) = \min_M \sum_M w_M,$$

Using Edmond's matching polytop theorem, Scheinerman and Ullman ([12]) derived

$$\chi'_f(G) = \max\{\Delta(G), \max_{H \subseteq G, |V(H)| \geq 2} \frac{e(H)}{\lfloor v(H)/2 \rfloor}\} \tag{4}$$

Thus, $\chi'_f(G) \leq \Omega(G) \leq \chi'(G)$, and for multigraphs with $\chi'_f > \Delta(G)$, $\Omega(F) = \lceil \chi'_f(G) \rceil$. Note that our proof of (3) is significantly simpler than that of (4).

The following notations are used in this paper. Given a set $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S. If $F \subseteq E(G)$, then $G[F]$ denotes the subgraph of G induced by F: the vertex set of $G[F]$ is the set of vertices incident to the edges in F, and the set of edges of $G[F]$ is set F. Unless otherwise specified, $deg(x)$ denotes the degree of a vertex x in G; given $S, T \subseteq V(G)$, $deg(S,T)$ denotes the number of edges xy such that $x \in S$ and $y \in T$; $deg_S(x)$ denotes the degree of x in the subgraph $G[S]$ induced on S; $\delta_S(x) = deg(x) - deg_{G[S]}(x)$; $\delta(S) = \sum_{x \in S} \delta_S(x) = deg(S, V(G) - S)$; $\nabla(x) = \Delta - deg(x)$; and $\nabla(S) = \sum_{x \in S} \nabla(x)$. See [17] for undefined notations.

2 Topology of minimal clusters.

The goal of this section is to establish several structural properties of the set of minimal clusters in a multigraph G with $\Gamma = \Gamma(G) > \Delta(G) = \Delta$. We also give an upper bound for the size of any cluster in a multigraph with $\Gamma(G) > \Delta + 1$; it turns out that for such multigraphs, the cluster size is bounded from the above by a function depending on Δ and Γ only. Throughout this section, G is a multigraph with $\Gamma(G) > \Delta(G)$. If $S \subseteq V(G)$, then $e(S)$ denotes $e(G[S])$. The first lemma is a simple extension of the standard inequality $2e(S) \leq \Delta|S|$.

Lemma 1 *For every subset $S \subseteq V(G)$,*

$$\delta(S) + \nabla(S) + 2e(S) = \Delta|S|. \tag{5}$$

Proof. The result follows from

$$\begin{aligned} 2e(S) &= \sum_{x \in V(S)} deg_S(x) = \sum_{x \in V(S)} (deg(x) - \delta_S(x)) \\ &= \sum_{x \in V(S)} (\Delta - \nabla(x) - \delta_S(x)) \\ &= |V(S)|\Delta - \nabla(S) - \delta(S). \qquad \blacksquare \end{aligned}$$

Lemma 2 *The cardinality of every cluster S in G is odd.*

Proof. If $|S|$ were even, then the defining inequality $(\Gamma - 1)\lfloor |S|/2 \rfloor < e(S)$ could be rewritten as

$$(\Gamma - 1)|S| < 2e(S).$$

Since $2e(S) \leq \Delta|S|$, it would imply

$$(\Gamma - 1)|S| < 2e(S) \leq \Delta|S|,$$

which contradicts our assumption $\Delta < \Gamma$. $\blacksquare$

Lemma 3 *For every cluster S,*

$$\delta(S) + \nabla(S) \leq \Delta - 2 - (\Gamma - \Delta - 1)(|S| - 1). \tag{6}$$

Proof. Since $|S|$ is odd,

$$(\Gamma - 1)\lfloor \frac{|S|}{2} \rfloor = (\Gamma - 1)\frac{|S| - 1}{2} < e(S).$$

This implies $(\Gamma - 1)\frac{|S|-1}{2} + 1 \leq e(S)$, which, in turn, yields

$$(\Gamma - 1)(|S| - 1) + 2 \leq 2e(S).$$

Combining the latter with $\Delta|S| - \nabla(S) - \delta(S) = 2e(S)$ (Lemma 1), we have

$$(\Gamma - 1)(|S| - 1) + 2 \leq \Delta|S| - \nabla(S) - \delta(S),$$

which is equivalent to (6). $\blacksquare$

Lemma 4 *For any two minimal clusters S_1 and S_2 of a multigraph G, if $S_1 \cap S_2 \neq \emptyset$, then $|S_1 \cap S_2|$ is odd.*

Proof. Let us assume that $|S_1 \cap S_2| = 2a$, where a is a positive integer. Let $A = S_1 \cap S_2$; $e_o = e(G[A])$; $2p_i + 1 = |S_i|$; $e_i = e(G[S_i - A])$, and $w_i = deg(A, S_i - A)$ $(i = 1, 2)$. By definition of a cluster, $(\Gamma - 1)p_i < e_i + w_i + e_0$ $(i = 1, 2)$, implying

$$(\Gamma - 1)(p_1 + p_2) < e_1 + e_2 + w_1 + w_2 + 2e_0. \tag{7}$$

Since $|S_i - A| = 2p_i - 2a + 1$, by the minimality of cluster S_i, $e_i \leq (p_i - a)(\Gamma - 1)$ $(i = 1, 2)$, hence

$$e_1 + e_2 \leq (p_1 + p_2 - 2a)(\Gamma - 1) = (p_1 + p_2)(\Gamma - 1) - 2a(\Gamma - 1). \tag{8}$$

By Lemma 1, $w_1 + w_2 + 2e_0 \leq 2a\Delta \leq 2a(\Gamma - 1)$. Plugging it into (8), we obtain

$$e_1 + e_2 + w_1 + w_2 + 2e_0 \leq (\Gamma - 1)(p_1 + p_2),$$

which contradicts inequality (7). ▮

It is easy to construct examples of minimal clusters that intersect. The multigraph in Figure 1 shows that the intersection of two minimal clusters can have more than one vertex.

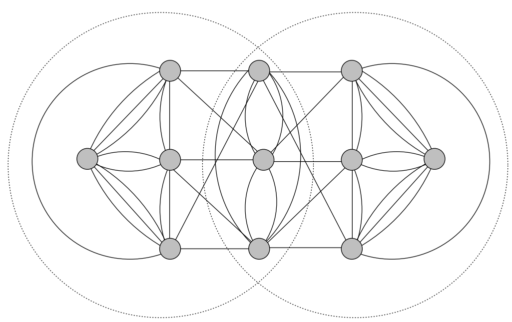

Figure 1: The intersection of two minimal clusters within dotted circles consists of three vertices; for the multigraph, $\Gamma = 9$; and $\Delta = 8$.

Theorem 1 *The set $\mathcal{T} = \{S_i\}_{i=1}^m$ of all minimal clusters in a multigraph G is a set-forest.*

Proof. Suppose that, contrary to the statement, there is a set-cycle $\{S_i\}_{i=1}^k$ all of whose sets are minimal clusters in G. Let $A_i = S_i \cap S_{i-1}$ and $B_i = S_i - A_i - A_{i+1}$ $(i \in [1, k])$. As before, we use indices "cyclically": $A_1 = S_1 \cap S_k$ and $B_k = S_k - A_k - A_1$.

Since $|S_i|$ and $|A_i|$ are odd (Lemmas 2 and 4), and $A_i \cap A_{i+1} = \emptyset$ (from the definition of a set-cycle), it follows that $|B_i| = |S_i| - |A_i| - |A_{i+1}|$ is also odd $(i \in [1, k])$.

Let $|A_i| = 2a_i + 1$, $|B_i| = 2b_i + 1$, $w_i^+ = deg(A_i, S_i - A_i)$, and $w_i^- = deg(A_i, S_{i-1} - A_i)$ $(i \in [1, k])$. Clearly,

$$e(S_i) \leq e(A_i) + e(B_i) + e(A_{i+1}) + w_i^+ + w_{i+1}^-.$$

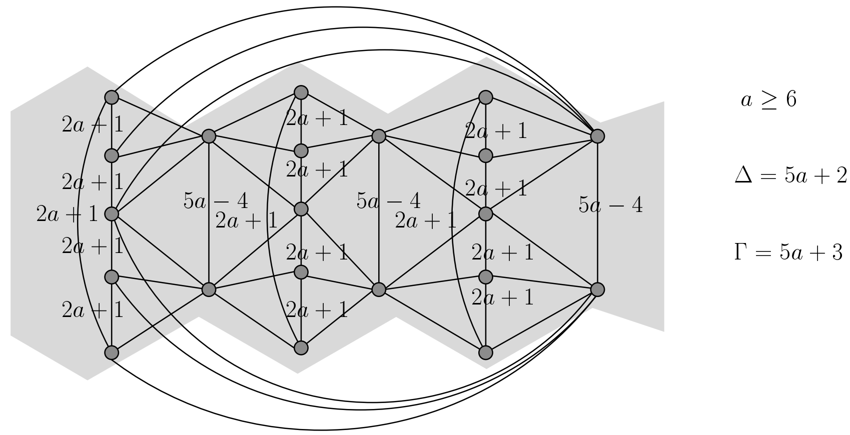

Figure 2: This multigraph has a set-cycle composed of non-minimal clusters. The labels on the edges indicate their multiplicities; the shaded 6-gons indicate clusters; the two right-most vertices belong to two clusters.

Since A_i and B_i are proper subsets of minimal clusters.

$$e(A_i) \leq (\Gamma - 1)a_i \text{ and } e(B_i) \leq (\Gamma - 1)b_i \ (i \in [1, k]).$$

Thus,

$$\begin{aligned} \sum_{i=1}^{k} e(S_i) &\leq \sum_{i=1}^{k}((\Gamma-1)a_i + (\Gamma-1)b_i + (\Gamma-1)a_{i+1}) + \sum_{i=1}^{k}(w_i^+ + w_{i+1}^-) \\ &= (\Gamma-1)\sum_{i=1}^{k}(2a_i + b_i) + \sum_{i=1}^{k}(w_i^- + w_{i+1}^+). \end{aligned}$$

Since, $w_i^- + w_{i+1}^+ \leq \delta(S_i)$ and, by Lemma 6, $\delta(S_i) \leq \Delta - 2 \leq \Gamma - 1$, after some simplifications we have

$$\sum_{i=1}^{k} e(S_i) \leq (\Gamma-1)\sum_{i=1}^{k}(2a_i + b_i + 1). \tag{9}$$

On the other hand, by the definition of clusters,

$$e(S_i) > (\Gamma-1)\lfloor \frac{2a_i + 1 + 2a_{i+1} + 1 + 2b_i + 1}{2} \rfloor = (\Gamma-1)(a_i + a_{i+1} + b_i + 1),$$

which implies

$$(\Gamma-1)\sum_{i=1}^{k}(2a_i + b_i + 1) < \sum_{i=1}^{k} e(S_i). \tag{10}$$

Inequality (10) contradicts inequality (9), proving the correctness of the theorem. ∎

Theorem 2 *For every cluster S of a multigraph with $\Gamma > \Delta + 1$,*

$$|S| \leq \frac{\Gamma - 3}{\Gamma - 1 - \Delta}.$$

Proof. Express inequality (6) with respect to $|S|$ and then use $0 \leq \delta(S) + \nabla(S)$. ∎

3 Lower bounds

Although, for some multigraphs, Ω is a stronger lower bound for χ' than Γ, it turns out that is it not stronger than Δ and Γ combined.

Lemma 5 *For any multigraph G,*

$$\max(\Gamma(G), \Delta(G)) \leq \Omega(G) \leq \chi'(G). \tag{11}$$

Proof. For any edge-coloring of G and any $F \subseteq E(G)$, the number of edges colored the same color does not exceed $m(H) \leq \lfloor |V(H)/2 \rfloor$, where $H = G[F]$. Hence,

$$\chi'(H) \geq \lceil \frac{|F|}{m(F)} \rceil \geq \lceil \frac{e(H)}{\lfloor \frac{|V(H)}{2} \rfloor} \rceil.$$

To complete the proof, notice that if x is a vertex of the maximal degree in G and F is the set of edges incident to x, then $m(F) = 1$, implying that $\Delta(G) \leq \Omega(G)$. ∎

Theorem 3 *For every multigraph G, $max(\Gamma(G), \Delta(G)) = \Omega(G)$.*

Proof. By Lemma 5, we only need to prove that $\max(\Gamma(G), \Delta(G)) \geq \Omega(G)$. Let F be a set of edges for which

$$\lceil \frac{|F|}{m(F)} \rceil = \Omega(G)$$

and let $H = G[F]$. If $m(F) \geq (|V(H)| - 1)/2$, the result follows immediately. Thus, we assume that

$$m(F) < \frac{|V(H)| - 1}{2} \text{ and } \Omega > \max(\Delta, \Gamma).$$

By Tutte's theorem ([17]), there is a subset $K \subseteq V(H)$ such that the number of odd connected components of $H - K$ is $q = k + n - 2m(F)$, where $n = |V(H)|$ and $k = |K|$.

Let $\{V_i, F_i\}_{i=1}^{q+t}$ be the connected components of $G - K$, where the first q of them are odd and the remaining t are even. Let $|V_i| = 2a_i + 1$, for $i \in [1, q]$ and $|V_i| = 2a_i$, for $i \in [q + 1, q + t]$. Let C_i be the set of edges of H with one endpoint in V_i and the other in K ($i \in [1, q + t]$). Finally, let $E(K)$ denote the set of edges in F with both end-points in K.

Using the assumption $\Omega > \max(\Delta, \Gamma)$, for every odd connected component,

$$\forall i \in [1, q], \quad |F_i| \leq \Gamma a_i \leq (\Omega - 1)a_i. \tag{12}$$

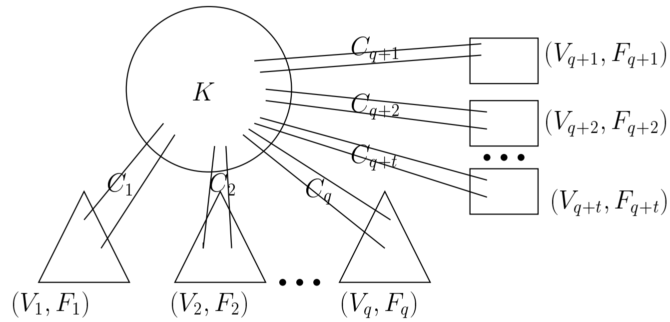

Figure 3: The triangles (resp. squares) represent odd (resp. even) components.

Since the maximum vertex degree is Δ,

$$\forall i \in [q+1, q+t], \quad |F_i| \leq \Delta a_i \tag{13}$$

$$|E(K)| + \sum_{i=1}^{q+t} |C_i| \leq 2|E(K)| + \sum_{i=1}^{q+t} |C_i| \leq \Delta\, k. \tag{14}$$

Combining (12), (13), (14), and using $\Omega - 1 \geq \Delta$ one more time, we get an upper bound for $|F|$:

$$|F| \leq |E(K)| + \sum_{i=1}^{q+t}(|F_i| + |C_i|) \leq \Delta k + (\Omega-1)\sum_{i=1}^{q} a_i + \Delta \sum_{i=q+1}^{q+t} a_i \leq (\Omega-1)(k + \sum_{i=1}^{q+t} a_i). \tag{15}$$

Since

$$n = \sum_{i=1}^{q}(2a_i + 1) + \sum_{i=q+1}^{q+t}(2a_i) + k,$$

we have

$$m(F) = \sum_{i=1}^{q+t} a_i + k.$$

Using this expression for $m(F)$ and inequality (15),

$$\Omega = \lceil \frac{|F|}{m(F)} \rceil \quad \leq \quad \frac{(\Omega-1)(\sum_{i=1}^{q+t} a_i + k)}{\sum_{i=1}^{q+t} a_i + k} = \Omega - 1.$$

The contradiction disproves the assumption and completes the proof. ∎

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References

[1] C. Berge. Fractional Graph Theory. *ISI Lecture Notes 1, Macmillan of India*, 1978.

[2] D. Cariolaro. On fans in multigraphs. *Journal of Graph Theory*, 51(4):301–318, 2006.

[3] A. Chetwynd and A. Hilton. Star multigraphs with three vertices of maximum degree. *Math. Proc. Cambridge Philos. Soc.*, 100:303–317, 1986.

[4] A. Chetwynd and A. Hilton. The edge-chromatic class of graphs with maximum degree at least $|v| - 3$. in: L.D. Andersen, I.T.Jakobsen, C. Thomassen, B. Toft, and P.D. Vestergaard, editors, Graph Theory in Memory of G. A. Dirac, *Annals of Discrete Mathematics*, 100:303–317, 1989.

[5] L. Favrholdt, M. Steibitz, and B. Toft. Graph colouring: Vizing's Theorem and Goldberg's Conjecture. *IMADA preprint: PP - 2006-20*, 2006.

[6] M. K. Goldberg. On multigraphs of almost maximal chromatic class. *Discret. Analiz*, 23:3–7, 1973. In Russian.

[7] M. K. Goldberg. Edge coloring of multigraphs: Recoloring technique. *J. Graph Theory*, 8:123–137, 1984.

[8] I. Holyer. The NP-completeness of edge coloring. *SIAM J. Comput.*, 10:718–720, 1981.

[9] T. Jensen and B. Toft. Graph coloring problems. *Wiley-Interscience Publications, John Wiley & Sons*, 1995.

[10] L. Kahn. Asymptotics of the chromatic index for multigraphs. *J. Combin. Theory Ser. B*, 68:195–235, 1996.

[11] T. Nishizeki and K. Kashiwagi. On the 1.1 edge-coloring of multigraphs. *SIAM J. Discrete Math.*, 3:391–410, 1990.

[12] E. R. Scheinerman and D. H. Ullman. Fractional Graph Theory: A Rational Approach to the Theory of Graphs. *Wiley Interscience, New York*, 1997.

[13] P. D. Seymour. Some unsolved problems on one-factorizations of graphs. *Graph Theory and Related Topics, Academic Press*, 1979. Bondy and Murty, eds.

[14] C. E. Shannon. A theorem on coloring the lines of a network. *J. Math. Phys.*, 28:148–151, 1949.

[15] V. A. Tashkinov. On an algorithm for the edge coloring of multigraphs. *Discretn. Anal. Issled. Oper.*, 1(7):72–85, 2000. in Russian.

[16] V. G. Vizing. Critical graphs with a given chromatic class. *Discret. Analiz*, 5:9–17, 1965. in Russian.

[17] D. B. West. Introduction to graph theory. *Prentice Hall, Upper Saddle River, NJ*, 2003.